PhD thesis

Axiomatization of the Mathias model in terms of games

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Abstract

The objective of this thesis is one of the most significant models considered in set theory, namely the Mathias model, which is cnstructed via the technique of iterated forcing. Although it is obtained by iteration of length ω_2 of Mathias forcing, we study its combinatorial structure using the framework of descriptive set theory. We present a series of axioms, modeled on the CPA axiom from [2], that describe the combinatorial core of the model. We prove their consistency and study their consequences. To this end, we give a descriptive set theoretical characterization of the iterated Mathias forcing. Our axioms are formulated in terms of Borel sets and functions, σ-ideals on Polish spaces, games and strategies. In this way we develope an axiomatization of the Mathias model, which gives a descriptive set theoretic insight into its structure, makes it more approachable, and leads to new results. As a byproduct, we obtain a few facts about V-ultrafilters[1] induced by reals from the generi extension via the iterated Mathias forcing.

Chapter 1 is an itroduction to the topic, gives some background and history. It contains an overview of the thesis, remarks about possible application of developed methods for investigating other models, and acknowledgement. In Chapter 2 we gather the neccesary preliminaries as well as fix the notation and interpretation of symbols and phrases. In Chapter 3 we reformulate the iteration of Mathias forcing. This approach was considered in [2] and its adaptation to our case uses ideas of [7].

The following chapters describe the axioms. In each case, we prove that the considered axiom holds true in the model as well as discuss its consequences. In Chapter 4 we present the basic axiom, CPA, which is analogous to the one from [2]. It implies that $\mathrm{cov}(\mathcal{J}) = \mathfrak{x} = \omega_1$, where $\mathcal{J}$ is the σ-ideal of meager, or null sets, and $\mathfrak{x}$ denotes the distributivity of Boolean algebras r.o.$(\mathbb{R}^*, \subseteq^*)$ or r.o.$(c_0 \backslash \ell^1, \leq^*)$ (see [3], [5]). Its modificatiom, the axiom CPAs, introduced in Chapter 5, proves such assertions as

- $\mathfrak{h} > \omega_1$, where $\mathfrak{h}$ is the distributivity of $(\mathcal{P}(\omega)/\operatorname{fin})$,
- Borel Conjecture (see [1]),
- the lack of rapid ultrafilters (see [6]),

[1]maximal filters on $\mathcal{P}(\omega) \cap V$, where V is the ground model

- the lack of far cut points in $\mathbb{I}_{\mathcal{U}}$ for $\mathcal{U} \in \omega^*$ (see [4]).

The latter statement is a new property of the Mathias model, so far it was known for the Laver model.

The motivation to search for a stronger version of CPA stems from the result of Shelah and Spinas from [7], which says that $\mathfrak{h}(2) = \omega_1$ in the Mathias model. The axiom SCPA$^-$, which implies this equality, is formulated in Chapter 6. It is a weaker, "tactic" version of the axiom SCPA (introduced in Chapter 8) but its expression is much less technical. The axiom SCPA$^-$ also proves that the distributivity of $((\omega)^\omega, \leq^*)$ equals ω_1. ($(\omega)^\omega$ is the set of all infinite partitions of ω with the ordering $X \leq^* Y$ iff all but finitely many elements of X are unions of elements of Y, see [8].)

Chapter 7 is devoted to the axiom $\Diamond$CPA, which is a natural modifaction of CPA capturing some combinatorics provided by the principle $\Diamond$, which holds in the intermediate generic extensions of cofinality ω_1. In Chapter 8 we discuss implications between the axioms introduced so far and formulate their generalizations. In particular, we present the full, "strategic" version of SCPA as well as the strongest axiom, called $\Diamond$SCPAs, which implies all the previously stated.

Two following chapters can be considered as an appendix to the main topic. In Chapter 9 we present some results concerning V-ultrafilters in the Mathias model. In particular, we give elementary proofs of two main Propositions from [7] and, in fact, we generalize them. (Original proofs are based on difficult and technically complicated methods, which make them hard to follow.) In Chapter 10 we modify $\Diamond$CPA obtaining the axiom $\Diamond$mCPA, which is strong enough to imply the $\clubsuit$ principle. This modification is quite technical. We present it for for the sake of the completeness of the research.

References

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