Report for: "Combinatorial Banach spaces" by Sebastian Jachimek

I have reviewed the doctoral thesis of Mr. Jachimek and recommend it for the award of the scientific degree of Doctor.

In what follows, I briefly describe the content of the thesis and its principal contributions, and I offer a small number of suggestions for minor edits and improvements. These comments should not detract from my overall very favorable assessment of the work. The work contain a lot of new mathematics, introduces new tools and directions, poses interesting questions, and solves several natural and at least one classical problem. It is a very welcome contribution the mathematical and Banach space literature.

In this thesis, the author undertakes a penetrating study of a class of Banach spaces known as combinatorial Banach spaces, originally named by Gowers, but which have been studied in some form for nearly 100 years. These spaces are defined in a simple and flexible way, depending only on a family of finite subsets of the natural numbers satisfying natural stability properties. In some sense they are discrete analogues of the $C(K)$ spaces. At one end of the spectrum, taking the family of all singletons yields the space c_0, while at the other end, taking all finite subsets of the natural numbers yields ℓ_1. This framework naturally raises questions about the isomorphic structure and subspace structure of general combinatorial Banach spaces. It turns out that structure can be quite rich: certain combinatorial spaces contain all ℓ_p or even more generally all spaces with an unconditional basis. This precludes results like only "every subspace must have subspace isomorphic to ℓ_1 or c_0."

A major strength of the thesis is its systematic treatment of combinatorial Banach spaces induced by non-compact families, an area that has received relatively little attention in the literature. The author carefully explains why non-compactness leads to genuinely new phenomena, particularly with respect to saturation properties and the behavior of bases.

In chapter one, the author constructs several new families of combinatorial Banach spaces, which he calls Farah spaces, g-Farah spaces and rapid Farah space. He proves rapid Farah spaces have the are ℓ_1 saturated while but fail the Schur property. This phenomenon had not previously been observed within the class of combinatorial Banach spaces. The constructions are delicate but not unnecessarily complicated. This work has already been published and deservedly so.

Also in this chapter, the author develops a method for constructing combinatorial Banach spaces with prescribed subspaces isomorphic to a given Banach space with a 1-unconditional basis. While any combinatorial space containing no copy of ℓ_1 must be c_0-saturated, he introduces examples of spaces that do not contain c_0, and yet contain ℓ_2 (and ℓ_1). These results significantly refine our understanding of the possible subspace structure of combinatorial Banach spaces. The final result of this section is the construction of a Pełczyński like universal spaces among that combinatoral spaces. This space and its properties answer question of Pełczyński from more than 50 years ago.

This aspect of the work significantly broadens the scope of combinatorial Banach space theory and I suspect will lead to further study of combinatorial space built on non-compact families.

The author devotes Chapter 3 to the study of dual spaces of combinatorial Banach spaces and introduces a closely related class of quasi-Banach spaces. While the norm of a combinatorial space is quite concrete, in general the norm of the dual is not easy to calculate and thus it is more difficult to extract various properties (like the Schur property for example).

This chapter contains results from a joint paper that appeared in *Mathematische Nachrichten*. The idea of studying the dual norm via an associated quasi-normed space is original and effective. Despite the lack of local convexity, these quasi-Banach spaces retain many essential features of the dual spaces, including saturation properties and the structure of extreme points. The author uses these touch points to prove many technical structure result that are a welcome contribution to the literature.

The final chapter, that contain possibly the most novel results, investigates the extreme points of the unit ball in combinatorial Banach spaces and related spaces. The author provides clear characterizations of extreme points in several important cases and successfully extends known results beyond the compact setting. Curiously the question of characterizing the extreme point of a "simple" combinatoral spaces like the Schreier's space is a difficult, and still unanswered, question. The author analyses spaces induced by graphs, and in particular the distinction between perfect and non-perfect graphs, reveals an interesting interaction between Banach space theory, convex geometry, and graph theory.

Throughout the thesis, the author demonstrates a high level of mathematical maturity, independence, and creativity. While parts of the work build on joint publications, the author delineates his own contributions. A substantial portion of the results, especially those in the later chapters, appear to be new and unpublished.

Here is a list of specific line comments the author should address. The copy is pretty clean.

- This is a stylistic question: It seems that the author is not indenting paragraphs. In my opinion this give a less appealing presentation.
- Proposition 1.3.9 needs some rewriting: Let X be a Banach spce with a Schauder basis (x_n) and let Y be an infinite dimensional subspace of X ...
- On page 18, R.C. James should be cited for the result about shrinking bases having boundedly complete duals.
- On page 23, I would take out the vague sentence "heuristic and informal intuition."
- On page 25, take out the phrase "One may deduce" since you really mean conjecture.
- On page 27, in the proof you refer to a functional as being not weakly null.
- On page 27, please recall the definition of $Z_{\mathbf{F}_g}$.
- On page 29 the word example is misspelled.
- In Lemma 4.0.3 make is more explicit that you are allow all changes of sign for $|\lambda_i \pm \alpha_i|$.

Kevin Beanland