PhD Thesis Review Report

Author: Krzysztof Kępczyński
Title: Asymptotics of functionals of Gaussian and Lévy processes with a view towards risk and queueing models
Reviewer: Enkelejd Hashorva
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Overview and Main Contributions

The dissertation, supervised by Professor Krzysztof Dębicki primarily focuses on the **asymptotic properties of extremes of stochastic processes**. This research is driven by both the theoretical aspects of extreme value theory and practical problems encountered in various fields such as finance, physics, risk theory and queueing theory. In recent years, the study of the asymptotic behavior of different functionals of stochastic processes has gained significant attention, particularly for **Gaussian and Lévy models**, which are widely used to describe a broad range of phenomena. Earlier research often concentrated on one-dimensional problems (e.g., supremum, infimum, sojourn-time functionals), but contemporary work focuses on developing tools and appropriate methods for more **general, often multidimensional, functionals** and for **extremes observed over random time horizons**.

This research work is technically very demanding and demonstrates the author's deep understanding of advanced probabilistic tools including Piterbarg/Pickands theory, Gaussian comparison techniques, multivariate regular variation, and large deviation principles. The thesis is composed of several chapters, each addressing a distinct yet thematically related problem. The style of exposition is clear and mathematically precise, with detailed proofs and a consistent notation system throughout.

Despite the technical nature of the subject, the thesis is clearly written. Definitions and assumptions are stated precisely, and the structure of each chapter is logical and well motivated. The results are supported by detailed proofs, and the bibliography is comprehensive, covering foundational literature as well as recent developments in extreme value theory, queing and risk theory.

In a more technical setting, a key problem with numerous applications analysed in this thesis can be formulated as the investigation of the following probability:

$$\mathbb{P}\{\Theta(X(t), t \in E) \in K_u\},$$

where $X(t), t \in E$ is a vector-valued stochastic process, E is a subset of $\mathbb{R}^d$, $\Theta(\cdot)$ is a functional, and K_u represents a family of sets indicating a rare event regime. It is noteworthy that the author has successfully extended the scope of this problem to include multidimensional cases, which is a significant advancement in the field. The thesis builds upon foundational contributions to extreme value theory, notably James Pickands III's work on Gaussian processes and various methodologies developed for Lévy processes.

The dissertation contributes significantly to various research fields by investigating specific problems related to risk and queueing models, introducing new techniques and ideas that can handel difficult settings of vector-valued processes. Here's a detailed breakdown of the key areas explored in each chapter:

- **Chapter 2: Sojourn Time for Correlated Brownian Motion with Drift**
  - This chapter analyzes the **sojourn-time functional of a two-dimensional correlated Brownian motion with drift** over a finite time horizon $[0, T]$. Specifically, it investigates the occoupation time
  $$\int_{[0,T]} I_{W(t)-ct>au} dt,$$
  where $W(t)$ is a two-dimensional Brownian motion with constant correlation $\rho \in (-1, 1)$, and a, c are drift and threshold vectors. This functional quantifies the time a bivariate correlated Brownian motion spends above a specific threshold within a given interval $[0, T]$.
  The research is highly relevant to **risk theory**, particularly for understanding the **cumulative Parisian ruin probability**. This probability describes the likelihood that ruin (where a risk process falls below zero) persists for a significant period of time, often referred to as "occupation time in red".
  - In the context of **risk theory**, this functional is directly relevant to the **cumulative Parisian ruin probability**. This probability measures the likelihood that financial ruin persists for a significant duration. The reason for letting $u \to \infty$ is related also to a "many-source regime," where an aggregate risk process $R(t)$ is formed from N independent and identically distributed (i.i.d.) Brownian Risk processes $R^{(k)}(t)$, representing individual companies. Rescaling by N explains the role of u.
  - The study derives the **exact asymptotic behavior** of this cumulative Parisian ruin probability as $u \to \infty$, particularly for $H(u) = zu^{-2}$. The analysis identifies two distinct regimes, based on the interplay between the two components of the Brownian motion:
    * **Dimension-reduction case** ($|I| = 1$): Here, one coordinate asymptotically dominates the other. This scenario is detailed in **Theorem 2.2.2**.
    * **Full-dimensional case** ($|I| = 2$): In this regime, both coordinates significantly influence the asymptotics. This case is presented in **Theorem 2.2.3**.

  The proofs for these theorems involve analyzing the properties of the function $q(t)$ near its minimizer $t_0^* = \min(t_0, T)$, as shown in **Lemma 2.3.1**. They also rely on bounding techniques for probabilities of excursions and analysing asymptotic behaviour of probabilities related to the sojourn time functional within small intervals.

The main difficulty is in the rigorous treatment, which requires all the results to be uniform approximation on intervals that depend on the threshold u. The author successfully navigates these challenges by employing advanced techniques and careful analysis of the underlying events.

- **Chapter 3: Stationary Gaussian Queues on a Random Time Interval** This chapter investigates a **stationary Gaussian queue** where the input is a **fractional Brownian motion**. The central focus is on the **buffer overflow probability**

$$\mathbb{P}\{\sup_{t\in[0,T_u]}(Q(t)) > u\}$$

over a **random time interval** T_u. The stationary buffer content process $Q(t)$ is represented as

$$\sup_{-\infty<s\leq t}(B_H(t) - B_H(s) - c(t-s)),$$

where $B_H(t)$ is fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $c > 0$ is the service rate.

Random Time Horizon

The random time interval is defined as $T_u = u^\gamma \sum_{k=1}^{u^\beta} T_k$, where T_k are i.i.d. non-negative random variables and are independent of the queueing process $\{Q(t) : t \geq 0\}$. This setup is motivated by models involving "resetting", where the system terminates at a random time.

Regimes of Random Variable T

Consider T_k's to be iid copies of $T \geq 0$ almost surely. The analysis distinguishes three regimes for the random variable T based on the heaviness of its tail distribution, each leading to qualitatively different asymptotic behavior and requiring distinct proof techniques:

- **D1: Integrable ($\mathbf{E\{T\}} < \infty$)**: In this scenario, the distribution of T influences the asymptotics via a constant and a polynomial factor, but it does not affect the logarithmic asymptotics. **Theorem 3.2.1** provides the exact asymptotics for both $\beta = 0$ and $\beta > 0$ cases. An explicit example for $H = 1/2$ and exponentially distributed T is given in **Example 3.2.3**.
- **D2: Regularly varying tail (index $\alpha \in (0,1)$)**: Here, the index of regular variation α critically affects the logarithmic asymptotics, resulting in heavier, exponentially decaying asymptotics. The asymptotics are dominated by the event that T exceeds a high threshold $N(u) = 1/P\{\sup_{t\in[0,1]}(Q(t)) > u\}$. **Theorem 3.2.5** provides the asymptotic behavior for this regime.
- **D3: Slowly varying tail**: This regime leads to significantly heavier asymptotic behavior, potentially resulting in a regularly varying probability. An additional condition (D4) on the tail distribution function of

T is required to guarantee specific asymptotic equivalence. **Theorem 3.2.7** details the asymptotics for this case.

Each regime requires distinct proof techniques due to their qualitatively different asymptotic behaviors.
The proofs for D1 utilize results over deterministic intervals and Karamata's theorem, while D2 and D3 necessitate an extension of a lemma to uniform convergence for "very long" intervals, as seen in **Lemma 3.3.1** and **Lemma 3.3.2**.

• Chapter 4: Tail asymptotics for functionals of stationary Lévy queues

This chapter delves into the **asymptotic behavior of general functionals** Θ of the sample paths of a **stationary queue fed by a Lévy process** $\{X(t) : t \in \mathbb{R}\}$. The Lévy process is assumed to satisfy the **Cramér condition** (A1), along with conditions (A2) ensuring non-monotone paths or non-lattice Lévy measure. The stationary buffer content process is defined as $Q(t) = \sup_{-\infty<s\leq t}(X(t) - X(s) - c(t-s))$, where $c > 0$ is the service rate.

Functional Properties

The functional $\Theta : D(E) \to \mathbb{R}$, defined on the space of real-valued càdlàg functions on a compact set $E \subset [0, \infty)$, must satisfy two key conditions:

- **F1**: $\Theta(f) \leq \sup_{t\in E}(f(t))$ for any $f \in D(E)$.
- **F2**: $\Theta(af + b) = a\Theta(f) + b$ for any $f \in D(E)$ and $a > 0, b \in \mathbb{R}$.

These conditions cover important examples like supremum, infimum, and convex combinations thereof.

Asymptotic Results

The chapter analyzes the tail probability

$$p_\Theta(E; u) := \mathbb{P}\{\Theta(\{Q(t) : t \in E\}) > u\}$$

as $u \to \infty$, for two regimes of the time horizon T_u (where $E = [0, T_u]$):

- **Fixed-interval case** ($T_u \equiv T > 0$): **Theorem 4.2.1** states that as $u \to \infty$

  $$p_\Theta(T; u) \sim H^{\Theta}_{\omega,X_c}[0, T]\mathbb{P}\{Q(0) > u\}.$$

  This result generalizes prior works which focused solely on the supremum functional on a constant time interval. The proof primarily uses **Breiman's Lemma (Lemma 4.2.2)** to reduce the tail analysis to the product of independent random components, specifically an expected value and the tail distribution function of two independent random variables.

- **Growing-interval case** ($T_u = n(u) \to \infty$ as $u \to \infty$): **Theorem 4.2.3** provides the asymptotics for the supremum functional when $n(u)$ grows sufficiently slowly ($n(u) = o(e^{\beta u})$, with $\beta \in (0, 1/2)$). The prefactor in the asymptotics of $p_{\sup}(n(u); u)$ is a generalised Pickands constant $H^{\sup}_{\omega, X_c}$ which if interests outside the scope of the thesis.

- # Chapter 5: Extended uniform Breiman's Lemma

This chapter introduces a significant methodological development: an **extension of Fougeres & Mercadier's version of Breiman's lemma** to products of random matrices and random vectors that are themselves indexed by a threshold parameter u (and potentially an additional parameter τ_u). This extension aims to analyse the uniform asymptotic behavior of $b_{u,\tau_u} \mathrm{P}\{M_{u,\tau_u} X_{u,\tau_u} \in a_{u,\tau_u} \cdot\}$ as $u \to \infty$, where X_{u,τ_u} and M_{u,τ_u} are families of random vectors and matrices. The uniformity is a critical point and requires very careful treatment. The extension is particularly useful for analyzing the asymptotic behavior of functionals of Gaussian and Lévy processes, as well as for various extreme value theory problems.

Results and Assumptions

The extended uniform Breiman's lemma stated in **Theorem 5.2.1** proves that under certain assumptions (A1, A2, A3), the vague convergence

$$b_{u,\tau_u} \mathrm{P}\{M_{u,\tau_u} X_{u,\tau_u} \in a_{u,\tau_u} \cdot\} \xrightarrow{v} \nu_G(\cdot)$$

holds uniformly with respect to $\tau_u \in K_u$. The assumptions are:

- **A1**: $b_{u,\tau_u} \mathrm{P}\left\{\left(\frac{X_{u,\tau_u}}{a_{u,\tau_u}}, M_{u,\tau_u}\right) \in \cdot\right\} \xrightarrow{v} (\nu \times G)(\cdot)$ uniformly with respect to $\tau_u \in K_u$, where ν is a homogeneous Radon measure and G is a probability measure. This implies X_{u,τ_u} has a multivariate regularly varying distribution.
- **A2**: A condition related to the moments of the scaled product, ensuring it remains bounded for small scaled values of X_{u,τ_u}.
- **A3**: The α-th moment of the matrix norm is finite, where α is the homogeneity index of the measure ν.

Applications

This generalized lemma proves to be a powerful tool for various extreme value theory problems, providing an alternative and direct proof for several Gaussian models.

- **Uniform Pickands Lemma for homogeneous functionals of Gaussian fields**: It offers an alternative proof for results like **Theorem 2.1** and the celebrated **Pickands lemma** (**Corollary 5.3.2**, generalizing these to a broader class of sets (Theorem 5.3.1). This allows for calculating tail asymptotics for cases beyond just the half-line $[u, \infty)$. A further generalization, the **Piterbarg lemma** (**Corollary 5.3.3**, is also discussed.

– **Supremum of self-standardized Gaussian processes**: The framework is applied to analyze the extremes of processes like the self-standardized Gamma process (Gamma bridge) and Gaussian processes with random variance. This includes two specific regimes:

  * **G1** (centered Gaussian process with variance function $\sigma_Y^2(t)$ and covariance $R_Y(s,t)$): **Theorem 5.3.5** provides the asymptotic behavior for the supremum of $(Y(t) + \mu t)/(Y(T) + \mu T)$ over $[0, T]$.

  * **G2** (centered Gaussian process with stationary increments and slowly varying variance function $\sigma_Y(t)$): **Theorem 5.3.6** details the asymptotics for the supremum of $Y(t)/Y(\delta(u))$ over $[0, \delta(u)]$.

The applications are important and completely novel contributions.

Methodological Strengths, Originality and Impact

The methodology employed in the thesis is solid and demonstrates a good balance between generality and tractability. The author successfully applies advanced probabilistic tools to derive both exact and logarithmic asymptotics. Particularly notable is the careful handling of situations where classical tools fail due to lack of smoothness, non-stationarity, or non-Gaussianity.

Moreover, the author often goes beyond direct application of known results by developing technical lemmas adapted to specific settings—especially in the presence of trend functions, irregular variance structures, Gaussian and Lévy models.

The dissertation includes several novel contributions, techniques and ideas. Among these are refined asymptotic estimates in non-stationary and Lévy settings, extensions of classical Gaussian results to non-Gaussian contexts, and new applications of comparison techniques.
Very original is also the idea of analysing the decay of the probability that the supremum funcitonal belongs to some threshold-scaled set uK. In the literature so far, only the case $K = [1, \infty)$ has been considered. Additionally, the investigation of heavy tailed self-normalised Gaussian processes is also a completely novel and original result. These contributions are going to be useful for further theoretical developments and for practical modeling of extremes in areas such as climate science, engineering reliability, and risk theory.

Conclusion

In summary, Mr. Kępczyński has produced a high-quality doctoral dissertation that makes significant contributions to the field of extreme value theory, particu-

larly in the context of stochastic processes by advancing analytical techniques for Gaussian and Lévy processes. The work is well-structured, technically sound, and demonstrates a deep understanding of both theoretical and practical aspects of the subject matter.
The results are original, and the methodologies employed are robust and innovative. This PhD thesis is a valuable addition to the literature and is likely to have a lasting impact on both theoretical research and practical applications in related fields.

Recommendation: I strongly recommend the thesis for acceptance and the award of the PhD degree.
Moreover, I believe that Mr. Kępczyński has the potential to become a leading researcher in the field of extreme value theory and stochastic models and therefore, I recommend awarding the PhD degree with distinction.

Sincerely,

Enkelejd Hashorva