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Evaluation of Maciej Tadej's PhD thesis "Analysis and simulations of the non-linear models with non-local dispersal of species considered in finite, flat habitats"

This document represents my evaluation of Maciej Tadej's PhD thesis "Analysis and simulations of the non-linear models with non-local dispersal of species considered in finite, flat habitats", submitted to the University of Wroclaw, Poland.

Brief synopsis

The thesis presents a comprehensive analysis of nonlocal dispersal operators on finite spatial domains under both zero Dirichlet and zero Neumann boundary conditions. Following an introduction (Chapter 1) which sets the scene from both a mathematical and ecological perspective, a novel spectral theory for the nonlocal operators under consideration are developed in Chapter 2. Chapter 3 analytically shows existence and uniqueness of equilibrium solutions to models equipped with the nonlocal operators and used newly developed and tested (in Chapter 5) numerical methods alongside established numerical continuation algorithms to complement the analysis. Conditions for convergence to equilibria is analytically and numerically considered in Chapter 4, which establishes global in time existence and uniqueness of solutions and addresses how initial conditions affect model dynamics. Throughout the thesis, the Klausmeier model for dryland vegetation patterns is used as an example model and thus novel ecological insights are developed to complement the new mathematical theory.

My expertise and relation to the topic of the thesis

I am an expert in the numerical bifurcation analysis of (systems of) PDEs, including those with nonlocal terms. In particular, I have worked extensively with the Klausmeier model for dryland vegetation patterns, a novel version of which features heavily in this thesis. My work often features a combination of numerical integration and numerical continuation techniques, both of which are used in this thesis. I have little expertise in other topics of the thesis, such as functional analysis and numerical analysis.

Overall assessment

This is a comprehensive thesis which demonstrates deep knowledge of the subject matter. It is particularly impressive that the candidate has achieved expertise in a wide range of subfields, in

particular functional analysis, numerical analysis and mathematical biology to answer the fundamental question on the impact of nonlocal dispersal operators on finite domains.

To the best of my knowledge, results are novel and of high significance, at least from the mathematical biology angle which is my main area of expertise. Traditionally, models of dryland vegetation patterns have been posed on infinite domains with local spatial movement terms. Both properties are simplifying assumptions taken for mathematical convenience, which are hard to justify ecologically. Both assumptions have previously been relaxed in isolation, but I am not aware of any other work that relaxes both assumptions simultaneously. However, both assumptions ought to be relaxed in tandem due to the links between dispersal and boundary conditions, and I highly commend the candidate for doing so and providing comprehensive insights into the resulting systems.

Part of the thesis (Chapter 2) has been published already as a monograph in a high-quality peer-reviewed journal [1], and I expect the remaining content to be of publishable quality too.

Detailed comments

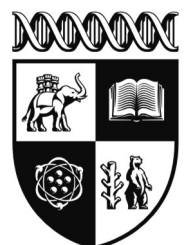
In this section, I state some detailed comments, grouped by chapters. I make many suggestions for improvements, but emphasise that these are all very minor problems. I hope that the candidate finds them useful when preparing the currently unpublished chapters for journal publications. The quantity of comments should not distract from my overwhelmingly positive general assessment of the thesis.

Chapter 1 -- Introduction:

The introduction is well written and sets the scene nicely for the chapters that follow without giving too much unnecessary detail. I particularly enjoyed reading the summary of the main contribution of the thesis at the end of the introduction. The introduction to the structure of the thesis and comments on notational conventions are also very helpful.

Below, I state some suggestions for minor improvements:

- I wonder what the relevance of Neumann boundary conditions for vegetation patterns is. I think this requires some further motivation.
- In the very first paragraph, the wrong Figure is referenced.
- In Definition 1.1, I disagree with the statement that a vegetation pattern is a “stationary solution”. While this may be true on flat ground, in general, vegetation patterns can also be represented by spatio-temporal patterns, such as periodic travelling waves.
- I was left a bit confused by the combination of Definitions 1.2 and 1.3. They are states as if critical patch size and critical aggregation threshold are independent of each other. However,



it appears that the critical patch size depends on the critical aggregation threshold and therefore on the initial condition. This should be clarified.

- In Assumption 1.2.1, it is assumed that the second moment of the kernel function is finite. In previous models, also kernels with infinite variance have been considered [2, 3]. I wonder if a short discussion on relaxing this assumption (here or elsewhere) would be beneficial.
- In (1.2.1), it could be worth explicitly stating that the domain is not $\mathbb{R}^n$. At first, I was confused why the second term does not simplify to $v(x)$.
- Statements such as those at the beginning of Section 1.4.2 should include references to the existing literature.

Chapter 2 – Nonlocal spectra theory

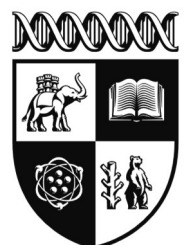
The chapter is well written and the analysis appears to be sound (I did not check all details as this chapter is far from my area of expertise). Previously, spectral theory for nonlocal dispersal operators focussed on infinite domains only. The characterisation of spectral properties on finite domains with either homogeneous Dirichlet or homogeneous Neumann boundary conditions is a highly valuable addition to the literature. The literature review at the beginning of the section is particularly helpful.

Below, I state some suggestions for minor improvements:

- There are some typos in this section, such as a missing full stop at the end of page 15, a missing “the” in the first sentence on page 17, and a missing “the” and “an” in the first sentence on page 18.
- The chapter investigates both the continuous and the discrete spectrum of the operator. Does the operator also feature a residual spectrum? Even if not, a short statement on this would be helpful. It would also be helpful if the notation and naming of different parts of the spectrum was clarified. For example, I was not quite sure whether “eigenvalues” are the same as the “point spectrum”, and whether the continuous spectrum, denoted by σ_c , is the same as the essential spectrum, denoted by σ_{ess} .
- In Assumption 2.3.2 and later, reference to the “interior sphere condition” is made. A clarification on the meaning of this term should be added.
- Given that theory exists for spectra of the operator on infinite domains, a brief discussion on the differences would be a valuable addition to the chapter.

Chapter 3 – Non-local stationary problems

This chapter is well-written and presents very comprehensive bifurcation analyses for both single-species and multi-species models. The combination of sophisticated analysis and numerical tools is highly commendable. In particular, it uses the Klausmeier model for vegetation patterns as a case study for numerical bifurcation analysis. As with Chapter 2, the analysis is far from



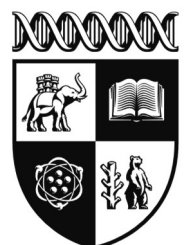
the area of expertise and I therefore did not check all details. Numerical bifurcation analysis is much closer to my expertise and therefore most comments focus on this part of the chapter.

My major confusion with this section relates to the applicability of results to the Klausmeier model for vegetation patterns. Analytical results (Lemma 3.3.3 and Proposition 3.3.2) require a sufficiently small value for the water diffusion parameter d_w . Consequently, some of the numerical analysis uses $d_w = 0.1$. I question the relevance of this parameter regime because, at least in the local Klausmeier model, the water diffusion coefficient is a large parameter that describes the ratio of the water diffusion coefficient to the plant diffusion coefficient (water “moves” much faster than plants). I think the connection between the analysis results and the Klausmeier model needs to be clarified. For Section 3.4.3, the parameter value suddenly changes to a much more realistic value of $d_w=80$. This makes more sense ecologically, but creates an inconsistency with the preceding analysis.

I also doubt the accuracy of the bifurcation diagrams shown in Fig. 3.4 and 3.5. First, it is not clear to me whether the diagrams show patterned solution branches (Conclusion 3.4.6 suggests so) or constant steady states (by contrast, Conclusion 3.4.5 suggests so). If the latter, I do not understand why the fold bifurcation does not occur exactly at $A=2B$. Moreover, in Figure 3.5, solution branches are self-intersecting, which requires further justification. More generally, I wonder whether there is multistability of different patterns for fixed rainfall. Conclusion 3.4.8 suggests that this is not the case. This would be a major difference to model dynamics on infinite domains and should be discussed.

Below, I state some suggestions for minor improvements:

- For the one-species model, there is some inconsistency in the presentation. (3.0.1) contains a dispersal coefficient, while (3.2.1) does not.
- At the end of page 35, a comment on “highly heterogeneous environments” is made. My understanding is that “heterogeneous” refers to heterogeneity that is induced through the pattern formation. The sentence however, suggests that this is some externally imposed heterogeneity. I suggest rephrasing.
- For the literature review, I wonder whether there is any relation to the vast literature on nonlocal advection models for animal movement, for example by Carillo (Oxford).
- The paragraph before Theorem 3.2.4 states that the probability of landing in the domain vanishes, but the theorem itself deals with the Neumann operator. Since the Neumann operator preserves mass, I do not understand the meaning of the preceding paragraph.
- For the multi-species models, it was not clear to me why the diffusing species was only considered under a Dirichlet boundary, but not a Neumann boundary.



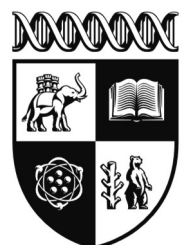
- Lemma 3.3.3. refers to the steady state (3.3.15). But (3.3.15) is the general definition of the steady states of the system rather than a specific steady state.
- When comparing the nonlocal model with its local counterpart (3.4.5), it was not clear to me how the Taylor expansion deals with the boundary of the domain. This should be clarified.
- In experiment 3.4.1, it is stated that the initial condition is the uniform state with an added small-amplitude perturbation. I suggest providing more detail, in particular on the wavenumber(s) of the perturbation.
- There is overlapping text in Figure 3.1.
- I suggest adding a definition of the terms “fat-tailed” and “thin-tailed”. I initially got confused because, to the best of my knowledge, all exponentially decaying distributions are thin-tailed. However, it appears as if in this context, the qualitative statement on the tail of the distribution compares the decay to that of a Gaussian. This is fine, but should be clarified.
- Experiment 3.4.2 and Fig 3.2: Why is the data bounded below by $L=1$? Table 3.1 gives $L=1$ as the lowest used L value. The figure suggests that even lower L values should be tested.
- In Experiment 3.4.4, I did not understand why a perturbation to the uniform state is needed to induce pattern formation. Does the pseudo arclength continuation not detect a bifurcation at which branch switching can be performed?

Chapter 4 – Non-local evolutionary problems

Chapter 4 follows naturally from Chapter 3 because it investigates the time-evolution dynamics in both a single nonlocal PDE and a system of 2 PDEs featuring a nonlocal dispersal operator. Like other sections, it is well written and features a combination of analytical and numerical work. As with Chapter 3, I did not check all steps of the analysis due to lack of expertise (except Section 4.3.2), and focus my review more on the numerical work. As a non-expert on the analysis techniques used, I particularly appreciated the clear interpretation of results (e.g. Remark 4.2.2). In particular, Conclusion 4.4.6 presents a very interesting ecological result, which rounds off a very good chapter.

Suggestions for minor improvements are:

- Minor typo close to the top of page 65: RadEmacher theorem
- In the proof of Lemma 4.2.2, should the integration domain be Ω rather than the generic “domain”?
- First sentence on page 67 should start with “The second..”.
- At the end of the proof of Proposition 4.3.1, it should be made clear that the word “this” corresponds to $B>2$.
- In Experiment 4.4.2, there is an unlabelled table splitting a sentence.



- It would be nice to add a visualisation of different solution types described in Conclusion 4.4.1.
- The description of the kernel in Section 4.4.1 already features in Chapter 3 and could be removed.
- Like in Chapter 3, the description of Experiment 4.4.1 does not fully clarify which wavenumber(s) featured in the initial condition.
- Section 4.4 only considers the Neumann case. At least, a short remark on why the Dirichlet case was not considered should be added.
- Minor typo in Conclusion 4.4.6: space after “Figure 4.3” missing.

Chapter 5 – Numerical methods

The final chapter provides information on the numerical methods used in the preceding chapters, including derivation, and convergence analysis. The careful consideration of the numerical schemes us a major strength of the thesis as numerical approximations of nonlocal terms on finite domains with boundary conditions are non-trivial, as discussed at the beginning of the chapter. I am not a numerical analyst and therefore have limited comments. My main comment is that, in the interest of open science, the code for these numerical methods (and the continuation code used for Chapter 3) should be published as part of the thesis.

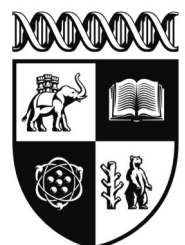
My suggestions for minor improvements are:

- In Experiments 5.4.1 and 5.4.3, references to an “exact” solution are made. However, the text also acknowledges that some of these “exact” solutions are obtained numerically. I suggest rephrasing. Also, there is a notational discrepancy between Experiment 5.4.1 and the figure caption of Figure 5.1.
- In Experiments 5.4.1 and 5.4.3, it is stated that N ranges from 10 to 500 nodes. However, in Figures 5.1 and 5.2, the N axis ranges from 1 to 100.
- Similar to the above, Experiment 5.4.3 states that the timestep varies between 10^{-1} and 10^{-4} , but the time step axis in Figure 5.3 ranges between 10^{-2} and 10^{-4} only.

Bibliography

There is a minor inconsistency in the bibliography: some references give the full name of the journal, while some use abbreviations.

Recommendation



I recommend award of a PhD. As outlined in the “Overall Assessment” section of my report, I regard this as a high-quality thesis that should result in the award of a PhD degree. I have made lengthy comments on all chapters, but emphasise that these are all very minor issues.

Plagiarism and academic integrity

While not checked with a software, I have no reason to suspect any cases of plagiarism or other academic misconduct.

References

1. Tadej M. Long time behaviour of solutions to non-local and non-linear dispersal problems. *Journal of Differential Equations* 2025;**416**:2043–2064. <https://doi.org/10.1016/j.jde.2024.10.046>
2. Eigentler L, Sherratt JA. Analysis of a model for banded vegetation patterns in semi-arid environments with nonlocal dispersal. *J Math Biol* 2018;**77**:739–763. <https://doi.org/10.1007/s00285-018-1233-y>
3. Bennett JJR, Sherratt JA. Long-distance seed dispersal affects the resilience of banded vegetation patterns in semi-deserts. *Journal of Theoretical Biology* 2019;**481**:151–161. <https://doi.org/10.1016/j.jtbi.2018.10.002>



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