

**Estimation of a block-structured covariance matrix**

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The thesis is based on five articles, denoted as A – E, four of which have already been published in quality international journals (3x Q2 [JSCS, 2x SP], 1x Q3 [AoM]), and one is under review process (possibly Q1 journal [LMA]). All articles have multiple authors, but the contribution of the applicant is substantial in all cases (3x 40% and 2x 50%). These numbers themselves speak about high quality of the presented thesis.

The thesis is divided into two parts, the first one containing general framework with commentaries on the articles and summary of the achieved results, and the second one containing full texts of all the articles. List of notations and brief introduction precedes the first chapter.

Part I, Chapter 1 contains the motivation and mathematical model studied in the thesis. Different types of special structures of variance matrices, which are the main object of interest of the thesis, are introduced. This is followed by description of various estimation techniques of structured matrices, and loss functions for measuring the distance between matrices. Chapter ends with a review of the special results of matrix analysis useful for the studied problems.

Part I, Chapter 2 is the core of the thesis. It contains detailed description of the results contained in the presented articles.

Article A solves the problem of optimal approximation of  $2 \times 2$  block matrix with compound symmetry (CS) or autoregression of the 1<sup>st</sup> order (AR(1)) matrix.

Article B provides a new method of estimation of block-structured matrices, named quasi-shrinkage, which preserves both special block structure and positive definiteness. General results are applied to CS blocks, for which explicit solution is derived.

Article C further develops the method from Article B, providing half-explicit solution for the AR(1) case via system of equations. Also, an alternative method of estimation of block-structured matrices is proposed here, based on another decomposition of original matrix.

Most valuable part are articles D and E. In article D, complete characterization of quadratic subspaces in partitioned matrices is derived. Moreover, important properties of the projection are proved (positive definiteness, decrease of condition number, BLUE property of the projection of variance matrix estimator S). There is also characterization of commutative quadratic subspaces.

Article E focuses on problem of estimation of a structured variance matrix, whose structure is linear in unknown parameters, but the structure does not necessarily form a quadratic subspace. It is proven that minimum quadratic risk is attained using quadratic subspace of the linear structure space, and explicit intensity of the shrinkage is derived. Consistency and asymptotic unbiasedness of the estimator are also proved. This is very important result, since all practically used variance matrix structures belong to some of the linear structure spaces.

Part II contains full texts of articles A – E with detailed specification of the contributions of all authors. It is clear that the applicant did a substantial part of the work, and mainly the harder parts.

Technical errors, misprints, suggestions and questions:

- p. 11, Definition 1.2.1: independent  $\rightarrow$  linearly independent
- p. 12, l. -3: What are the conditions for  $\alpha_i, \beta_i$  to form the cone of p.d. matrices?
- p. 14, formula (1.10) and l. 5: Matrix  $\Psi$  is here defined as lower triangular; at some places it is used as upper triangular (e.g.  $\Psi_{12}$  at the same page), and at others as a symmetric one (p. 33, formula (C.1)). This is inconsistent.
- p. 19, l. 3 and many other places: entropy loss  $\rightarrow$  cross-entropy loss
- p. 21, l. -7: missing transposition at  $K_{p,q}$
- p. 24, Theorem A.1 and many other places: Expressions of the type  $\mathbf{1}'\mathbf{A}\mathbf{1}$  are scalars, and it is better to take them out of complicated matrix formulas to simplify the resulting structure.
- p. 34, Theorem C.5: Here again  $\mathbf{H}_i$  is symmetric, contrary to the (1.10).
- p. 37, Proposition D.1: Matrix  $\Psi_i$  here denotes any symmetric matrix, which is confusing.
- p. 159, l. -1: In the proof, you have exchanged the order of limit and integral. Justify this step!
- p. 160, l. 2: Why do you use convergence in probability to non-random matrices? Here I doubt if convergence in  $L_1$  can be justified by convergence in probability.

To conclude, the thesis is well written, with only minor drawbacks (I would recommend especially improving the English), and brings original and valuable scientific results. The progress of the work is clearly visible, each article extending previous results, and finally resulting in very general and useful theorems. These results have been published in several respected international journals, which shows candidate's ability for scientific work. As a result, the thesis corresponds to requirements for a PhD-thesis and I strongly recommend awarding the degree to the author.

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