Report on the doctoral thesis *Combinatorial Banach Spaces* by Sebastian Jachimek

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Combinatorial Banach spaces constitute a class of Banach spaces whose norm is defined as a kind of mixture of the minimal and maximal possibilities: the supremum and the sum of absolute values. One is allowed to sum only over sets of coordinates from a given family $\mathcal{F}$. These spaces have played an important role in the development of Banach space theory as a source of counterexamples. In spite of having been around for many decades and having been thoroughly used by outstanding mathematicians (like Gowers, Maurey, Alspach, Argyros... among many others), there are still intriguing questions about them. This thesis is making substantial contributions about the fundamental structure of combinatorial Banach spaces, both by providing new results and by pointing to new interesting questions. Some of the results of the thesis are contained in two articles by Jachimek and coauthors published in two well-respected journals in mathematics, Studia Mathematica and Mathematische Nachrichten. Other results are still unpublished.

Chapter 2 contains a number of new examples. It was proved by Borodulin-Nadzieja and Farkas that one could transfer some ideas by Farah into producing combinatorial spaces with the Schur property. The thesis shows that this construction can be modified to produce a variety of new non-isomorphic combinatorial spaces both with the Schur property and without the Schur property (but ℓ_1-saturated).

Chapter 3 deals with the problem of identifying the dual space of a combinatorial Banach space $X_{\mathcal{F}}$. A natural candidate for this dual, $X^{\mathcal{F}}$, resulting from the inversion of the norm formula, is proposed. This happens to be a quasi Banach space rather than a Banach space. But there is a procedure, called the Banach envelope, to turn that quasi Banach space into a Banach space. The main result of this chapter is that the Banach envelope of $X^{\mathcal{F}}$ is indeed isometric to the dual space of $X_{\mathcal{F}}$ in the case when $\mathcal{F}$ is a compact hereditary family. This is already a non trivial result and it covers the most commonly used combinatorial spaces in the literature. It is still an open question what happens for more general families.

Chapter 4 treats the description of the extreme points of the ball of a combinatorial space and its dual. The dual case is easier, the thesis provides a precise description. For the space itself, besides general facts already present in the literature, the thesis finds a description for the Farah spaces considered in Chapter 2. The last results of the thesis provide insight into this question for a new interesting class of combinatorial spaces arising from graphs. Even in this class, the description of the extreme points presents serious difficulties, but the thesis presents advances, particularly in relation to graphs that are constructed from odd holes or antiholes.

From a formal point of view, the manuscript is carefully written. Nevertheless, I mention a few minor remarks for the benefit of future readers. There is a typo *exmaple* on page 29, $Ext(X_{\mathcal{F}})$ should be $Ext(X^*_{\mathcal{F}})$ in Proposition 4.1.1, v_2m should be v_{2m} on the second paragraph of page 80, and it would be more clear if the inequality displayed in Lemma 4.0.3 is written as

$$\sum_{i\leq n} |\lambda_i + \alpha_i| \leq 1 \quad \text{and} \quad \sum_{i\leq n} |\lambda_i - \alpha_i| \leq 1$$

or similar, since the interpretation of $\pm$ could be ambiguous.

As a conclusion, I consider this as an excellent doctoral thesis. I recommend awarding the degree of doctor with distinction.

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