

Analiza i symulacje nieliniowych modeli z nielokalną dyspersją gatunków w ograniczonych, płaskich siedliskach

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Analysis and simulations of the non-linear models with non-local dispersal of species considered in finite, flat habitats

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*If you can't love yourself, how in the hell you gonna love somebody else?
Amen!*

Streszczenie

Niniejsza rozprawa przedstawia wszechstronną analizę matematyczną i numeryczną nieliniowych modeli dynamiki populacji, uwzględniających nielokalne rozprzestrzenianie się (dyspersję) w ograniczonych dziedzinach przestrzennych. Motywowane złożoną dynamiką ekosystemów półsuchych, taką jak formowanie się wzorów roślinności i nagłe pustynnienie, badania zastępują klasyczną dyfuzję lokalną (operator Laplace’a) nielokalnymi operatorami całkowymi. Praca rozpatruje zarówno skalarne równania reakcji-dyspersji, jak i hybrydowe układy reakcji-dyfuzji-dyspersji, na przykładach nielokalnego sformułowania modelu Logistycznego z efektem Allego oraz Klausmeiera-Graya-Scotta, z uwzględnieniem warunków brzegowych odpowiadających środowisku zamkniętemu (typu Neumanna) i otoczonego niebezpiecznym obszarem (typu Dirichleta).

W dziedzinie analizy funkcjonalnej rozprawa ustanawia rygorystyczną teorię spektralną dla nielokalnych operatorów dyspersji. Zapewnia jawną charakteryzację widma ciągłego, dowodzi istnienia ściśle dodatniej przerwy spektralnej przy określonych założeniach geometrycznych i założeniach dotyczących jądra całkowego, a także formułuje dokładną charakteryzację typu max-min dla widma dyskretnego i głównej wartości własnej dla obu rodzajów warunków brzegowych. Opierając się na tych podstawach spektralnych, w pracy zbadano niezależne od czasu problemy stacjonarne, ustalając kryteria istnienia i jednoznaczności przestrzennie niejednorodnych stanów stacjonarnych. Rygorystycznie udowodniono powstawanie nieciągłych wzorów przestrzennych w reżimach silnie nielokalnej dyspersji oraz wykorzystano teorię bifurkacji do konstrukcji ciągłych rozwiązań. Ponadto wykazano poprawność sformułowania, istnienie rozwiązań globalnych w czasie oraz ciągłą zależność od warunków początkowych dla powiązanych ewolucyjnych problemów Cauchy’ego. Konstruując obszary niezmiennicze, autor podał ostre warunki analityczne dla długookresowego zachowania asymptotycznego populacji, rozróżniając wykładnicze wymieranie od jednostajnego przetrwania (persystencji).

Aby przezwyciężyć trudności obliczeniowe związane z gęstymi macierzami splotu, które nieodłącznie wynikają z nielokalnych operatorów, opracowano wysoce wydajny pseudospektralny schemat numeryczny. Metoda ta wykorzystuje rozwinięcia w wielomianach ortogonalnych (takich jak wielomiany Legendre’a), co drastycznie zmniejsza złożoność obliczeniową i umożliwia przeprowadzanie symulacji w wysokiej rozdzielczości w 1D i 2D. Rozprawa zawiera rygorystyczną analizę zbieżności, ustanawiającą dokładne oszacowania błędów dla aproksymacji ciągłego operatora splotu, lokalnych funkcji multiplikatywnych oraz błędów całkowania w czasie.

Ostatecznie, ramy matematyczne i numeryczne zostały zastosowane do kluczowych pytań ekologicznych. Badanie formalnie tłumaczy zasady biologiczne na język matematyki poprzez zdefiniowanie Krytycznego Rozmiaru Płata (ang. *Critical Patch Size*) oraz Krytycznego Progu Agregacji (ang. *Critical Aggregation Threshold*) dla modeli nielokalnych. Szeroko zakrojona numeryczna analiza kontynuacji i bifurkacji dowodzi, że ekosystemy zdominowane przez długozasięgowe jądra dyspersji o grubych ogonach wykazują znacznie większą odporność, zdolność do przetrwania w mniejszych i bardziej pofragmentowanych siedliskach w porównaniu do tych opartych na lokalnej dyfuzji. Badania odkrywają również unikalne morfologie przestrzenne indukowane przez interakcje nielocalne, w szczególności zjawiska „spadku brzegowego” (ang. *boundary drop-off*) i „grzbietu brzegowego” (ang. *boundary ridge*), które są całkowicie nieobecne w klasycznych, lokalnych modelach reakcji-dyfuzji, co dostarcza głębokiego wglądu w strukturalną samoorganizację gatunków w pofragmentowanych środowiskach.

Abstract

This dissertation presents a comprehensive mathematical and numerical analysis of non-linear population dynamics models incorporating non-local dispersal in bounded spatial domains. Motivated by the complex dynamics of semi-arid ecosystems, such as vegetation pattern formation and abrupt desertification, the study replaces classical local diffusion (the Laplace operator) with non-local integral operators. The research considers both scalar reaction-dispersal equations and hybrid reaction-diffusion-dispersal systems, exemplified by a non-local formulations of the Logistic model with Allee effect and Klausmeier-Gray-Scott model, under closed (Neumann-type) and hostile (Dirichlet-type) boundary conditions.

The author's contributions are categorized into three main areas: theoretical analysis, numerical methods, and theoretical ecology.

In the realm of functional analysis, the dissertation establishes a rigorous spectral theory for non-local dispersal operators. It provides an explicit characterization of the continuous spectrum, proves the existence of a strictly positive spectral gap under specific geometric and kernel assumptions, and formulates an exact max-min characterization of the discrete spectrum and the principal eigenvalue for both boundary conditions. Building upon this spectral foundation, the study investigates time-independent stationary problems, establishing criteria for the existence and uniqueness of spatially heterogeneous stationary states. It rigorously proves the emergence of discontinuous spatial patterns under highly non-local dispersal regimes and utilizes bifurcation theory to construct continuous patterned solutions. Furthermore, the well-posedness, global-in-time existence, and continuous dependence on initial data for the associated evolutionary Cauchy problems are established. By constructing invariant regions, the author provides sharp analytical conditions for the long-time asymptotic behavior of populations, distinguishing between exponential extinction and uniform persistence.

To overcome the computational intractability of dense convolution matrices inherently arising from non-local operators, an efficient pseudo-spectral numerical scheme is developed. This method exploits orthogonal polynomial expansions (such as Legendre polynomials) to drastically reduce computational complexity, enabling high-resolution 1D and 2D simulations. The dissertation includes a rigorous convergence analysis, establishing exact error bounds for the approximation of the continuous convolution operator, the local multiplicative functions, and the temporal integration errors.

Finally, the mathematical and numerical frameworks are applied to pressing ecological questions. The study formally translates biological principles into mathematics by defining the Critical Patch Size and Critical Aggregation Threshold for non-local models. Extensive numerical continuation and bifurcation analysis demonstrate that ecosystems governed by long-range, heavy-tailed dispersal kernels exhibit significantly enhanced resilience, persisting in smaller and more fragmented habitats compared to those restricted by local diffusion. The research also uncovers unique spatial morphologies induced by non-local interactions, notably the “boundary drop-off” and “boundary ridge” phenomena, which are entirely absent in classical local reaction-diffusion models, providing profound insights into the structural self-organization of species in fragmented habitats.

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Chapter 1

Introduction

Understanding the dynamics of ecosystems, specifically the conditions that dictate whether a population will persist, face extinction, or settle into complex intermediate states, remains one of the most pressing challenges in ecology and biology. Example of such striking visual manifestation of these complex dynamics can be observed in the Parque Regional de la Sierra de Gredos. As illustrated in Figure 1.2, a dense, thriving forested area can exist in astonishingly close proximity to a barren region undergoing severe desertification. To rigorously investigate what drives a population towards survival or collapse, and to understand the transitional states between these extremes, we must rely on advanced mathematical modeling. Therefore, while this work is deeply motivated by critical biological and environmental questions, it is fundamentally a dissertation in applied mathematics. The primary focus of this research is to develop, analyze, and apply mathematical frameworks to model these spatial phenomena, offering both qualitative and quantitative insights into the delicate balance of ecological systems.



Figure 1.1: *Aerial images of vegetated and deserted areas, presented as screenshots from Google Maps ©Airbus, Maxar Technologies. In all panels, vegetated regions are darker and bare-soil regions lighter. Both screenshots show areas in Parque Regional de la Sierra de Gredos. The left panel shows forested area ($40^{\circ}07'40.00''$ N; $4^{\circ}38'57.48''$ W) while the right panel shows area undergoing desertification ($40^{\circ}07'46.58''$ N; $4^{\circ}39'19.80''$ E).*

1.1 Crucial concepts from biology and ecology

Water-limited ecosystems often display striking regular vegetation patterns arising from the interplay of water–vegetation feedbacks at multiple scales [48, 59, 35, 10, 38]. These self-organized structures not only signal transitions toward desertification, hexagonal spot patterns emerging under extreme aridity serve as early warning indicators, but may also enhance ecosystem resilience by redistributing biomass and avoiding catastrophic collapse [32, 65, 39, 47, 46]. Because vegetation patterns evolve over long timescales, mathematical models have become indispensable for uncovering the mechanisms of pattern formation and for predicting ecosystem responses under changing climatic stresses [37, 34].

To rigorously study these phenomena, we first formalize the notion of a pattern within the context of population (biomass, vegetation) dynamics.

Definition 1.1 (Vegetation Pattern). A **Vegetation Pattern** is a spatially heterogeneous stationary solution $V(x)$ to the biomass evolution equation. It is characterized by the segregation of the spatial domain into *patches* (*clusters*, *habitats*), e.g. regions of high biomass density where vegetation facilitates its own growth.

Real-world examples of such self-organization are documented in various semi-arid regions, as illustrated in Figure 1.2.

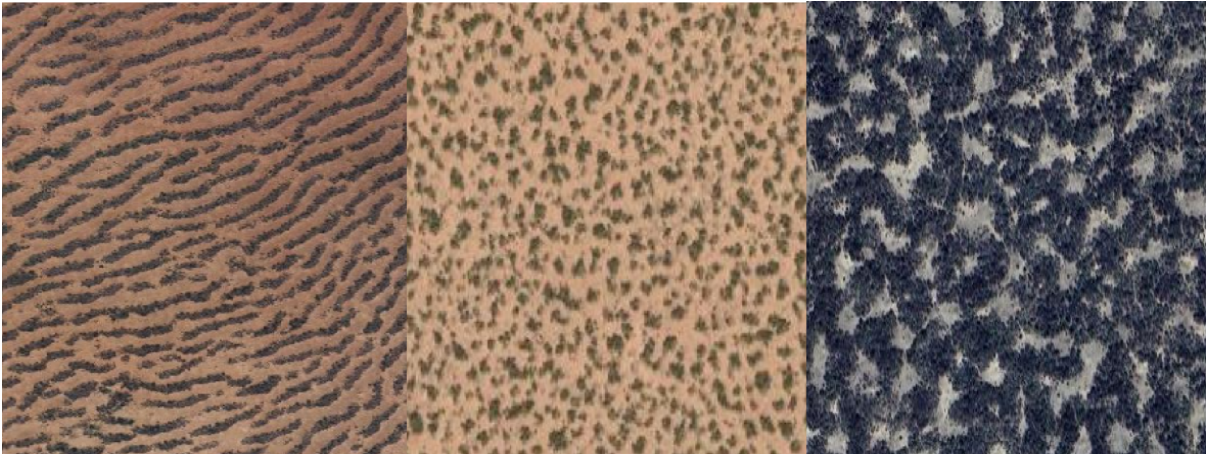


Figure 1.2: *Aerial images of self-organized vegetation patterns, presented as screenshots from Google Maps ©Airbus, Maxar Technologies. In all panels, vegetated regions are darker and bare-soil regions lighter. Top left panel shows stripe pattern in Sudan ($11^{\circ}34'55.2''$ N; $27^{\circ}54'47.52''$ E), central panel shows spot pattern in Niger ($11^{\circ}1'12''$ N; $28^{\circ}10'48''$ E) and the last panel shows irregularly structured pattern in Sudan ($11^{\circ}3'0''$ N; $28^{\circ}17'24''$ E).*

While classical modeling approaches often assume idealized, infinite or periodic domains to study these patterns, realistic ecosystems are inherently bounded. Vegetation often exists in fragmented habitats surrounded by hostile environments, such as hyper-arid deserts or human-developed landscapes. In such finite habitats, the survival of these biological structures depends strictly on the domain dimensions.

The trade-off between local biomass growth inside the habitat and the dispersal of biomass across the hostile boundaries leads to the concept of a minimum necessary domain size.

Definition 1.2 (Critical Patch Size). The **Critical Patch Size**, denoted by ω_{crit} , is the threshold of the domain size $|\Omega|$ required for population persistence.

- **Extinction Regime:** If the habitat size satisfies $|\Omega| < \omega_{crit}$, the losses at the boundaries overwhelm reproduction, leading to inevitable extinction ($v \rightarrow 0$).
- **Persistence Regime:** If the habitat size satisfies $|\Omega| > \omega_{crit}$, the population can persist, potentially forming the stable patterns sketched in Figure 1.3.

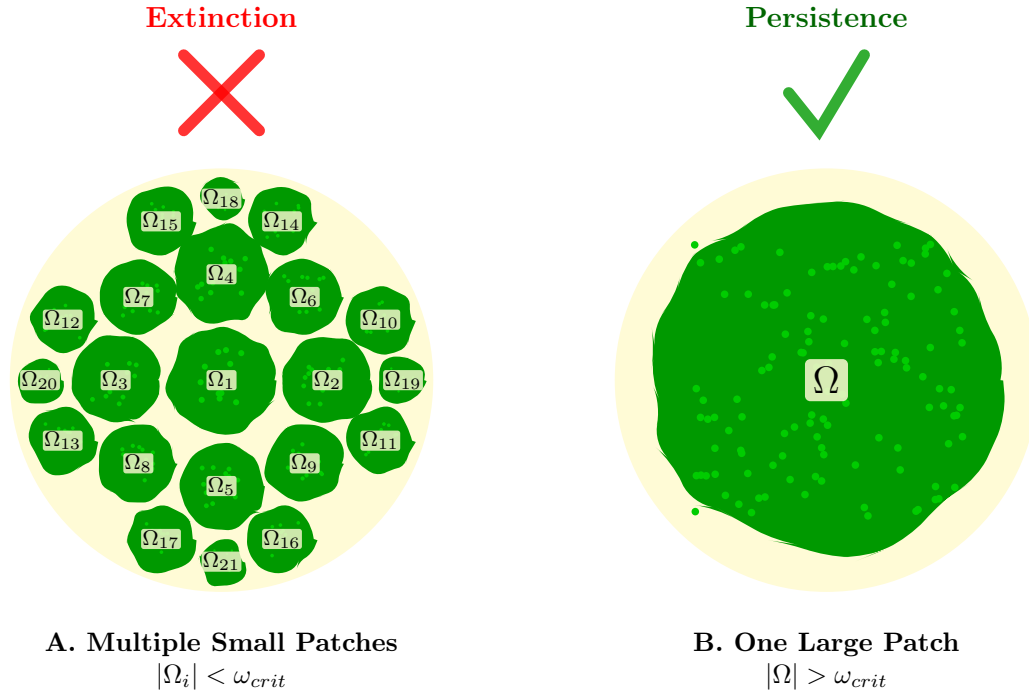


Figure 1.3: Visual representation of the Critical Patch Size concept. **Panel A:** A domain smaller than ω_{crit} leads to fragmentation and extinction, represented by small, unsustainable clusters. **Panel B:** A domain larger than ω_{crit} supports the formation of a large, stable vegetation cluster, even if the initial total biomass density is comparable.

However, a sufficiently large habitat is not the sole condition for survival. In ecosystems governed by non-linear reactions, such as the infiltration feedback in the Klausmeier model, vegetation must also exceed a specific local density to initiate self-facilitation. If plants patches are too sparse, they cannot retain enough resource to overcome mortality, regardless of the patch size.

This observation leads to the concept of a threshold on the initial biomass concentration.

Definition 1.3 (Critical Aggregation Threshold). The **Critical Aggregation Threshold**, denoted by v_{crit} , is the minimum local biomass density required to trigger the positive feedback mechanism necessary for survival.

- **Extinction Regime:** If the initial biomass density v_0 satisfies $\max_{x \in \Omega} v_0(x) < v_{crit}$, the facilitation effects are too weak to counter mortality. Consequently, the system converges to the bare-soil state ($v \rightarrow 0$) even on large domains.

- **Persistence Regime:** If the initial biomass is aggregated such that $\max_{x \in \Omega} v_0(x) > v_{crit}$, the vegetation has a chance to self-organize into stable pattern (either spatially uniform or non-uniform).

This phenomenon highlights that *spatial organization* is as crucial as total biomass. A population that would perish if spread uniformly can survive if aggregated into a dense cluster, as illustrated in Figure 1.4.

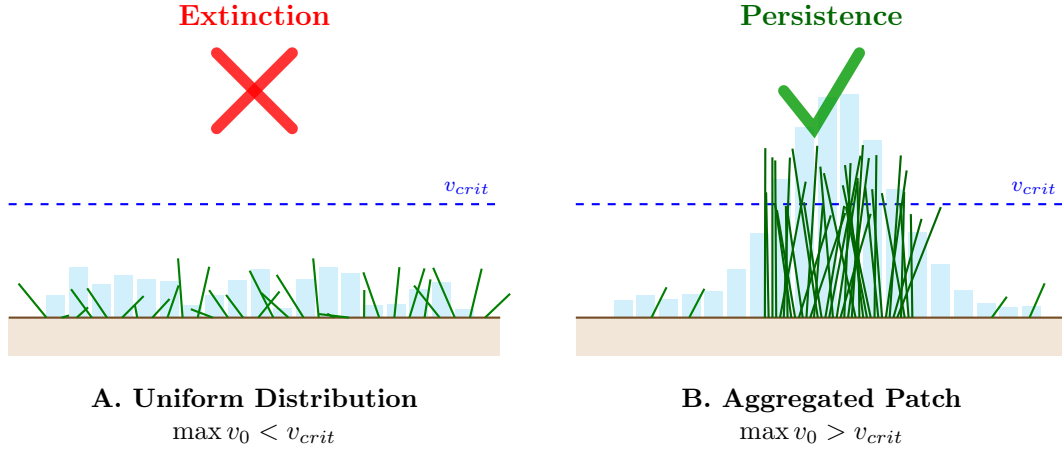


Figure 1.4: Side-profile visualization of the impact of biomass aggregation. The light blue bars represent the local biomass density histogram, while the green curves represent individual plants. **Panel A:** Vegetation spread uniformly stays below the critical threshold v_{crit} (dashed blue line), failing to trigger facilitation. **Panel B:** The biomass is aggregated into a central patch, exceeding v_{crit} locally, which allows the vegetation to survive and grow.

1.2 Modeling with non-local dispersal

Classical models of population dynamics typically rely on the Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ to describe the spatial spread of organisms. This approach assumes that movement is a local phenomenon, analogous to Brownian motion, where the flux of biomass depends solely on the local density gradient.

However, in many biological contexts, dispersal is inherently non-local. Plants disperse seeds over long distances via wind or animals; insects and animals can travel significant distances in short time intervals compared to their reproduction cycles. To capture these long-range interactions, we replace the local diffusion operator Δ with a non-local dispersal operator $\mathcal{L}$ defined using the integral kernel J .

The kernel $J : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ represents the probability distribution of the jump varying from the origin to a given point. We typically assume the following properties of this kernel

Assumption 1.2.1. The integral kernel $J \in C^0(\mathbb{R}^n)$ satisfies

(J1) **Positivity:** $J(z) \geq 0$ for all $z \in \mathbb{R}^n$, and $J(0) > 0$.

(J2) **Symmetry:** $J(z) = J(-z)$ for all $z \in \mathbb{R}^n$.

(J3) **Normalization:** $\int_{\mathbb{R}^n} J(z) dz = 1$.

(J4) **Finite Second Moment:** $\int_{\mathbb{R}^n} J(z)|z|^2 dz < \infty$.

1.2.1 Derivation and properties of the non-local dispersal operator

To motivate the mathematical structure of the operator $\mathcal{L}$, let us consider the net balance of individuals at a specific location x within the habitat. In a discrete framework, where the population is distributed among a set of locations $\{x_i\}_{i=1}^N$, the change in density is determined by the difference between the total influx from all other patches and the total outflux. If $\mathbb{P}(x_j \rightarrow x_i)$ denotes the transition rate from location x_j to x_i , the balance can be expressed as follows

$$\begin{aligned} \text{Total Balance at } x_i &= \text{Inflow}(x_i) - \text{Outflux}(x_i) \\ &= \sum_{j=1}^N \mathbb{P}(x_j \rightarrow x_i) v(x_j) - \sum_{j=1}^N \mathbb{P}(x_i \rightarrow x_j) v(x_i) \\ &\xrightarrow{N \rightarrow \infty} \int_{\text{domain}} J(x, y) v(y) dy - \int_{\text{domain}} J(y, x) v(x) dy. \end{aligned}$$

By assuming a spatially homogeneous environment, the kernel $J(x, y)$ depends only on the relative distance between locations, $J(x - y)$. This leads to the general definition of the nonlocal dispersal operator

$$\mathcal{L}v(x) = \underbrace{\int_{\text{domain}} J(x - y) v(y) dy}_{\text{individuals arriving at } x} - \underbrace{\int_{\text{domain}} J(y - x) v(x) dy}_{\text{individuals leaving } x}. \quad (1.2.1)$$

The choice of the domain of integration and the behavior of individuals at the boundary of the habitat leads to operators corresponding to the classical Dirichlet and Neumann boundary conditions. In the Proposition below, we gather the fundamental mathematical properties of $\mathcal{L}$ that are independent of these specific boundary formulations.

Proposition 1.2.1 (Properties of $\mathcal{L}$). *Let J be a dispersal kernel satisfying Assumption 1.2.1. For a sufficiently regular function v defined on the bounded domain Ω (and suitably extended if required by the chosen boundary conditions), the nonlocal operator $\mathcal{L}$ enjoys the following properties*

(P1) **Linearity and Boundedness:** $\mathcal{L}$ is a linear operator. It is bounded in all of the $L^p(\Omega)$ norms with $p \geq 1$ and the C^0 (supremum) norm.

(P2) **Compact Perturbation:** The integral component of the operator, $\mathcal{K}v(x) = \int_{\text{domain}} J(x - y) v(y) dy$, is a compact operator, hence the full operator $\mathcal{L}$ is a compact perturbation of the multiplication operator $\mathcal{M}v(x) = - \int_{\text{domain}} J(y - x) v(x) dy$.

(P3) **Self-Adjointness:** $\mathcal{L}$ is a self-adjoint operator on $L^2(\Omega)$. In particular for all functions $u, v \in L^2(\Omega)$ is satisfies

$$\langle \mathcal{L}u, v \rangle = \langle u, \mathcal{L}v \rangle. \quad (1.2.2)$$

(P4) **Dirichlet Form Symmetrization:** The quadratic form associated with the operator can be expressed as a symmetric double integral over the corresponding integration domains

$$\langle \mathcal{L}v, v \rangle = -\frac{1}{2} \int_{\text{domain}} \int_{\text{domain}} J(x-y) (v(y) - v(x))^2 dx dy. \quad (1.2.3)$$

(P5) **Sign at Extrema:** If v attains a global maximum at $x_{\max} \in \Omega$, then $\mathcal{L}v(x_{\max}) \leq 0$. Conversely, if v attains a global minimum at $x_{\min} \in \Omega$, then $\mathcal{L}v(x_{\min}) \geq 0$.

Proof. Proofs of all these properties are well known and either straightforward or even trivial, thus we skip it. $\square$

1.2.2 Hostile environment (Dirichlet-type)

When the habitat Ω is surrounded by a completely hostile environment (e.g., an island surrounded by the ocean, or an oasis surrounded by hyper-arid bedrock), individuals that leave the domain die immediately and do not return.

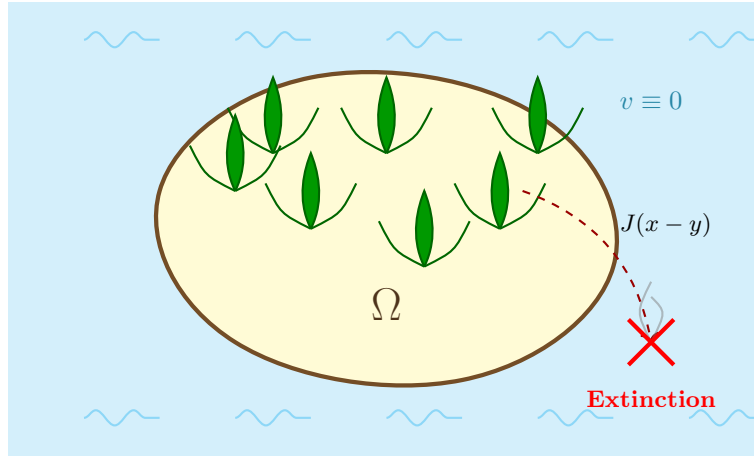


Figure 1.5: Visual interpretation of the Dirichlet boundary condition. The habitat Ω (island) is surrounded by a hostile environment (ocean). Individuals that disperse outside the domain (represented by the dashed arrow) cannot survive ($v = 0$), resulting in a net loss of biomass for the system.

Mathematically, we consider the dispersal process over the whole space $\mathbb{R}^n$, but we impose that the population density is zero outside the habitat, e.g. $v \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. Hence we need to carefully choose the proper Banach or Hilbert space we are working with. We suggest the following choices

$$C_{0,\Omega}(\mathbb{R}^n) = \{v \in C(\Omega) \mid v(x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \Omega\}, \quad (1.2.4)$$

$$L^2_{0,\Omega}(\mathbb{R}^n) = \{v \in L^2(\Omega) \mid v(x) = 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}. \quad (1.2.5)$$

In such spaces the dispersal operator takes the form

$$\mathcal{L}_D v(x) = \int_{\Omega} J(x-y) v(y) dy - v(x) \quad x \in \Omega. \quad (1.2.6)$$

Here, the first term represents the mass arriving at x from inside the domain. The second term, $-v(x) = -v(x) \int_{\mathbb{R}^n} J(y-x) dy$, represents the total mass leaving x . Note that the mass leaving x and landing outside Ω is effectively removed from the system. This operator is non-conservative, thus the "boundary loss" mentioned in Definition 1.2 is strictly positive.

1.2.3 Closed habitat (Neumann-type)

In many ecological experiments or fragmented landscapes, the domain boundaries act as barriers. Individuals may sense the boundary and refrain from crossing it, or they may bounce back. This is often referred to as a "no-flux" condition.

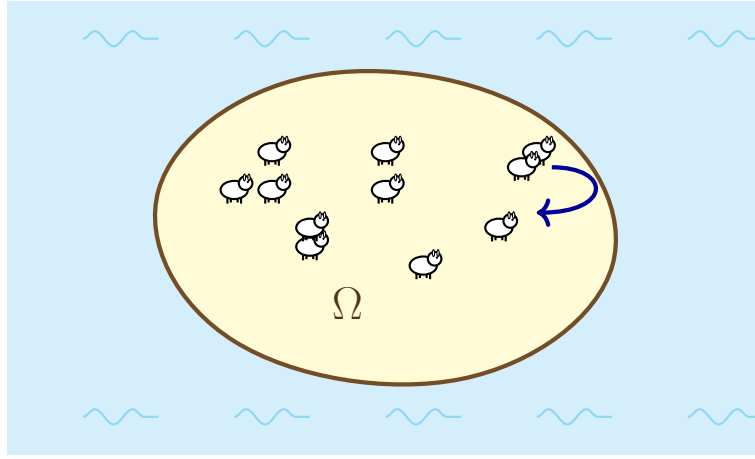


Figure 1.6: *Visual interpretation of the Neumann boundary condition. Unlike the passive seed dispersal in the Dirichlet case, here individuals (represented as animals) sense the hostile boundary or are physically confined. They refrain from crossing into the ocean, resulting in zero net flux across the boundary and conservation of total biomass.*

To model this, we assume that dispersal is restricted entirely to Ω . Individuals at x only jump to locations $y \in \Omega$. Here we can work in standard function spaces such as $C(\overline{\Omega})$ or $L^2(\Omega)$. In these spaces, the corresponding nonlocal operator is defined as

$$\mathcal{L}_N v(x) = \int_{\Omega} J(x-y)(v(y) - v(x)) dy \quad x \in \Omega. \quad (1.2.7)$$

By integrating over the domain and applying Fubini's theorem (given the symmetry of J), we can easily see that $\int_{\Omega} \mathcal{L}_N v(x) dx = 0$, which reflects the strict conservation of mass. This operator is used when modeling populations in closed environments, such as tanks, fenced reserves, or islands, or when assuming that the timescale of evolution is short enough that migration in and out of the domain is negligible. Unlike the Dirichlet case, any spatially constant solution is always a stationary state of the nonlocal heat equation governed by this operator.

1.3 Reaction-dispersal models

The study of spatial ecology fundamentally relies on understanding how movement and local population dynamics interact to shape the distribution of species. While the previous sections

focused on the mathematical derivation and properties of the nonlocal dispersal operator $\mathcal{L}$, we now integrate this spatial component with local growth kinetics. This synthesis leads to the class of reaction-dispersal models, which provide a robust framework for investigating the persistence, spread, and spatial patterning of populations in bounded habitats.

1.3.1 General formulation

Before investigating complex multi-species interactions, it is essential to understand the dynamics of a single population evolving in a bounded habitat. By combining the nonlocal dispersal operators introduced in the previous section with local population growth kinetics, we obtain the scalar reaction-dispersal equation:

$$\frac{\partial v}{\partial t} = d_v \mathcal{L}v + f(v), \quad x \in \Omega, t > 0, \quad (1.3.1)$$

where $v := v(x, t)$ represents the population density, $d_v > 0$ is the dispersal rate, and $f(v)$ describes the local reproduction and mortality rates.

1.3.2 Motivating example: Nonlocal Logistic model with Allee effect

An ecologically vital extension of the scalar reaction-dispersal system is the nonlocal logistic model incorporating the Allee effect. It is obtained by specifying the reaction term $f(v)$ in equation (1.3.1) to account for reduced per capita growth rates at low population densities. The governing equation takes the following form:

$$\frac{\partial v}{\partial t} = d_v \mathcal{L}v + \underbrace{rv \left(1 - \frac{v}{K}\right) \left(\frac{v}{A} - 1\right)}_{\text{logistic growth with Allee effect}}, \quad (1.3.2)$$

where $r > 0$ is the intrinsic growth rate, $K > 0$ denotes the carrying capacity of the environment, and $A \in (0, K)$ represents the critical Allee threshold.

Historically, while classical logistic models (like the Fisher-KPP equation) assume maximum per capita growth at near-zero densities, the Allee effect captures the reality of many biological populations where a minimum local density A is required to overcome challenges such as finding a mate, cooperative defense, or altering the local micro-environment. If the local density $v(x, t)$ drops below A , the growth rate becomes negative, driving the local population toward extinction. Incorporating a nonlocal dispersal operator $\mathcal{L}$ into this framework is particularly important, as the survival of the population heavily depends not only on the total biomass but on its spatial aggregation. The interplay between the homogenizing nature of the dispersal operator and the bistable nature of the Allee reaction kinetics dictates whether a spatially heterogeneous initial population will merge and survive, fragment or perish.

1.4 Reaction-diffusion-dispersal models

While single equation models provide fundamental insights into population survival and spread, real ecosystems are rarely composed of isolated entities. They are typically characterized by complex interactions such as consumer-resource dynamics, predator-prey cycles, or competitive hierarchies. From a mathematical perspective, capturing these interactions requires systems

of coupled evolution equations. A particularly compelling class of problems arises when the interacting components exhibit fundamentally different transport mechanisms. In this work, we focus on hybrid *reaction-diffusion-dispersal* systems, where one component diffuses locally while another disperses nonlocally.

1.4.1 General formulation

Consider a system describing two interacting components, represented by densities $v(x, t)$ and $w(x, t)$, evolving within a bounded habitat Ω . The general mathematical structure of such a reaction-diffusion-dispersal system is given by

$$\begin{cases} \frac{\partial v}{\partial t} = d_v \mathcal{L}v + f(v, w), & x \in \Omega, \ t > 0, \\ \frac{\partial w}{\partial t} = d_w \Delta w + g(v, w), & x \in \Omega, \ t > 0. \end{cases} \quad (1.4.1)$$

Here, parameters $d_v, d_w > 0$ denote their respective dispersal and diffusion rates, while the non-linear reaction terms f and g capture the local interaction kinetics between the two components.

1.4.2 Motivating example: Nonlocal Klausmeier-Gray-Scott model

The primary ecological motivation for studying such hybrid systems in this dissertation originates in the dynamics of semi-arid ecosystems. A canonical framework for understanding these environments is the Klausmeier model, which describes the continuous feedback between vegetation biomass (v) and soil water availability (w). In classical formulations, both vegetation and water are assumed to spread via local diffusion. However, empirical observations strongly suggest that while soil moisture diffusion is indeed a local process driven by capillary forces, plant reproduction via seed dispersal is inherently nonlocal, often mediated by wind or animals over distances far exceeding the immediate vicinity of the parent plant.

To accurately capture this multi-scale spatial reality, we consider a generalized, nonlocal variant of the Klausmeier model on a flat landscape. In its dimensionless form, the governing system is formulated as

$$\begin{cases} \frac{\partial v}{\partial t} = d_v \mathcal{L}v + \overbrace{v^2 w}^{\text{growth}} - \overbrace{Bv}^{\text{mortality}}, \\ \frac{\partial w}{\partial t} = d_w \Delta w - \underbrace{v^2 w}_{\text{uptake}} - \underbrace{w}_{\text{evaporation}} + \underbrace{A}_{\text{rainfall}}, \end{cases} \quad (1.4.2)$$

subject to appropriate boundary conditions (e.g., lethal exterior conditions for plants and zero-flux boundaries for water). Here, the nonlinear term $v^2 w$ models the strong facilitative feedback: established vegetation enhances soil permeability, increasing water infiltration, which in turn accelerates biomass growth. The parameter $B > 0$ represents the linear plant mortality rate, $A > 0$ denotes the constant precipitation influx, and the linear decay of w accounts for natural evaporation and drainage. This nonlocal model serves as a vital mathematical tool for investigating the emergence of vegetation patterns and assessing the resilience of ecosystems against abrupt desertification.

1.5 Author's main contributions

The overarching goal of this dissertation is to bridge the gap between abstract functional analysis, numerical methods, and theoretical ecology in the context of non-local dispersal. To this end, the original results presented in this work can be grouped into three main categories: theoretical analysis, numerical analysis, and theoretical biology.

1.5.1 Theoretical analysis

In the realm of pure and applied mathematical analysis, this dissertation rigorously establishes the functional properties of non-local dispersal operators and the well-posedness of the associated differential equations.

Contribution 1 (Spectral Theory of Non-local Operators). *We provide a comprehensive spectral characterization of the non-local dispersal operator $\mathcal{L}$ on bounded domains. Specifically, we explicitly determine the continuous spectrum for the Neumann-type operator $\mathcal{L}_N$ (Theorem 2.2.1) and its Dirichlet counterpart (Remark 2.2.1). We establish geometric and kernel-dependent conditions that guarantee the existence of a strictly positive spectral gap (Proposition 2.3.1 and Lemma 2.3.1). Furthermore, we prove the existence of a discrete spectrum and formulate the exact max-min characterization of the principal eigenvalue for both Neumann and Dirichlet boundary conditions (Theorem 2.4.1 and Theorem 2.4.2).*

Contribution 2 (Existence and Structure of Stationary Solutions). *For the time-independent non-local problems, we establish the criteria for the existence and uniqueness of spatially heterogeneous stationary states (Theorem 3.2.1). We rigorously prove the emergence of discontinuous spatial patterns in regimes of highly non-local dispersal (Theorem 3.2.4) and apply bifurcation theory to construct continuous patterned solutions near the uniform steady states (Theorem 3.2.3).*

Contribution 3 (Well-posedness and Long-time Behavior of Evolutionary Problems). *We establish the global-in-time existence, uniqueness, and continuous dependence on initial data for the Cauchy problems associated with non-local evolution equations (Theorems 4.2.1, 4.3.1 and Remark 4.2.1). By constructing invariant regions (Theorems 4.2.2 and 4.3.2) and utilizing the spectral properties of $\mathcal{L}$, we provide sharp analytical conditions for the exponential extinction or uniform persistence of solutions as $t \rightarrow \infty$ (Theorems 4.2.3, 4.2.4, and 4.2.5).*

1.5.2 Numerical analysis

To bypass the analytical intractability of complex, multi-component non-local systems (such as the non-local Klausmeier-Gray-Scott model), a robust numerical framework was developed.

Contribution 4 (Development of a Pseudo-Spectral Scheme). *We develop and implement a highly efficient pseudo-spectral numerical method (Section 5.3) based on Legendre polynomials, designed specifically to handle the dense convolution matrices arising from non-local dispersal operators.*

Contribution 5 (Convergence and Error Analysis). *We rigorously prove the strong convergence of the finite-dimensional kernel approximation in $L^2(\Omega)$ (Theorem 5.3.1). We also study the numerical error for the approximation of the continuous convolution operator $\mathcal{K}$ and the local multiplicative functions. Crucially, we demonstrate that the proposed pseudo-spectral method*

may achieve exponential (spectral) accuracy for sufficiently regular kernels. Furthermore, we successfully benchmark the temporal integration errors to ensure numerical reliability.

1.5.3 Theoretical biology

The mathematical and numerical tools developed in this work are utilized to answer pressing ecological questions regarding the survival and self-organization of species in fragmented habitats.

Contribution 6 (Formalization of Ecological Thresholds). *We provide a rigorous mathematical formalization of the Critical Patch Size (Definition 1.2 and Remark 4.2.2) and the Critical Aggregation Threshold (Definition 1.3) concepts for non-local models. We demonstrate analytically and numerically how the interplay between the principal eigenvalue of the dispersal operator and local reaction kinetics dictates the survival or extinction of a population in a finite habitat.*

Contribution 7 (Impact of Heavy-Tailed Dispersal on Ecosystem Resilience). *Through extensive numerical continuation and bifurcation analysis (Sections 3.4.2 and 3.4.3), we quantify the resilience premium provided by long-range (heavy-tailed) dispersal strategies. We show that ecosystems governed by fat-tailed dispersal kernels can persist in significantly smaller and more fragmented habitats compared to those restricted by thin-tailed or purely local diffusion regimes (Figures 3.2 and 3.5), effectively shifting the tipping points of the system.*

Contribution 8 (Morphology of Non-local Patterns and Fragmentation). *We uncover fundamentally different spatial morphologies induced by non-local interactions. Notably, we document and analyze the "boundary drop-off" phenomena, where populations aggregate heavily near the edges of a habitat (Figures 3.3 and 3.9)—a structural feature entirely absent in classical local reaction-diffusion models. Furthermore, we demonstrate that under slow dispersal, the lack of spatial homogenization allows for the emergence of sharp 2D fragmented survival zones (Figure 4.3), rescuing populations from extinction driven by the Allee effect.*

1.6 Notation and structure

1.6.1 Structure

The remainder of this dissertation is organized into four main chapters, guiding the reader from abstract functional analysis of non-local operators, through stationary and evolutionary problems, to numerical methods and simulations.

Chapter 2: Non-local spectral theory. This chapter establishes the rigorous functional-analytic foundation for the non-local dispersal operators. We deeply analyze the spectral properties of both Neumann-type ($\mathcal{L}_N$) and Dirichlet-type ($\mathcal{L}_D$) operators. It includes the explicit formulation of the continuous spectrum, the mathematical proof of the existence of a spectral gap under specific geometric and kernel assumptions, and the characterization of the discrete spectrum (eigenvalues and eigenfunctions) using variational techniques.

Chapter 3: Non-local stationary problems. Building upon the spectral theory, this chapter investigates the time-independent (steady-state) solutions of non-local equations. We analyze scalar reaction-dispersal models to find uniform solutions and explore the conditions for the emergence of continuous and discontinuous spatial patterns. The chapter concludes with the

theoretical and bifurcation analysis of multi-component systems, focusing on the hybrid non-local Klausmeier-Gray-Scott model, revealing how different dispersal strategies dictate spatial self-organization.

Chapter 4: Non-local evolutionary problems. This chapter shifts the focus to the transient, time-dependent dynamics of populations. We prove the well-posedness (existence, uniqueness, and continuous dependence on initial data) of the Cauchy problems associated with non-local evolution equations. By constructing invariant regions and utilizing the principal eigenvalues derived in Chapter 2, we establish sharp criteria for the long-time asymptotic behavior of the system, specifically proving conditions under which a population faces exponential extinction versus when it achieves uniform persistence and stabilizes.

Chapter 5: Numerical methods. The final chapter is dedicated to the computational framework necessary to simulate and visualize the complex dynamics discussed theoretically. We review classical quadrature methods (Trapezoidal, Simpson's) and then introduce a highly efficient pseudo-spectral method tailored for non-local convolution operators. The chapter provides a rigorous convergence analysis, error bounding, and benchmarking of the numerical scheme.

1.6.2 Notation

To maintain consistency and clarity throughout the dissertation, we adopt the standard mathematical notation outlined in Table 1.1. While most symbols follow standard conventions in functional analysis and partial differential equations, some are specific to the non-local framework discussed in this work.

Symbol	Description / Definition
<i>Domains and Boundaries</i>	
$\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{C}$	Real, natural, integer and complex numbers.
$\mathbb{R}^n, \mathbb{T}^n$	n -dimensional Euclidean space and torus.
Ω	A bounded, open spatial domain in $\mathbb{R}^n$ representing the habitat (patch, cluster).
$\bar{\Omega}, \partial\Omega$	The closure and the boundary of the domain Ω , respectively.
$ \Omega $	The Lebesgue measure (volume or area) of the domain Ω .
<i>Function Spaces</i>	
$L^p(\Omega)$	Lebesgue space of p -integrable functions on Ω (typically $p \in \{2, \infty\}$).
$L^2_{0,\Omega}(\mathbb{R}^n)$	Space of $L^2(\mathbb{R}^n)$ functions that vanish almost everywhere outside Ω .
$W^{k,p}(\Omega)$	Sobolev space of functions with $k \in \mathbb{N}$ weak derivatives in $L^p(\Omega)$.
$W^{k,2}_0(\Omega)$	Sobolev space of functions with $k \in \mathbb{N}$ weak derivatives in $L^2(\Omega)$ and vanishing almost everywhere outside Ω .
$C^k(\Omega), C_0(\Omega)$	Spaces of $k \in \mathbb{N}$ times continuously differentiable functions in Ω , and continuous functions in $\bar{\Omega}$ and vanishing on $\partial\Omega$ respectively.
$C_{0,\Omega}(\mathbb{R}^n)$	Space of functions continuous in Ω and vanishing on the complement $\mathbb{R}^n \setminus \Omega$.
$\ \cdot\ _p, \ \cdot\ _{k,p}, \ \cdot\ _{C^0}$	The standard norm in the spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$ and $C^0(\Omega)$, respectively.
<i>Operators and Kernels</i>	
$J(x-y), K(x,y)$	The non-local dispersal kernel, e.g. probability distribution of jumps.
$\mathcal{L}$	General non-local dispersal operator.
$\mathcal{L}_N, \mathcal{L}_D$	Non-local operators with Neumann-type (closed) and Dirichlet-type (hostile) boundary conditions, respectively.
$\mathcal{K}$	The integral part of dispersal operator: $\mathcal{K}v(x) = \int J(x-y)v(y)dy$.
$\mathcal{M}, b(x)$	The multiplicative part of dispersal operator $\mathcal{M}v(x) = b(x)v(x)$.
Δ	The classical Laplace operator $\sum_{i=1}^n \partial_{x_i}^2$.
<i>Spectral Theory</i>	
$\sigma(\mathcal{L})$	The full spectrum of the operator $\mathcal{L}$.
$\sigma_c(\mathcal{L})$	The continuous (essential) spectrum of the operator $\mathcal{L}$.
β_k	The k -th eigenvalue of the non-local operator $\mathcal{L}$.
v_k	The corresponding eigenfunction associated with the eigenvalue β_k .
<i>Ecological Notation</i>	
ω_{crit}	The Critical Patch Size required for population persistence in a finite habitat.
R_{crit}	The Critical Boundary Drop-off, e.g. maximal density of population at the boundary
v_{crit}	The Critical Aggregation Threshold required to trigger positive feedback.

Table 1.1: Summary of the principal mathematical and ecological notation used in this dissertation.

Chapter 2

Non-local spectral theory

We begin with the detailed spectral analysis of the non-local dispersal operators. We will focus our attention on the Neumann-type operator $\mathcal{L}_N$ (closed environment) and, for complementarity, we will also study the spectrum of the Dirichlet-type operator $\mathcal{L}_D$ (hostile environment). For that, we decompose the Neumann dispersal operator $\mathcal{L}_N : L^2(\Omega) \rightarrow L^2(\Omega)$ as follows

$$\mathcal{L}_N u(x) = \int_{\Omega} J(x-y)u(y) dy + b(x)u(x), \quad (2.0.1)$$

where we denoted the multiplicative function

$$b(x) = - \int_{\Omega} J(x-y) dy. \quad (2.0.2)$$

It is also very convenient to introduce the notation for range of b , namely we denote the set

$$b[\overline{\Omega}] = \left\{ - \int_{\Omega} J(x-y) dy \mid x \in \overline{\Omega} \right\} \quad (2.0.3)$$

Now, we discuss the state-of-art knowledge about the spectrum σ of the dispersal operator $\mathcal{L}$.

2.1 Literature review

A central challenge in the study of non-local dispersal is the lack of compactness, which fundamentally changes the spectral structure. The implications of this property for nonautonomous, periodic, and random environments were extensively investigated by Shen, Vickers, and Hutson [50, 28]. Their work established specific conditions under which a principal eigenvalue is guaranteed to exist, and demonstrated that temporal fluctuations strictly increase the principal Lyapunov exponent, yielding the ecological conclusion that heterogeneous environments mathematically facilitate species invasion.

Foundational efforts to understand the eigenvalue problem for such operators were made by Coville, Rossi, and their collaborators (see e.g., [13, 14, 23]), as well as comprehensively detailed in the monograph by Andreu-Vaillo et al. [3]. This line of research primarily focused on establishing maximum principles, exploring the asymptotic behavior of solutions, and analyzing the criteria for the existence of principal eigenvalues, often highlighting the fundamental differences between local and non-local operators under various boundary conditions. We highlight the following Theorems that are crucial to this work

Theorem 2.1.1 (Principle eigenvalue of the Neumann type dispersal). *Let β_1 be the first non-trivial eigenvalue of $\mathcal{L}_N$. This value is well defined and satisfies the following bounds*

$$\max_{x \in \bar{\Omega}} b(x) \leq \beta_1 < 0. \quad (2.1.1)$$

Proof. For the detailed proofs of lower and upper bounds of β_1 we refer the reader to Proposition 3.4 and Lemma 3.5 in [3]. $\square$

Note, that even though Theorem 2.1.1 states that the candidate for eigenvalue β_1 is well defined, it is not generally known whether there exists a corresponding eigenfunction. In fact, it might be that such a function does not exist, see [14, 52] for examples. On the other hand, situation in the hostile environment case (Dirichlet) is much more optimistic.

Theorem 2.1.2 (Principal eigenvalue of the Dirichlet type dispersal). *Let $\mathcal{L}_D$ be the non-local Dirichlet operator. There exists a principal eigenvalue β_1 associated with $\mathcal{L}_D$, and a corresponding non-negative eigenfunction $v_0 \geq 0$. Moreover, v_0 is strictly positive in Ω (with a positive continuous extension to $\bar{\Omega}$) and vanishes strictly outside Ω , meaning the boundary value is not taken in the usual classical sense and a discontinuity occurs on $\partial\Omega$.*

Proof. For the detailed proof of the existence of the principal eigenvalue and the exact spatial properties of the corresponding non-negative eigenfunction v_0 , we refer the reader to Section 2.1.1, Proposition 2.3, and Remark 2.4 in [3]. $\square$

In more recent work, Foss, Radu, and Wright [21] established a robust framework for minimizing non-local energy functionals. Their work demonstrates how to derive non-local Euler-Lagrange equations and prove the existence and regularity of minimizers using joint convexity and non-local Poincaré inequalities, notably without relying on the restrictive growth conditions required in classical Sobolev spaces.

Subsequent research focused on generalized, variational characterizations of the principal eigenvalue. Berestycki, Coville, and Vo [9] established the equivalence of various spectral definitions and analyzed kernel scaling asymptotics. Li, Coville, and Wang [33] further utilized the strong maximum principle to derive exact max-min characterizations of the principal eigenvalue, rigorously proving its algebraic simplicity and isolation. Recently, Wang and Zhang [66] extended this framework to cooperative nonlocal systems. By approximating the generalized principal eigenvalue through monotonic control matrices, they demonstrated that this spectral bound strictly dictates the threshold dynamics and global stability of the coupled systems.

Parallel to the development of variational formulations, extensive research has been devoted to the parametric dependence of the principal spectrum point. Shen and Xie [51] systematically analyzed the impact of spatial inhomogeneity, dispersal rate, and dispersal distance across Dirichlet, Neumann, and periodic boundary conditions, establishing rigorous asymptotic limits and persistence criteria. Bao and Shen [7] subsequently extended this framework to time-periodic cooperative systems, deriving explicit geometric conditions, such as the vanishing inhomogeneity condition and small dispersal distance limits, that guarantee the existence and algebraic simplicity of the principal eigenvalue. More recently, Vo [64] addressed the distinct analytical challenges posed by time-periodic nonlocal Neumann operators. Because the principal eigenvalue in the nonlocal Neumann setting lacks domain monotonicity, Vo developed new techniques to determine its precise asymptotic limits with respect to varying dispersal scales and established the exact spectral criteria required for the validity of the associated maximum principle.

2.2 Continuous spectrum

This section is based on author's results published in [58]. Firstly we show that the set $b[\overline{\Omega}]$ defined by equation (2.0.3), belongs to the spectrum of $\mathcal{L}_N$.

Lemma 2.2.1. *Let J satisfy Assumptions 1.2.1 and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then for b defined as in formula (2.0.2) we obtain the inclusion*

$$b[\overline{\Omega}] \subseteq \sigma(\mathcal{L}_N).$$

Proof. Let $\lambda \in b[\overline{\Omega}]$. Suppose that $\lambda \notin \sigma(\mathcal{L}_N)$. Then for all $v \in L^2(\Omega)$ there is a solution $u \in L^2(\Omega)$ to the equation

$$\mathcal{L}_N u(x) - \lambda u(x) = v(x).$$

As a consequence of bounded inverse theorem, there exists $C > 0$ such that

$$C\|u\|_2 \leq \|\mathcal{L}_N u - \lambda u\|_2. \quad (2.2.1)$$

By the continuity of J , there exists $x_\lambda \in \Omega$ such that $b(x_\lambda) = \lambda$. Hence we define $u_R^\lambda(x) = c_n^{-1/2} R^{-n/2} \mathbf{1}_{B(x_\lambda, R)}(x)$ with $c_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ and compute

$$\begin{aligned} \|\mathcal{L}_N u_R^\lambda - \lambda u_R^\lambda\|_2 &\leq \left\| \int_{\Omega} J(\cdot - y) u_R^\lambda(y) dy \right\|_2 + \|(b - \lambda) u_R^\lambda\|_2 \\ &= c_n^{-1/2} \left(\int_{\Omega} \left(\int_{B(x_\lambda, R)} J(x - y) R^{-n/2} dy \right)^2 dx \right)^{1/2} \\ &\quad + \left(\frac{c_n^{-1}}{R^n} \int_{B(x_\lambda, R)} (b(x) - \lambda)^2 dx \right)^{1/2} \\ &\leq \|J\|_\infty |\Omega|^{1/2} R^{n/2} + \left(\frac{c_n^{-1}}{R^n} \int_{B(x_\lambda, R)} (b(x) - \lambda)^2 dx \right)^{1/2}. \end{aligned}$$

The first component goes to zero as $R \rightarrow 0$. The latter one is estimated by

$$\inf_{x \in B(x_\lambda, R)} |b(x) - \lambda|^2 \leq \frac{c_n^{-1}}{R^n} \int_{B(x_\lambda, R)} (b(x) - \lambda)^2 dx \leq \sup_{x \in B(x_\lambda, R)} |b(x) - \lambda|^2,$$

thus it also vanishes as $R \rightarrow 0$. But this contradicts (2.2.1) since $\|u_R^\lambda\|_2 = 1$. That yields $\lambda \in \sigma(\mathcal{L}_N)$. We conclude Lemma 2.2.1 by the fact that $\sigma(\mathcal{L}_N)$ is a spectrum of a bounded linear operator and hence it is a closed set. $\square$

Next, we show that every other element of the spectrum is an eigenvalue of $\mathcal{L}_N$.

Lemma 2.2.2. *Let J satisfy Assumption 1.2.1 and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then every $\lambda \in \sigma(\mathcal{L}_N) \setminus b[\overline{\Omega}]$ is an eigenvalue of the operator $\mathcal{L}_N$.*

Proof. Denote a set $D = \mathbb{C} \setminus b[\overline{\Omega}]$. Since b is continuous and $\overline{\Omega}$ is compact we obtain that $b[\overline{\Omega}]$ is a compact subset of negative real semiaxis and thus D is open and connected subset of $\mathbb{C}$. Now we re-write the eigenvalue problem using formula (2.0.1) to obtain equation

$$\int_{\Omega} J(x-y)u(y) dy = (\beta - b(x)) u(x). \quad (2.2.2)$$

We divide both sides of equation (2.2.2) to get the fixed point problem and denote the analytic, operator valued function $F : D \rightarrow \mathcal{L}(L^2(\Omega))$ as follows

$$F(\beta)[u](x) := \int_{\Omega} \frac{J(x-y)}{\beta - b(x)} u(y) dy = u(x).$$

Now, let us verify the convergence of the difference quotient

$$\begin{aligned} \frac{F(\beta) - F(\beta_0)}{\beta - \beta_0}[u](x) &= \frac{1}{\beta - \beta_0} \left(\int_{\Omega} \frac{J(x-y)}{\beta - b(x)} u(y) dy - \int_{\Omega} \frac{J(x-y)}{\beta_0 - b(x)} u(y) dy \right) \\ &= - \int_{\Omega} \frac{J(x-y)}{(\beta - b(x))(\beta_0 - b(x))} u(y) dy \\ &\xrightarrow{\beta_0 \rightarrow \beta} - \int_{\Omega} \frac{J(x-y)}{(\beta - b(x))^2} u(y) dy, \end{aligned}$$

which proves that $F : D \rightarrow \mathcal{L}(L^2(\Omega))$ is an analytic operator, as claimed.

We emphasize that since β never intersects the range of b , the integral kernel $K(x, y) = \frac{J(x-y)}{\beta - b(x)}$ has a finite $L^2(\Omega \times \Omega)$ norm and so it is a Hilbert-Schmidt operator. In particular it is a compact operator.

Now, we verify that F is a contraction for some $\beta \in D$. For instance we can take $\beta_0 \in D$ such that

$$\Re(\beta_0) = - \inf_{x \in \Omega} -b(x) + 2\|J\|_{\infty}|\Omega|^{\frac{1}{2}}, \quad \Im(\beta_0) = 0.$$

Having that we can bound the operator norm of $F(\beta_0)$ to obtain the following

$$\|F(\beta_0)\| = \sup_{\|u\|_2=1} \|F(\beta_0)[u]\|_2 \leq \sup_{\|u\|_2=1} \left(\int_{\Omega} \int_{\Omega} \frac{\|J\|^2 u^2(y)}{(\beta_0 - b(x))^2} dy dx \right)^{\frac{1}{2}} \leq \frac{1}{2}$$

Hence $(I - F(\beta_0))^{-1}$ exists. But now, by the Analytical Fredholm Theorem [45], we know that $(I - F(\beta))^{-1}$ exists for all $\beta \in D \setminus \{\beta_k\}_{k=0}^M$, with $M \geq 0$. On the other hand, for any element $\beta_k \in \{\beta_k\}_{k=0}^M$ there exists a corresponding function $\varphi_k \in L^2(\Omega)$ such that (β_k, φ_k) solves the eigenvalue equation. Thus β_k is an eigenvalue. $\square$

We characterise the spectrum $\sigma(\mathcal{L}_N)$ on an arbitrary bounded domain Ω .

Theorem 2.2.1 (General case). *Let J satisfy Assumption 1.2.1 and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the spectrum of $\mathcal{L}_N$ is given by*

$$\sigma(\mathcal{L}_N) = b[\overline{\Omega}] \cup \{\beta_k\}_{k=0}^M, \quad (2.2.3)$$

where $M \geq 0$ and the set $b[\overline{\Omega}]$ is given by formula (2.0.3), we recall it for purpose of readability

$$b[\overline{\Omega}] = \left\{ - \int_{\Omega} J(x-y) dy \mid x \in \overline{\Omega} \right\}.$$

Moreover, the sequence $\{\beta_k\}_{k=0}^M$ is a finite or infinite set of isolated eigenvalues of $\mathcal{L}_N$.

Proof. Combining the results obtained in Lemma 2.2.1 and Lemma 2.2.2, we finish the proof. $\square$

Remark 2.2.1 (Continuous spectrum for the Dirichlet case). *The spectral analysis conducted above extends directly to the Dirichlet-type operator $\mathcal{L}_D$. Using the analogous decomposition $\mathcal{L}_D u(x) = \int_{\Omega} J(x-y)u(y)dy - u(x)$, the multiplicative function simply becomes the constant $b(x) \equiv -1$. Consequently, the proofs of Lemmas 2.2.1 and 2.2.2 apply identically. The essential difference is that the continuous spectrum of $\mathcal{L}_D$ collapses to a singleton, namely $\sigma_{\text{ess}}(\mathcal{L}_D) = \{-1\}$, rather than an interval.*

2.3 Spectral gap

While Theorem 2.1.1 guarantees that β_1 cannot be below the continuous spectrum, it does not prevent β_1 from collapsing into σ_c (i.e., $\beta_1 = \sup \sigma_c$), which would result in the non-existence of the first non-trivial eigenfunction. To ensure the existence of an isolated eigenvalue β_1 , we need to assume on of the supplementary conditions regarding the geometry of Ω and the kernel J .

Assumption 2.3.1. *The domain $\Omega \subset \mathbb{R}^n$ is open and bounded. There exist $\Omega_1, \Omega_2 \subset \Omega$ such that $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 \cup \Omega_2 = \Omega$ and the following inequality holds true*

$$\frac{4}{|\Omega|} \iint_{\Omega_1 \times \Omega_2} J(x-y) dx dy < \min_{x \in \overline{\Omega}} \int_{\Omega} J(x-y) dy. \quad (2.3.1)$$

Assumption 2.3.2. *The domain $\Omega \subset \mathbb{R}^n$ is open, bounded and satisfies interior sphere condition. We assume that Ω is large enough so the following inequality holds true*

$$\frac{1}{|\Omega|^{2/n}} \frac{m_2(J)}{2 I_{\max}} < \min_{x \in \overline{\Omega}} \int_{\Omega} J(x-y) dy, \quad (2.3.2)$$

where we denoted respectively the second moment of dispersal and a maximal moment of inertia by $m_2(J) = \int_{\mathbb{R}^n} J(z)|z|^2 dz$, $I_{\max} = \max_{1 \leq i \leq n} \int_{\Omega_0} x_i^2 dx$ respectively.

Having stated that, we begin with the justification of the Assumption 2.3.1.

Proposition 2.3.1. *Let Ω and J satisfy Assumptions 1.2.1 and 2.3.1. Then the following strict lower bound holds true*

$$\sup \sigma_c < \beta_1.$$

Proof. We construct a test function $v \in A_1$ given by $v(x) = 1/\sqrt{|\Omega|}$ for $x \in \Omega_1$ and $v(x) = -1/\sqrt{|\Omega|}$ for $x \in \Omega_2$. Since $|\Omega_1| = |\Omega_2|$, we have $\int_{\Omega} v(x) dx = 0$ and $\|v\|_{L^2} = 1$. Using the symmetrization identity, we compute the Dirichlet form associated to $\mathcal{L} = \mathcal{L}_N$

$$\begin{aligned} \langle \mathcal{L}v, v \rangle &= -\frac{1}{2} \iint_{\Omega \times \Omega} J(x-y)(v(x) - v(y))^2 dx dy \\ &= -2 \iint_{\Omega_1 \times \Omega_2} J(x-y) \left(\frac{2}{\sqrt{|\Omega|}} \right)^2 dx dy \\ &= -\frac{4}{|\Omega|} \iint_{\Omega_1 \times \Omega_2} J(x-y) dx dy. \end{aligned}$$

On the other hand, $\sup \sigma_c = \max(-\int_{\Omega} J(x-y)dy) = -\min \int_{\Omega} J(x-y)dy$. By the assumption (2.3.1), multiplying both sides by -1 yields $\langle \mathcal{L}v, v \rangle > \sup \sigma_c$. Taking the supremum over all $v \in L^2(\Omega)$ with $\|v\|_2 = 1$ and $\int_{\Omega} v(x) dx = 0$ concludes the proof. $\square$

For the sake of exposition, we provide an example of the kernel J defined in the one-dimensional domain Ω satisfying Assumptions 1.2.1 and 2.3.1.

Example 2.1. Let $\Omega = (-L, L)$ and consider the family of symmetric, generalized exponential dispersal kernels given by

$$J_p(z) = C_p e^{-\lambda|z|^p}, \quad \lambda > 0, \quad p > 0, \quad (2.3.3)$$

where C_p is the normalization constant ensuring $\int_{\mathbb{R}} J_p(z) dz = 1$. The fulfillment of the sufficient condition for the spectral gap (2.3.1) strongly depends on the shape parameter p :

- If $p \leq 1$, the condition holds true for any domain length $L > 0$ and any dispersal parameter $\lambda > 0$.
- If $p > 1$, the condition holds true provided that the non-dimensional parameter $\eta = L\lambda^{1/p}$ is sufficiently large.

Proof. We split the domain into $\Omega_1 = (-L, 0)$ and $\Omega_2 = (0, L)$. Evaluating the right-hand side of Assumption 2.3.1, the minimum of the local potential $b(x) = \int_{-L}^L J_p(x-y) dy$ occurs at the boundaries $x = \pm L$, yielding

$$\min_{x \in \overline{\Omega}} \int_{-L}^L J_p(x-y) dy = \int_{-L}^L J_p(L-y) dy = \int_0^{2L} J_p(z) dz.$$

For the cross-term on the left-hand side of (2.3.1), substituting $z = y - x$ and noticing that $|\Omega| = 2L$, we yield

$$\frac{2}{L} \iint_{\Omega_1 \times \Omega_2} J_p(x-y) dy dx = \frac{2}{L} \int_0^L z J_p(z) dz + \frac{2}{L} \int_L^{2L} (2L-z) J_p(z) dz.$$

Let $\Delta(L, \lambda)$ denote the difference between the right-hand side and the left-hand side of (2.3.1). Splitting the integral over $(0, 2L)$ into two intervals $(0, L)$ and $(L, 2L)$, we rearrange the terms as follows:

$$\begin{aligned} \Delta(L, \lambda) &= \int_0^{2L} J_p(z) dz - \left(\frac{2}{L} \int_0^L z J_p(z) dz + \frac{2}{L} \int_L^{2L} (2L-z) J_p(z) dz \right) \\ &= \int_0^L J_p(z) dz + \int_L^{2L} J_p(z) dz - \frac{2}{L} \int_0^L z J_p(z) dz - \frac{2}{L} \int_L^{2L} (2L-z) J_p(z) dz \quad (2.3.4) \\ &= \int_0^L \left(1 - \frac{2z}{L} \right) J_p(z) dz + \int_L^{2L} \left(1 - \frac{2(2L-z)}{L} \right) J_p(z) dz. \end{aligned}$$

Applying the substitution $u = 2L - z$ to the second integral and renaming the dummy variable z to u in the first integral, we obtain

$$\begin{aligned} \Delta(L, \lambda) &= \int_0^L \left(1 - \frac{2u}{L} \right) J_p(u) du + \int_0^L \left(1 - \frac{2u}{L} \right) J_p(2L-u) du \\ &= \int_0^L \left(1 - \frac{2u}{L} \right) [J_p(u) + J_p(2L-u)] du \quad (2.3.5) \\ &= \int_0^L \left(1 - \frac{2u}{L} \right) F(u) du, \end{aligned}$$

where we denoted $F(u) := J_p(u) + J_p(2L - u)$. Using the translation $u = L/2 - t$, we shift the domain of integration to $[-L/2, L/2]$ and note that $du = -dt$

$$\begin{aligned}\Delta(L, \lambda) &= \int_{-L/2}^{-L/2} \left(1 - \frac{2(L/2 - t)}{L}\right) F(L/2 - t) (-dt) \\ &= \int_{-L/2}^{L/2} \left(1 - 1 + \frac{2t}{L}\right) F(L/2 - t) dt \\ &= \int_{-L/2}^{L/2} \frac{2t}{L} F(L/2 - t) dt.\end{aligned}\tag{2.3.6}$$

By splitting the integration domain into two symmetric halves and applying the change of variable $t \mapsto -t$ on the negative half, we obtain

$$\begin{aligned}\Delta(L, \lambda) &= \int_{-L/2}^0 \frac{2t}{L} F(L/2 - t) dt + \int_0^{L/2} \frac{2t}{L} F(L/2 - t) dt \\ &= \int_{L/2}^0 \frac{-2t}{L} F(L/2 + t) (-dt) + \int_0^{L/2} \frac{2t}{L} F(L/2 - t) dt \\ &= - \int_0^{L/2} \frac{2t}{L} F(L/2 + t) dt + \int_0^{L/2} \frac{2t}{L} F(L/2 - t) dt \\ &= \int_0^{L/2} \frac{2t}{L} [F(L/2 - t) - F(L/2 + t)] dt.\end{aligned}$$

Since $t > 0$ strictly inside the integration domain, the inequality $\Delta(L, \lambda) > 0$ holds if $F(z)$ is strictly decreasing on $(0, L)$. We compute the derivative

$$F'(z) = J'_p(z) - J'_p(2L - z) = C_p \lambda p \left[(2L - z)^{p-1} e^{-\lambda(2L-z)^p} - z^{p-1} e^{-\lambda z^p} \right].$$

To determine the sign of $F'(z)$, we define $h(z) = z^{p-1} e^{-\lambda z^p}$ and analyze its derivative for $z > 0$

$$h'(z) = z^{p-2} e^{-\lambda z^p} [(p-1) - \lambda p z^p].$$

Case 1: $p \leq 1$. Since $z > 0$, $\lambda > 0$, and $p-1 \leq 0$, the bracket $[(p-1) - \lambda p z^p]$ is strictly negative. Thus, $h'(z) < 0$ globally on $(0, \infty)$, meaning $h(z)$ is strictly decreasing. Because $z < 2L - z$ for all $z \in (0, L)$, we immediately obtain $h(2L - z) < h(z)$, which implies $F'(z) < 0$. Therefore we have shown that $\Delta(L, \lambda) > 0$ unconditionally for any $L > 0$ and $\lambda > 0$.

Case 2: $p > 1$. Here, $p-1 > 0$, meaning $h(z)$ initially increases before decaying. This causes $F'(z) > 0$ for small arguments. Consequently, Assumption 2.3.1 fails for sufficiently small domains.

However, we can establish a rigorous global bound for the cross-term to find the sufficient conditions for the gap. Returning to the exact expression for the cross-term, we note that for the second integral in (2.3.4) where $z \in [L, 2L]$, the algebraic inequality $2L - z \leq z$ holds. Applying this inequality yields

$$\frac{2}{L} \int_L^{2L} (2L - z) J_p(z) dz \leq \frac{2}{L} \int_L^{2L} z J_p(z) dz.\tag{2.3.7}$$

Summing the integrals over $(0, L)$ and $(L, 2L)$, the entire cross-term is bounded by the first moment of the right half of the kernel $m_1(\lambda) = \int_{-\infty}^{\infty} J_p(z)|z| dz$

$$\frac{2}{L} \iint_{\Omega_1 \times \Omega_2} J_p(x-y) dy dx \leq \frac{2}{L} \int_0^{2L} z J_p(z) dz < \frac{m_1(\lambda)}{L}.$$

To examine the influence of the parameter λ , which controls the variance of the kernel, we introduce a scaling argument. Utilizing the change of variables $u = \lambda^{1/p} z$, we can express the kernel as $J_p(z) = \lambda^{1/p} \tilde{J}_p(\lambda^{1/p} z)$, where $\tilde{J}_p(u) = \tilde{C}_p e^{-|u|^p}$ is the reference unscaled kernel. Consequently, the first moment scales as

$$m_1(\lambda) = \lambda^{-1/p} \int_{-\infty}^{\infty} u \tilde{J}_p(u) du = \lambda^{-1/p} \tilde{m}_1,$$

where $\tilde{m}_1$ is a finite constant dependent only on p .

Applying the same variable change to the local part (the right-hand side of (2.3.1)), we obtain

$$\int_0^{2L} J_p(z) dz = \int_0^{2L\lambda^{1/p}} \tilde{J}_p(u) du.$$

We now define the non-dimensional parameter $\eta := L\lambda^{1/p}$. Using η , we can re-write the strict lower bound for our difference $\Delta(L, \lambda)$ as

$$\Delta(L, \lambda) > \int_0^{2\eta} \tilde{J}_p(u) du - \frac{\tilde{m}_1}{\eta}.$$

Since the reference kernel is normalized, the integral $\int_0^{2\eta} \tilde{J}_p(u) du$ converges strictly to $1/2$ as $\eta \rightarrow \infty$. Conversely, the penalty term $\frac{\tilde{m}_1}{\eta}$ decays to 0. Thus, there exists a threshold $\eta_0 > 0$ such that for all $\eta > \eta_0$, the inequality $\Delta(L, \lambda) > 0$ is satisfied. $\square$

Now, we show that as a domain is uniformly scaled to infinity, the number $\sup \sigma_c$ remains bounded strictly away from zero.

Proposition 2.3.2. *Let $\Omega_0 \subset \mathbb{R}^n$ be any open, bounded, and non-empty domain satisfying the uniform interior sphere condition. For a scaling factor $\lambda > 0$, define the scaled domain $\Omega_\lambda = \{\lambda x : x \in \Omega_0\}$. Assume the dispersal kernel $J \in C^0(\mathbb{R}^n)$ satisfies Assumption 1.2.1.*

Then, the minimum retained mass of the kernel on Ω_λ is asymptotically bounded strictly away from zero

$$\liminf_{\lambda \rightarrow \infty} \left(\min_{x \in \overline{\Omega_\lambda}} \int_{\Omega_\lambda} J(x-y) dy \right) \geq \frac{1}{2} > 0. \quad (2.3.8)$$

Proof. Let $x_\lambda \in \overline{\Omega_\lambda}$ be a point where the minimum retained mass is attained for a given scale λ

$$\min_{x \in \overline{\Omega_\lambda}} \int_{\Omega_\lambda} J(x-y) dy = \int_{\Omega_\lambda} J(x_\lambda - y) dy. \quad (2.3.9)$$

By performing a change of variables $z = x_\lambda - y$, we shift the domain of integration. Let $D_\lambda = x_\lambda - \Omega_\lambda = \{x_\lambda - y : y \in \Omega_\lambda\}$. The integral in (2.3.9) becomes:

$$\int_{\Omega_\lambda} J(x_\lambda - y) dy = \int_{D_\lambda} J(z) dz. \quad (2.3.10)$$

We analyze the geometric properties of the shifted domain D_λ as $\lambda \rightarrow \infty$. By definition, the origin $0 \in \overline{D_\lambda}$ for all $\lambda > 0$, because $x_\lambda \in \overline{\Omega_\lambda}$. Furthermore, D_λ is an open, bounded set derived from scaling Ω_0 by λ and shifting it.

Consider an arbitrary sequence of scaling factors $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$, and let x_{λ_k} be the corresponding sequence of minimum points. We can write $x_{\lambda_k} = \lambda_k \tilde{x}_k$, where $\tilde{x}_k \in \overline{\Omega_0}$. Since $\overline{\Omega_0}$ is compact, we can extract a subsequence (still denoted by λ_k) such that $\tilde{x}_k \rightarrow x^* \in \overline{\Omega_0}$.

We define the shifted reference domain $U_0 = x^* - \Omega_0$. Since $x^* \in \overline{\Omega_0}$, the origin $0 \in \overline{U_0}$. Because Ω_0 is open and non-empty, and we assumed that the domain satisfies the uniform interior sphere condition, we can choose an open ball $B(p, r) \subset U_0$ such that the origin lies exactly on its boundary, meaning $|p| = r$. As $k \rightarrow \infty$, the scaled and shifted domain $D_{\lambda_k} = \lambda_k(\tilde{x}_k - \Omega_0)$ converges geometrically to the scaled domain $\lambda_k U_0$. Since $B(p, r) \subset U_0$, it follows that D_{λ_k} asymptotically contains the expanding ball $B(\lambda_k p, \lambda_k r)$.

Consider an arbitrary point z in the open half-space $H_p = \{z \in \mathbb{R}^n : z \cdot p > 0\}$ and analyze the condition $z \in B(\lambda_k p, \lambda_k r)$. This is equivalent to

$$|z - \lambda_k p|^2 < (\lambda_k r)^2. \quad (2.3.11)$$

Expanding the squared norm and dividing by $\lambda_k > 0$ yields

$$\frac{|z|^2}{\lambda_k} - 2(z \cdot p) + \lambda_k |p|^2 < \lambda_k r^2 \iff \frac{|z|^2}{\lambda_k} - 2(z \cdot p) < \lambda_k (r^2 - |p|^2). \quad (2.3.12)$$

Since we specifically chose the ball such that its boundary touches the origin ($|p| = r$), the right-hand side is exactly zero for all k : $\lambda_k (r^2 - |p|^2) = 0$. For the left-hand side, since $z \in H_p$, we have strictly $z \cdot p > 0$. As $k \rightarrow \infty$, the term $|z|^2/\lambda_k \rightarrow 0$. Thus, the left-hand side converges to a strictly negative value

$$\lim_{k \rightarrow \infty} \left(\frac{|z|^2}{\lambda_k} - 2(z \cdot p) \right) = -2(z \cdot p) < 0. \quad (2.3.13)$$

Consequently, for any fixed $z \in H_p$, there exists an index $K \in \mathbb{N}$ such that inequality (2.3.12) holds strictly for all $k \geq K$. This implies $z \in B(\lambda_k p, \lambda_k r) \subset D_{\lambda_k}$, and thus $\mathbb{1}_{D_{\lambda_k}}(z) = 1$ for all $k \geq K$. Taking the limit infimum confirms pointwise convergence on H_p

$$\liminf_{k \rightarrow \infty} \mathbb{1}_{D_{\lambda_k}}(z) \geq \mathbb{1}_{H_p}(z) \quad \text{for all } z \in \mathbb{R}^n. \quad (2.3.14)$$

Since the integral kernel $J(z) \geq 0$ and $\int_{\mathbb{R}^n} J(z) dz = 1$, we can apply Fatou's Lemma to obtain

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} J(z) \mathbb{1}_{D_{\lambda_k}}(z) dz \geq \int_{\mathbb{R}^n} \liminf_{k \rightarrow \infty} (J(z) \mathbb{1}_{D_{\lambda_k}}(z)) dz \geq \int_{H_p} J(z) dz. \quad (2.3.15)$$

By Assumption 1.2.1, $J(z)$ is symmetric. Thus, integrating it over the half-space H_p we get

$$\int_{H_p} J(z) dz = \frac{1}{2}. \quad (2.3.16)$$

Since this holds for any sequence $\lambda_k \rightarrow \infty$, we conclude that

$$\liminf_{\lambda \rightarrow \infty} \left(\min_{x \in \overline{\Omega_\lambda}} \int_{\Omega_\lambda} J(x - y) dy \right) \geq \frac{1}{2}. \quad (2.3.17)$$

□

Lemma 2.3.1. *Let the dispersal kernel $J \in C^0(\mathbb{R}^n)$ satisfy Assumption 1.2.1. Let $\Omega_0 \subset \mathbb{R}^n$ be any open, bounded domain. There exists a scaling factor $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the non-local operator $\mathcal{L}$ on the scaled domain $\Omega_\lambda = \{\lambda x : x \in \Omega_0\}$ admits a strictly positive spectral gap*

$$\sup \sigma_c(\Omega_\lambda) < \sup_{v \in A_1(\Omega_\lambda)} \langle \mathcal{L}v, v \rangle =: \beta_1(\Omega_\lambda). \quad (2.3.18)$$

Proof. Let $\bar{x} = \frac{1}{|\Omega_0|} \int_{\Omega_0} y dy$ be the geometric barycenter of the reference domain Ω_0 . The barycenter of the scaled domain Ω_λ is exactly $\lambda\bar{x}$. We define a linear test function centered at the barycenter of the domain Ω

$$v_\lambda(x) = c_\lambda(x_k - \lambda\bar{x}_k), \quad (2.3.19)$$

where the coordinate axis $k \in \{1, \dots, n\}$ corresponds to the maximum moment of inertia of Ω_0 , defined as

$$I_k(\Omega_0) = \max_{1 \leq i \leq n} \int_{\Omega_0} (y_i - \bar{x}_i)^2 dy. \quad (2.3.20)$$

The zero-mean condition required for $v_\lambda \in A_1(\Omega_\lambda)$ is satisfied by the definition of the barycenter

$$\int_{\Omega_\lambda} v_\lambda(x) dx = c_\lambda \int_{\Omega_\lambda} (x_k - \lambda\bar{x}_k) dx = 0. \quad (2.3.21)$$

We determine the constant c_λ by enforcing the L^2 normalization constraint $\|v_\lambda\|_{L^2}^2 = 1$. Applying the change of variables $x = \lambda y$

$$1 = \int_{\Omega_\lambda} c_\lambda^2 (x_k - \lambda\bar{x}_k)^2 dx = c_\lambda^2 \lambda^{n+2} \int_{\Omega_0} (y_k - \bar{x}_k)^2 dy = c_\lambda^2 \lambda^{n+2} I_k(\Omega_0). \quad (2.3.22)$$

This yields $c_\lambda^2 = (\lambda^{n+2} I_k(\Omega_0))^{-1}$.

We evaluate the quadratic form of the operator $\mathcal{L}$ on the test function using the symmetrization identity

$$\langle \mathcal{L}v_\lambda, v_\lambda \rangle = -\frac{1}{2} \iint_{\Omega_\lambda \times \Omega_\lambda} J(x-y) (v_\lambda(x) - v_\lambda(y))^2 dx dy. \quad (2.3.23)$$

The difference evaluates to $v_\lambda(x) - v_\lambda(y) = c_\lambda(x_k - y_k)$. The barycenter terms completely cancel out. Substituting this difference and enlarging the integration domain for the inner integral from Ω_λ to the entire space $\mathbb{R}^n$, we establish a strict lower bound

$$\begin{aligned} \langle \mathcal{L}v_\lambda, v_\lambda \rangle &\geq -\frac{c_\lambda^2}{2} \int_{\Omega_\lambda} \int_{\mathbb{R}^n} J(x-y) (x_k - y_k)^2 dy dx \\ &= -\frac{c_\lambda^2}{2} \int_{\Omega_\lambda} \int_{\mathbb{R}^n} J(z) z_k^2 dz dx \\ &\geq -\frac{c_\lambda^2}{2} |\Omega_\lambda| \int_{\mathbb{R}^n} J(z) |z|^2 dz \\ &= -\frac{c_\lambda^2}{2} \lambda^n |\Omega_0| m_2(J). \end{aligned} \quad (2.3.24)$$

The finiteness of the second moment $m_2(J) = \int_{\mathbb{R}^n} J(z) |z|^2 dz < \infty$ is guaranteed by Assumption 1.2.1. Substituting the normalization constant c_λ^2 into (2.3.24), we obtain

$$\langle \mathcal{L}v_\lambda, v_\lambda \rangle \geq -\frac{\lambda^n |\Omega_0| m_2(J)}{2 \lambda^{n+2} I_k(\Omega_0)} = -\frac{|\Omega_0| m_2(J)}{2 I_k(\Omega_0)} \cdot \frac{1}{\lambda^2}. \quad (2.3.25)$$

The term $\frac{|\Omega_0|m_2(J)}{2I_k(\Omega_0)}$ is a strictly positive geometric constant entirely independent of the scaling factor λ . As $\lambda \rightarrow \infty$, the lower bound of the energy strictly approaches 0 as $\mathcal{O}(\lambda^{-2})$.

From Proposition 2.3.2 it follows that there exists $\delta > 0$ such that $\sup \sigma_c < -\delta$ for all $\lambda > \lambda_0$. Thus, there must exist a sufficiently large scaling factor $\lambda^* \geq \lambda_0$ such that for all $\lambda > \lambda^*$:

$$\langle \mathcal{L}v_\lambda, v_\lambda \rangle \geq -\frac{|\Omega_0|m_2(J)}{2I_k(\Omega_0)} \cdot \frac{1}{\lambda^2} > -\delta \geq \sup \sigma_c(\Omega_\lambda). \quad (2.3.26)$$

Since the supremum over the admissible space $A_1(\Omega_\lambda)$ must be at least as large as the energy of this specific test function, we conclude $\beta_1(\Omega_\lambda) \geq \langle \mathcal{L}v_\lambda, v_\lambda \rangle > \sup \sigma_c(\Omega_\lambda)$, establishing the existence of the strictly positive spectral gap. $\square$

2.4 Discrete spectrum

The main objective of this section is to construct pairs $(\beta, v) \in \mathbb{R} \times L^2(\Omega)$ solving the non-local eigenvalue problem given below

$$\mathcal{L}v = \beta v. \quad (2.4.1)$$

We aim to achieve this goal by means of the calculus of variations. In particular we seek for solutions of the following optimization problem

$$\beta = \sup_{v \in L^2(\Omega)} \langle \mathcal{L}v, v \rangle. \quad (2.4.2)$$

As we already mentioned, the main difficulty lies in the lack of compactness for Neumann-type dispersal operator $\mathcal{L}_N$.

2.4.1 Closed environment case

Recall the fundamental mass conservation property of the non-local dispersal operator $\mathcal{L}_N$.

Remark 2.4.1. *For any integral kernel J satisfying Assumption 1.2.1, the dispersal operator $\mathcal{L}_N$ defined in (1.2.7) admits a trivial principal eigenvalue $\beta_0 = 0$. The corresponding eigenfunction is given by the constant function*

$$v_0(x) = \frac{1}{\sqrt{|\Omega|}}. \quad (2.4.3)$$

We start, by constructing first non-trivial solution $(\beta_1, v_1) \in \mathbb{R} \times L^2(\Omega)$ of the non-local eigenvalue problem (2.4.1). The solution is obtained variationally, that is we consider the optimization problem

$$\beta_1 = \sup_{v \in A_1} \langle \mathcal{L}v, v \rangle, \quad (2.4.4)$$

where the closed set of admissible functions $A_1 \subset L^2(\Omega)$ is defined by the formula

$$A_1 = \left\{ v \in L^2(\Omega) : \int_{\Omega} v(x) dx = 0, \int_{\Omega} v^2(x) dx = 1 \right\} \quad (2.4.5)$$

Lemma 2.4.1 (Galerkin Approximation). *Let $V_N = \text{span}\{\varphi_0, \dots, \varphi_N\} \subset L^2(\Omega)$ be a finite-dimensional subspace. The projection of the non-local eigenvalue problem (2.4.1) onto V_N is equivalent to the algebraic system*

$$(\mathbf{A} - \mathbf{B})\mathbf{c} = \beta_1^N \mathbf{c}, \quad (2.4.6)$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{(N+1) \times (N+1)}$ are symmetric matrices and $\mathbf{c} \in \mathbb{R}^{N+1}$ is the vector of coefficients. Under the assumption $\varphi_0 = \frac{1}{\sqrt{|\Omega|}}$, the condition $v \in A_1 \cap V_N$ holds if and only if

$$c_0 = 0, \quad \mathbf{c}^T \mathbf{c} = 1. \quad (2.4.7)$$

Proof. Given an orthonormal set $\{\varphi_i\}_{i=0}^\infty$ in $L^2(\Omega)$, since $L^2(\Omega^2) = L^2(\Omega) \otimes L^2(\Omega)$, we expand function of two variables $K(x, y) = J(x - y)$ into series with symmetric coefficients $a_{k,l}$

$$K(x, y) = \sum_{k,l=0}^{\infty} a_{k,l} \varphi_k(x) \varphi_l(y). \quad (2.4.8)$$

Plugging (2.4.8) into the equation (2.4.1) we get

$$\sum_{k,l=0}^{\infty} a_{k,l} \varphi_k(x) \int_{\Omega} \varphi_l(y) v_1(y) dy - \int_{\Omega} J(x - y) dy v_1(x) = \beta_1 v_1(x) \quad (2.4.9)$$

For convenience, denote the function $b(x) = \int_{\Omega} J(x - y) dy$. We seek for a finite dimensional solution of the form

$$v_1^N(x) = \sum_{m=0}^N c_m \varphi_m(x). \quad (2.4.10)$$

Substituting formula (2.4.10) into equation (2.4.9), we obtain

$$\sum_{k,l=0}^{\infty} \sum_{m=0}^N a_{k,l} c_m \varphi_k(x) \int_{\Omega} \varphi_l(y) \varphi_m(y) dy - \sum_{m=0}^N c_m b(x) \varphi_m(x) = \beta_1^N \sum_{m=0}^N c_m \varphi_m(x). \quad (2.4.11)$$

We take arbitrary $j \in \{0, \dots, N\}$, multiply equation (2.4.11) by φ_j and integrate in Ω . Re-indexing and using orthonormality of $\{\varphi_i\}_{i=0}^N$, this yields the simplified form

$$\sum_{l=0}^N (a_{j,l} - b_{l,j}) c_l = \beta_1^N c_j, \quad (2.4.12)$$

where we denoted the real, symmetric matrices $\mathbf{A}, \mathbf{B}$ of size $(N+1) \times (N+1)$ given by

$$(\mathbf{A})_{i,j} = a_{i,j} = \int_{\Omega^2} J(x - y) \varphi_i(x) \varphi_j(y) dx dy, \quad (\mathbf{B})_{i,j} = b_{i,j} = \int_{\Omega} b(x) \varphi_i(x) \varphi_j(x) dx,$$

and the coefficients vector $\mathbf{c} = (c_0, c_1, \dots, c_N)^T$. Then, we re-write equation (2.4.12) in the vector form

$$(\mathbf{A} - \mathbf{B})\mathbf{c} = \beta_1^N \mathbf{c}. \quad (2.4.13)$$

Before we solve the eigenvalue problem (2.4.13), we need to impose the constraints as in the definition (2.4.5). Namely, we get that the vector of coefficients $\mathbf{c}$ need to satisfy

$$c_0 = 0, \quad \mathbf{c}^T \mathbf{c} = 1. \quad (2.4.14)$$

□

We provide the following Proposition regarding solvability of the problem (2.4.13)-(2.4.14).

Proposition 2.4.1. *Let $\mathcal{C} = \{\mathbf{c} \in \mathbb{R}^{N+1} : c_0 = 0, \|\mathbf{c}\| = 1\}$ be the constraint set. The variational problem*

$$\beta_1^N := \sup_{\mathbf{c} \in \mathcal{C}} \langle (\mathbf{A} - \mathbf{B})\mathbf{c}, \mathbf{c} \rangle \quad (2.4.15)$$

admits a maximizer $\mathbf{c}^* \in \mathcal{C}$. Moreover, the pair $(\beta_1^N, \mathbf{c}^*)$ is a solution to the algebraic system (2.4.13)-(2.4.14).

Proof. Let $\mathbf{M} = \mathbf{A} - \mathbf{B}$. Since $\mathbf{A}$ and $\mathbf{B}$ are symmetric, $\mathbf{M}$ is a real symmetric matrix. The constraint $c_0 = 0$ implies that we are effectively looking for the spectrum of the submatrix $\tilde{\mathbf{M}} \in \mathbb{R}^{N \times N}$ with indices $1 \dots N$.

Since the set $\mathcal{C}$ is compact and the map $\mathbf{c} \mapsto \langle \mathbf{M}\mathbf{c}, \mathbf{c} \rangle$ is continuous, the supremum in (2.4.15) is attained. Let β_1^N denote this maximum value and $\mathbf{c}^*$ be a corresponding maximizer. By the min-max principle restricted to the subspace $\mathbb{R}^N \cong \{\mathbf{c} \in \mathbb{R}^{N+1} : c_0 = 0\}$, β_1^N coincides with the largest eigenvalue of $\tilde{\mathbf{M}}$, and $\mathbf{c}^*$ is the associated eigenvector. Consequently, the pair satisfies $(\mathbf{A} - \mathbf{B})\mathbf{c}^* = \beta_1^N \mathbf{c}^*$ with the required constraints. $\square$

Finally, we show that the sequence (β_1^N, v_1^N) of solutions to the finite dimensional problem (2.4.13)-(2.4.14), converges to the solution (β_1, v_1) of the original problem (2.4.1). We begin with the convergence of eigenvalues.

Lemma 2.4.2. *Let β_1^N be the principle eigenvalue of the discrete problem (2.4.15). Then*

$$\beta_1^N \rightarrow \beta_1 \quad \text{as } N \rightarrow \infty.$$

Proof. Let $V_N = \text{span}\{\varphi_0, \dots, \varphi_N\}$ and A_1 be defined as in (2.4.5). Since $V_N \cap A_1 \subset V_{N+1} \cap A_1 \subset A_1$, we have

$$\beta_1^N = \sup_{u \in V_N \cap A_1} \langle \mathcal{L}u, u \rangle \leq \sup_{u \in V_{N+1} \cap A_1} \langle \mathcal{L}u, u \rangle \leq \dots \leq \beta_1. \quad (2.4.16)$$

The sequence $\{\beta_1^N\}_{N \geq 1}$ is non-decreasing and bounded from above, thus it converges to a limit $\beta^* \leq \beta_1$. To prove that $\beta^* = \beta_1$, we utilize the density of the basis $\{\varphi_k\}$ in $L^2(\Omega)$. For any $u \in A_1$, there exists a sequence of projections $u_N \in V_N \cap A_1$ such that $u_N \rightarrow u$ in $L^2(\Omega)$. The continuity of the bilinear form associated with $\mathcal{L}$ yields

$$\lim_{N \rightarrow \infty} \beta_1^N = \lim_{N \rightarrow \infty} \sup_{u \in V_N \cap A_1} \langle \mathcal{L}u, u \rangle \geq \lim_{N \rightarrow \infty} \langle \mathcal{L}u_N, u_N \rangle = \langle \mathcal{L}u, u \rangle. \quad (2.4.17)$$

Taking the supremum over $u \in A_1$ on the right-hand side confirms that $\lim_{N \rightarrow \infty} \beta_1^N = \beta_1$. $\square$

Now, we prove the strong convergence of eigenfunctions.

Lemma 2.4.3. *Let v_1^N be the function defined by formula (2.4.10) with vector of coefficients $\mathbf{c}$ being the maximizer of the problem (2.4.15). Then there exists $v_1 \in A_1$ such that*

$$v_1^N \rightarrow v_1 \quad \text{strongly in } L^2(\Omega) \text{ as } N \rightarrow \infty.$$

Proof. Recall that $\|v_1^N\|_2 = 1$. By the Banach-Alaoglu theorem, there exists a subsequence, still denoted by v_1^N , and a function $v \in L^2(\Omega)$ such that $v_1^N \rightharpoonup v_1$ weakly in $L^2(\Omega)$. The zero mass constraint is preserved under weak convergence

$$\int_{\Omega} v_1(x) dx = \lim_{N \rightarrow \infty} \int_{\Omega} v_1^N(x) dx = 0. \quad (2.4.18)$$

To identify the limit, consider an arbitrary test function $\psi \in V_M$. For $N \geq M$, the discrete equation reads $\langle \mathcal{L}v_1^N, \psi \rangle = \beta_1^N \langle v_1^N, \psi \rangle$. Passing to the limit $N \rightarrow \infty$, and using the self-adjointness of $\mathcal{L}$, we obtain

$$\langle v_1, \mathcal{L}\psi \rangle = \lim_{N \rightarrow \infty} \langle v_1^N, \mathcal{L}\psi \rangle = \lim_{N \rightarrow \infty} \langle \mathcal{L}v_1^N, \psi \rangle \quad (2.4.19)$$

$$= \lim_{N \rightarrow \infty} \beta_1^N \langle v_1^N, \psi \rangle = \beta_1 \langle v_1, \psi \rangle. \quad (2.4.20)$$

Since $\bigcup_M V_M$ is dense in $L^2(\Omega)$, this implies $\mathcal{L}v_1 = \beta_1 v_1$ in the weak sense. It remains to prove that the weak convergence $v_1^N \rightharpoonup v_1$ is strong in $L^2(\Omega)$. We decompose the non-local operator $\mathcal{L}$ into a compact integral part $\mathcal{K}$ and a local multiplication part $\mathcal{M}$:

$$\mathcal{L}u(x) = (\mathcal{K}u)(x) - (\mathcal{M}u)(x) = \int_{\Omega} J(x-y)u(y) dy - b(x)u(x), \quad (2.4.21)$$

where $b(x) = \int_{\Omega} J(x-y)dy$. Theorem 2.2.1 states that the essential spectrum is generated by the multiplication operator, yielding $\sup \sigma_c = \max_{x \in \bar{\Omega}} (-b(x))$. From the Galerkin approximation, we evaluate the energy sequence

$$\langle \mathcal{L}v_1^N, v_1^N \rangle = \beta_1^N \rightarrow \beta_1 \quad \text{as } N \rightarrow \infty. \quad (2.4.22)$$

Using our decomposition of $\mathcal{L}$, we can rewrite this energy equation as:

$$\langle \mathcal{K}v_1^N, v_1^N \rangle - \int_{\Omega} b(x)(v_1^N(x))^2 dx = \beta_1^N. \quad (2.4.23)$$

Rearranging the terms in equation (2.4.23) to group the L^2 norms, we get

$$\int_{\Omega} (\beta_1^N + b(x))(v_1^N(x))^2 dx = \langle \mathcal{K}v_1^N, v_1^N \rangle. \quad (2.4.24)$$

Because $J \in C^0(\mathbb{R}^n)$ and Ω is bounded, the integral operator $\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator. A fundamental property of compact operators is that they map weakly convergent sequences into strongly convergent ones. Since $v_1^N \rightharpoonup v_1$ weakly in $L^2(\Omega)$, it follows that $\mathcal{K}v_1^N \rightarrow \mathcal{K}v_1$ strongly. This guarantees the convergence of the inner product

$$\lim_{N \rightarrow \infty} \langle \mathcal{K}v_1^N, v_1^N \rangle = \langle \mathcal{K}v_1, v_1 \rangle. \quad (2.4.25)$$

Next, from Lemma 2.4.2 we have that $\lim_{N \rightarrow \infty} \beta_1^N = \beta_1$ thus we write

$$\begin{aligned}
\lim_{N \rightarrow \infty} \int_{\Omega} (\beta_1^N + b(x))(v_1^N(x))^2 dx &= \lim_{N \rightarrow \infty} \left(\beta_1^N \int_{\Omega} (v_1^N(x))^2 dx + \int_{\Omega} b(x)(v_1^N(x))^2 dx \right) \\
&= \lim_{N \rightarrow \infty} \left(\beta_1^N \cdot 1 + \int_{\Omega} b(x)(v_1^N(x))^2 dx \right) \\
&= \beta_1 + \lim_{N \rightarrow \infty} \int_{\Omega} b(x)(v_1^N(x))^2 dx \\
&= \lim_{N \rightarrow \infty} \left(\beta_1 \int_{\Omega} (v_1^N(x))^2 dx + \int_{\Omega} b(x)(v_1^N(x))^2 dx \right) \\
&= \lim_{N \rightarrow \infty} \int_{\Omega} (\beta_1 + b(x))(v_1^N(x))^2 dx. \tag{2.4.26}
\end{aligned}$$

Applying formulas (2.4.25) and (2.4.26) into equation (2.4.24) we obtain

$$\int_{\Omega} (\beta_1 + b(x))v_1^N(x)^2 dx = \langle \mathcal{K}v_1, v_1 \rangle. \tag{2.4.27}$$

Now, recall that the weak limit v_1 satisfies the equation $\mathcal{L}v_1 = \beta_1 v_1$. By rewriting this exact relation using the operator decomposition and taking the inner product with v_1 , we obtain:

$$\langle \mathcal{K}v_1, v_1 \rangle - \int_{\Omega} b(x)v_1(x)^2 dx = \beta_1 \int_{\Omega} v_1(x)^2 dx. \tag{2.4.28}$$

Rearranging this yields

$$\int_{\Omega} (\beta_1 + b(x))v_1(x)^2 dx = \langle \mathcal{K}v_1, v_1 \rangle. \tag{2.4.29}$$

Combining equations (2.4.27) and (2.4.29), we have rigorously proven that

$$\lim_{N \rightarrow \infty} \int_{\Omega} (\beta_1 + b(x))(v_1^N(x))^2 dx = \int_{\Omega} (\beta_1 + b(x))v_1(x)^2 dx. \tag{2.4.30}$$

Finally, we utilize the crucial spectral gap condition. From Proposition 2.1, we established that $\beta_1 > \sup \sigma_c = \max_{x \in \overline{\Omega}} (-b(x))$. This implies that for all $x \in \overline{\Omega}$:

$$\beta_1 + b(x) \geq \beta_1 - \sup \sigma_c > 0. \tag{2.4.31}$$

Because the weight function $w(x) := \beta_1 + b(x)$ is strictly positive and bounded, it induces an equivalent norm on the Hilbert space $L^2(\Omega)$, defined by

$$\|u\|_* := \left(\int_{\Omega} (\beta_1 + b(x))u(x)^2 dx \right)^{1/2}. \tag{2.4.32}$$

Equation (2.4.30) demonstrates exactly that the integral of the squared sequence with this weight converges to the integral of the squared limit.

To rigorously prove that v_1^N converges strongly to v_1 in the standard $L^2(\Omega)$ norm, we evaluate the limit of the weighted squared difference $v_1^N - v_1$. Because $\beta_1 + b(x) \geq c > 0$, we can bound

the standard L^2 norm by this weighted integral. Expanding the square inside the integral yields:

$$\begin{aligned}
0 \leq c \|v_1^N - v_1\|_{L^2}^2 &= c \int_{\Omega} (v_1^N(x) - v_1(x))^2 dx \\
&\leq \int_{\Omega} (\beta_1 + b(x))(v_1^N(x) - v_1(x))^2 dx \\
&= \int_{\Omega} (\beta_1 + b(x))(v_1^N(x))^2 dx \\
&\quad - 2 \int_{\Omega} (\beta_1 + b(x))v_1^N(x)v_1(x) dx \\
&\quad + \int_{\Omega} (\beta_1 + b(x))(v_1(x))^2 dx.
\end{aligned} \tag{2.4.33}$$

We now take the limit as $N \rightarrow \infty$ for each of the three terms on the right-hand side. The limit of the first was already established in equation (2.4.30). For the second term, notice that since the weight function is bounded and $v_1^N \rightharpoonup v_1 \in L^2(\Omega)$, we have that

$$\lim_{N \rightarrow \infty} \int_{\Omega} v_1^N(x) [(\beta_1 + b(x))v_1(x)] dx = \int_{\Omega} v_1(x) [(\beta_1 + b(x))v_1(x)] dx. \tag{2.4.34}$$

The third term is independent of N . Combining these limits yields that

$$\begin{aligned}
\lim_{N \rightarrow \infty} \int_{\Omega} (\beta_1 + b(x))(v_1^N(x) - v_1(x))^2 dx &= \int_{\Omega} (\beta_1 + b(x))(v_1(x))^2 dx \\
&\quad - 2 \int_{\Omega} (\beta_1 + b(x))(v_1(x))^2 dx \\
&\quad + \int_{\Omega} (\beta_1 + b(x))(v_1(x))^2 dx \\
&= 0.
\end{aligned} \tag{2.4.35}$$

Since $0 \leq c \lim_{N \rightarrow \infty} \|v_1^N - v_1\|_{L^2}^2 \leq 0$ and $c > 0$, it immediately follows by the squeeze theorem that:

$$\lim_{N \rightarrow \infty} \|v_1^N - v_1\|_{L^2}^2 = 0. \tag{2.4.36}$$

Thus, $v_1^N \rightarrow v_1$ strongly in $L^2(\Omega)$. Because strong convergence preserves the standard L^2 -norm, we finally conclude $\|v_1\|_{L^2} = \lim_{N \rightarrow \infty} \|v_1^N\|_{L^2} = 1$, confirming that $v_1 \in A_1$ is a valid, non-trivial normalized eigenfunction. $\square$

Naturally, we are now interested in existence of subsequent eigenpairs $(\beta_k, v_k) \in \mathbb{R} \times A_k$ solving equation (2.4.1), with the k -th set of admissible functions A_k defined by relation

$$A_k = \left\{ v \in L^2(\Omega) : v \perp \text{span}\{v_0, \dots, v_{k-1}\}, \int_{\Omega} v^2(x) dx = 1 \right\} \quad k > 1. \tag{2.4.37}$$

To do that, we still need to guarantee existence of strict spectral gap, which still remains an open problem for general kernels J .

Lemma 2.4.4. *Suppose that there exist k pairs of eigenvalues and normalized orthonormal eigenfunctions $(\beta_j, v_j) \in \mathbb{R} \times A_j$ for $j = 1, \dots, k$ solving problem (3.1). Assume that*

$$\sup \sigma_c < \sup_{v \in A_{k+1}} \langle \mathcal{L}v, v \rangle =: \beta_{k+1}. \quad (2.4.38)$$

Then, there exists an eigenfunction $v_{k+1} \in A_{k+1}$ corresponding to the eigenvalue β_{k+1} such that

$$\mathcal{L}v_{k+1} = \beta_{k+1}v_{k+1}. \quad (2.4.39)$$

Proof. Let $k \geq 1$. We proceed using the approximation technique analogous to Lemmas 2.4.1, 2.4.2 and 2.4.3. Let us consider the truncated problem on the subspace $A_{k+1} \cap V_N$ with $V_N = \text{span}\{\varphi_0, \dots, \varphi_N\}$ and $N \geq k$. The discrete counterpart of the operator $\mathcal{L}$ again corresponds to the symmetric matrix $\mathbf{M} = \mathbf{A} - \mathbf{B}$ of size $(N+1) \times (N+1)$ restricted to the subspace $\mathcal{C} = \{\mathbf{c} \in \mathbb{R}^{N+1} : c_0 = \dots = c_k = 0, \|\mathbf{c}\| = 1\}$. We know that $\mathbf{M}$ has a complete orthonormal set, composed of eigenvectors $\mathbf{c}_1, \dots, \mathbf{c}_N$ corresponding to eigenvalues $\beta_1^N \geq \beta_2^N \geq \dots \geq \beta_N^N$.

Firstly, the sequence of eigenvalues $\{\beta_{k+1}^N\}_{N \geq k+1}$ is non-decreasing and bounded by β_{k+1} . Moreover, for every $u \in A_{k+1}$ there exists a sequence $u_i \rightarrow u$ converging strongly in $L^2(\Omega)$ as $i \rightarrow \infty$. Testing the expression 2.4.38 with u_i , we get that $\beta_{k+1}^N \geq \langle \mathcal{L}u_i, u_i \rangle$. Now passing to the limit and taking supremum, we get that $\beta_{k+1}^N \rightarrow \beta_{k+1}$ as $N \rightarrow \infty$.

Denote the function constructed from the vector $\mathbf{c}_{k+1} = (c_0^{k+1}, c_1^{k+1}, \dots, c_N^{k+1})$ by the formula

$$v_{k+1}^N(x) = \sum_{m=0}^N c_m^{k+1} \varphi_m(x). \quad (2.4.40)$$

Note that $\{v_{k+1}^N\}_{N \geq k+1}$ is bounded in $L^2(\Omega)$, thus up to a subsequence, it converges weakly to a function $v_{k+1} \in L^2(\Omega)$. Our goal is to show, that this sequence converges strongly, and $v_{k+1} \in A_{k+1}$.

For any $j \in \{0, 1, \dots, k\}$, utilizing the strong convergence $v_j^N \rightarrow v_j$ and the discrete orthogonality $\langle v_{k+1}^N, v_j^N \rangle = 0$ in the finite-dimensional subspace V_N , we can strictly bound the inner product

$$\begin{aligned} |\langle v_{k+1}, v_j \rangle| &= \lim_{N \rightarrow \infty} |\langle v_{k+1}^N, v_j \rangle| \\ &= \lim_{N \rightarrow \infty} |\langle v_{k+1}^N, v_j^N \rangle + \langle v_{k+1}^N, v_j - v_j^N \rangle| \\ &\leq \lim_{N \rightarrow \infty} (0 + \|v_{k+1}^N\|_2 \|v_j - v_j^N\|_2) \\ &= \lim_{N \rightarrow \infty} \|v_j - v_j^N\|_2 = 0. \end{aligned} \quad (2.4.41)$$

Now, we verify that v_{k+1} is a weak solution of the eigenvalue problem. Fix $\psi \in A_{k+1}$, we get

$$\begin{aligned} \langle v_{k+1}, \mathcal{L}\psi \rangle &= \lim_{N \rightarrow \infty} \langle v_{k+1}^N, \mathcal{L}\psi \rangle = \lim_{N \rightarrow \infty} \langle \mathcal{L}v_{k+1}^N, \psi \rangle \\ &= \lim_{N \rightarrow \infty} \beta_{k+1}^N \langle v_{k+1}^N, \psi \rangle = \beta_{k+1} \langle v_{k+1}, \psi \rangle. \end{aligned} \quad (2.4.42)$$

Now, to prove strong convergence of v_{k+1}^N to v_{k+1} , we use the same technique, as in the proof of strong convergence in Lemma 2.4.3. In particular, using compactness of the integral part of $\mathcal{L}$ we obtain

$$\lim_{N \rightarrow \infty} \int_{\Omega} (\beta_{k+1} + b(x))(v_{k+1}^N)^2 dx = \int_{\Omega} (\beta_{k+1} + b(x))v_{k+1}^2 dx. \quad (2.4.43)$$

As we assumed that $\sup \sigma_c < \beta_{k+1}$, there exists a constant $c > 0$, such that

$$0 < c < \beta_{k+1} + \sup \sigma_c \leq \beta_{k+1} + b(x) \quad (2.4.44)$$

Hence, using the weak convergence of v_{k+1}^N and formula (2.4.43), we get that

$$\begin{aligned} 0 &\leq c \lim_{N \rightarrow \infty} \|v_{k+1}^N - v_{k+1}\|_{L^2}^2 = \lim_{N \rightarrow \infty} c \int_{\Omega} (v_{k+1}^N(x) - v_{k+1}(x))^2 dx \\ &\leq \lim_{N \rightarrow \infty} \int_{\Omega} (\beta_{k+1} + b(x))(v_{k+1}^N(x) - v_{k+1}(x))^2 dx \\ &= \lim_{N \rightarrow \infty} \int_{\Omega} (\beta_{k+1} + b(x))(v_{k+1}^N(x))^2 dx \\ &\quad - 2 \lim_{N \rightarrow \infty} \int_{\Omega} (\beta_{k+1} + b(x))v_{k+1}^N(x)v_{k+1}(x) dx \\ &\quad + \int_{\Omega} (\beta_{k+1} + b(x))(v_{k+1}(x))^2 dx \\ &= 0. \end{aligned}$$

This proves that $v_{k+1}^N \rightarrow v_{k+1}$ strongly in $L^2(\Omega)$ as $N \rightarrow \infty$. This concludes the Lemma. $\square$

Hence, working in the framework of Assumption 1.2.1 together with Assumption 2.3.1 or 2.3.2, we obtained the following spectral theorem for the non-local dispersal operators

Theorem 2.4.1 (Non-local Spectral Theorem). *Assume that the kernel J satisfies Assumption 1.2.1 and that domain Ω satisfies either Assumption 2.3.1 or 2.3.2. Then there exists a finite or infinite sequence of pairs $(\beta_k, v_k) \in \mathbb{R} \times L^2(\Omega)$ solving equation (2.4.1). Moreover, the sequence of eigenvalues $\{\beta_k\}_{k=0}^M$ satisfies*

$$-\min_{x \in \bar{\Omega}} \int_{\Omega} J(x-y) dy < \beta_M < \dots < \beta_0 = 0, \quad (2.4.45)$$

where $M \geq 1$.

Proof. The existence of the trivial eigenpair (β_0, v_0) with $\beta_0 = 0$ follows directly from the mass-conserving property of the non-local operator $\mathcal{L}_N$, as stated in Remark 2.4.1.

Under Assumption 2.3.1 or 2.3.2, Proposition 2.3.1 and Lemma 2.3.1 guarantee the existence of a strictly positive spectral gap. This ensures that the first non-trivial eigenvalue β_1 is strictly separated from the continuous spectrum, whose supremum is exactly $\sup \sigma_c = -\min_{x \in \bar{\Omega}} \int_{\Omega} J(x-y) dy$.

The first non-trivial eigenpair (β_1, v_1) is constructed variationally via finite-dimensional Galerkin approximations. The algebraic solvability of the discrete problem is guaranteed by Proposition 2.4.1, and the strong $L^2(\Omega)$ -convergence to the exact normalized eigenfunction is rigorously established in Lemmas 2.4.2 and 2.4.3.

For $k > 1$, the existence of subsequent eigenpairs is verified by the inductive argument detailed in Lemma 2.4.4, provided the spectral gap condition holds. $\square$

To the author's best knowledge, this is the first, rigorous construction of eigenfunctions corresponding to non-trivial eigenvalues on a bounded domains in the Neumann-type case.

2.4.2 Hostile environment case

For the non-local Dirichlet operator, the eigenvalue problem (2.4.1) inside the domain Ω reads

$$\mathcal{L}_D v(x) = \int_{\Omega} J(x-y)v(y)dy - v(x) = \beta v(x), \quad x \in \Omega. \quad (2.4.46)$$

By re-arranging the terms, we can re-write this problem in terms of the integral operator $\mathcal{K}$

$$\mathcal{K}v(x) := \int_{\Omega} J(x-y)v(y)dy = (\beta + 1)v(x). \quad (2.4.47)$$

By substituting $\mu = \beta + 1$, the problem reduces to finding the spectrum of the operator $\mathcal{K}$. Because the kernel J is continuous and the domain Ω is bounded, $J \in L^2(\Omega \times \Omega)$. Consequently, $\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a Hilbert-Schmidt operator, which implies it is strictly compact. Furthermore, by Assumption 1.2.1, J is symmetric ($J(x-y) = J(y-x)$), making $\mathcal{K}$ a self-adjoint operator.

These properties allow us to completely characterize the discrete spectrum for the hostile environment case using standard functional analysis tools.

Theorem 2.4.2 (Discrete Spectrum for the Dirichlet operator). *Let J satisfy Assumption 1.2.1 and $\Omega \subset \mathbb{R}^n$ be a bounded domain. The non-local Dirichlet operator $\mathcal{L}_D$ admits a countable sequence of eigenvalues $\{\beta_k\}_{k=1}^{\infty}$ such that*

$$-1 < \dots \leq \beta_2 < \beta_1 < 0 \quad (2.4.48)$$

The sequence of eigenvalues converges exactly to the essential spectrum, that is

$$\lim_{k \rightarrow \infty} \beta_k = -1. \quad (2.4.49)$$

Moreover, the corresponding eigenfunctions $\{v_k\}_{k=1}^{\infty}$ form a complete orthonormal basis of $L^2(\Omega)$, and the principal eigenfunction v_1 is strictly positive in Ω .

Proof. As established in equation (2.4.47), the eigenvalue problem $\mathcal{L}_D v = \beta v$ is equivalent to $\mathcal{K}v = \mu v$ with $\mu = \beta + 1$. Since $\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact, self-adjoint operator, the Mercer's Theorem (see [42, 31, 20]) for compact self-adjoint operators applies directly.

Therefore, $\mathcal{K}$ admits a countable sequence of real eigenvalues $\{\mu_k\}_{k=1}^{\infty}$ (counting multiplicities) such that $\mu_k \rightarrow 0$ as $k \rightarrow \infty$, and an associated orthonormal basis of eigenfunctions $\{v_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$.

Translating this back to the spectrum of $\mathcal{L}_D$, we obtain the sequence of eigenvalues $\beta_k = \mu_k - 1$. Because $\mu_k \rightarrow 0$, it immediately follows that $\lim_{k \rightarrow \infty} \beta_k = -1$. This ends proof of the Theorem 2.4.2

□

Remark 2.4.2 (Variational characterization). *Unlike the Neumann case, where we had to employ Galerkin approximations to handle the spatial heterogeneity of the function $b(x)$, the eigenvalues in the Dirichlet case can be obtained directly via the standard Rayleigh-Ritz quotient for the compact operator $\mathcal{K}$*

$$\beta_k = \sup_{\substack{v \in L^2(\Omega) \\ v \perp \{v_1, \dots, v_{k-1}\}}} \frac{\langle \mathcal{K}v, v \rangle}{\|v\|_{L^2}^2} - 1. \quad (2.4.50)$$

Chapter 3

Non-local stationary problems

In the previous chapter, we established the fundamental spectral properties of the non-local dispersal operators $\mathcal{L}_N$ and $\mathcal{L}_D$, emphasizing the role of the principal eigenvalue and the existence of the spectral gap. Building upon that functional-analytic framework, this chapter shifts the focus to the study of non-linear stationary (steady-state) problems. From an ecological and biological perspective, understanding stationary solutions is very important, as they represent the long-term, stable configurations of ecosystems, such as persistent vegetation patches, self-organized spatial patterns, or endemic states of a disease.

Mathematically, we distinguish between two main classes of stationary problems in this work. The first class consists of scalar reaction-dispersal models, which describe the equilibrium density of a single species.

$$d_v \mathcal{L}V(x) + f(V(x)) = 0, \quad x \in \Omega, \quad (3.0.1)$$

where $V(x)$ denotes the stationary density of the population, $d_v > 0$ is the dispersal rate, $\mathcal{L}$ is the non-local dispersal operator (incorporating either Neumann or Dirichlet-type boundary conditions), and f represents the local reaction kinetics, which may be spatially heterogeneous.

The second class extends this framework to multi-component systems, capturing complex biological interactions such as vegetation-water feedbacks or host-pathogen dynamics. A particularly challenging and ecologically relevant case arises when distinct components exhibit fundamentally different transport mechanisms. This leads us to hybrid reaction-diffusion-dispersal systems. The stationary states of such systems are governed by a coupled set of equations

$$\begin{cases} d_v \mathcal{L}V(x) + f(V(x), W(x)) = 0, & x \in \Omega, \\ d_w \Delta W(x) + g(V(x), W(x)) = 0, & x \in \Omega, \end{cases} \quad (3.0.2)$$

subject to appropriate boundary conditions for the locally diffusing component $W(x)$ and sometimes for the non-locally dispersing component $V(x)$ (non-local Dirichlet case). Here, $V(x)$ typically represents the species dispersing non-locally (such as plant seeds or viral particles), while $W(x)$ denotes a resource or an interacting population spreading via classical local diffusion (such as soil water or susceptible host cells), and $d_w > 0$ is its diffusion coefficient.

In this chapter, we rigorously investigate the existence, uniqueness, and spatial structures of these stationary solutions for both scalar and coupled systems. We place a particular emphasis on pattern formation, bifurcation analysis, and the precise mathematical characterization of asymptotic profiles under extreme dispersal limits within highly heterogeneous environments.

3.1 Literature review

The mathematical analysis of stationary solutions for equations driven by non-local dispersal operators has evolved significantly over the past few decades. Early studies primarily focused on establishing the foundational theory of existence and uniqueness, which has recently been extended to highly complex, coupled systems, spatially degenerate environments, and discontinuous configurations.

The historical roots of incorporating non-local effects into stationary pattern formation can be traced back to early works such as Furter and Grinfeld [22], who demonstrated that non-local crowding interactions can induce stable, non-homogeneous stationary solutions even when the underlying physical dispersal is modeled by classical local diffusion. However, the foundational groundwork for the stationary problem driven purely by non-local dispersal operators was laid later by Bates and Zhao [8]. In their seminal paper, they investigated a population dispersal model with a general reaction term $f(x, u)$ on a bounded domain. By employing super- and sub-solution methods, they established the conditions for the existence and uniqueness of a strictly positive steady state. Crucially, their work demonstrated that the global dynamics and the existence of such a stationary solution are strictly determined by the sign of the principal eigenvalue of the linearized non-local operator, establishing a trichotomy of global asymptotics that mirrors the classical local reaction-diffusion theory. Shortly after, Zhang, Li, and Sun [68] extended this framework to non-local dispersal equations subject to Dirichlet boundary conditions, representing a hostile exterior environment. They proved that if the intrinsic growth rate exceeds the principal eigenvalue of a competition-free "refuge" subregion ($\lambda \geq \lambda_1(\Omega_0)$), the stationary solution ceases to be bounded and blows up, highlighting the drastic effects of spatial degeneracy.

A fundamental feature distinguishing non-local dispersal from classical local diffusion is the lack of a smoothing effect, which profoundly impacts the qualitative nature of stationary solutions. Rodríguez-Bernal and Sastre-Gomez [49] conducted a deep functional-analytic study on metric measure spaces, proving the existence of maximal and minimal bounded extremal equilibria. Strikingly, they showed that the lack of regularizing properties allows for the existence of non-isolated, discontinuous, piecewise-constant steady states. This phenomenon of discontinuous patterns was recently rigorously explored by Bai, Li, and Wang [6] in the context of semilinear equations with no-flux boundary conditions. Extending the classic local diffusion results of Casten, Holland, and Matano [11], they proved that stable *smooth* patterns cannot exist in convex domains (such as intervals or balls) under non-local diffusion. However, for bistable kinetics, they successfully constructed stable *discontinuous* patterns, revealing a crucial ecological principle: strong non-local dispersal tends to homogenize the space and destabilize patterns, whereas weak non-local dispersal preserves them, allowing for persistent, discontinuous spatial heterogeneity regardless of the domain's geometry.

Further exploring the boundaries of spatial heterogeneity, significant attention has been directed toward the asymptotic profiles of stationary solutions under extreme dispersal rates. Sun [55] extensively analyzed a non-local logistic equation with Neumann boundary conditions in an environment where the intrinsic growth rate $m(x)$ changes sign, successfully characterizing the exact limiting behavior of steady states when the dispersal rate μ tends to zero (small dispersal) and to infinity (large dispersal). Building upon this, Sun [56] investigated the formation of "sharp patterns" in semilinear non-local dispersal equations. By examining the limit of vanishing parameter $\delta \rightarrow 0$, the study revealed that the asymptotic profiles of the positive solutions are

highly sensitive to the geometric structure of the degenerate sets of the equation's coefficients. Notably, it was proven that the sharp profiles generated by non-local dispersal operators are fundamentally different from those observed in classical local reaction-diffusion equations, where sharp patterns typically appear only within the degenerate domain itself.

As the theoretical understanding of scalar equations matured, researchers began applying non-local operators to complex, multi-component biological systems. A notable contribution in this direction was made by Alfaro, Mimura, and their collaborators [2, 1], who proposed a hybrid reaction-diffusion-dispersal system to model vegetation in drylands. In their three-component model, surface and soil water spread via local diffusion, while plants disperse non-locally through a redistribution kernel for seeds. Beyond proving the well-posedness of the system via the Schauder fixed-point theorem, they conducted a profound numerical bifurcation analysis of the stationary problem, demonstrating how variations in environmental parameters and the specific shape of the non-local redistribution kernel drive the transitions between different self-organized vegetation patterns (spots, stripes, and gaps).

In the most recent literature, this coupled-system approach has been extensively utilized to study the asymptotic profiles of endemic steady states in epidemiological models. Feng, Li, and Yang [19] investigated a non-local dispersal Susceptible-Infected-Susceptible (SIS) epidemic model with Dirichlet boundary conditions, establishing the existence and uniqueness of endemic steady states based on the basic reproduction number $\mathcal{R}_0$. Their analysis demonstrated a critical difference between local and non-local models: in the presence of a hostile exterior, large dispersal rates inevitably drive the disease to extinction. Parallel to this, Guenad, Djilali, and Bentout [25] studied a non-local host-pathogen system with Neumann boundary conditions, incorporating distinct dispersal kernels for susceptible and infected populations alongside a logistic source. Overcoming the severe lack of compactness in the non-local dispersion operator, they proved the existence of a global compact attractor and the uniform persistence of the disease when $\mathcal{R}_0 > 1$. Furthermore, they provided a rigorous characterization of the asymptotic profiles of the endemic steady state as one or both diffusion coefficients tend to infinity, shedding light on the interplay between varied spatial dispersal strategies and disease dynamics in heterogeneous habitats.

3.2 Reaction-dispersal models

Recall the following non-linear, non-local stationary problem for population dispersal

$$\mathcal{L}V(x) + f(V(x)) = 0, \quad x \in \Omega. \quad (3.2.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain. We construct solution $V \in X$, where the choice of space X depends on the assumed boundary effects. In particular, we consider

- **Closed environment (Neumann case):** we take $X = W^{k,2}(\Omega)$ or $X = C(\overline{\Omega})$.
- **Hostile environment (Dirichlet case):** we take $X = W_{0,\Omega}^{k,2}(\mathbb{R}^n)$ or $X = C_{0,\Omega}(\mathbb{R}^n)$.

Having established this framework, we now construct non-constant solutions of equation (3.2.1).

3.2.1 Uniform solutions

The simplest class of stationary states to consider are the spatially uniform solutions. However, the existence of such solutions depends strictly on the chosen boundary conditions.

Closed environment (Neumann case): Since the non-local Neumann operator $\mathcal{L}_N$ conserves mass, it annihilates constant functions. That is, for any constant $V_* \in \mathbb{R}$, we have $\mathcal{L}_N(V_*) = 0$. Consequently, the steady-state problem (3.2.1) reduces to the algebraic equation

$$f(V_*) = 0. \quad (3.2.2)$$

Thus, in a closed environment, any root of the local reaction kinetics represents an exact, spatially uniform stationary solution across the entire domain Ω .

Hostile environment (Dirichlet case): The situation is fundamentally different. Due to the boundary losses, the non-local Dirichlet operator $\mathcal{L}_D$ does not annihilate non-zero constants, especially near the boundary $\partial\Omega$. Therefore, an exact constant solution $V(x) \equiv V_* > 0$ cannot exist. However, if the dispersal rate is sufficiently small, the system admits an "almost constant" solution that remains close to the kinetic root V_* in the interior of the domain, exhibiting a sharp drop-off at the boundaries. We formalize this existence in the following Theorem.

Theorem 3.2.1 (Almost constant equilibrium). *Let $\mathcal{L}_D$ be the non-local Dirichlet operator on a bounded domain Ω , and let $\beta_1 < 0$ be its principal eigenvalue. Let $f \in C^2(\mathbb{R})$ be the reaction kinetics. Assume there exists a constant state $V_* > 0$, sufficiently small $\epsilon > 0$, and positive numbers $m > 0, M > 0$ such that*

$$\|f(V_*)\|_\infty \leq \epsilon, \quad f'(V_*) \leq -m < 0, \quad \text{and} \quad \|f''\|_\infty \leq M. \quad (3.2.3)$$

Then there exist critical values $R_* > 0$ and $D_v > 0$ such that for all dispersal rates $d_v < D_v$, the steady-state equation

$$d_v \mathcal{L}_D V + f(V) = 0, \quad x \in \Omega, \quad (3.2.4)$$

admits a unique stationary solution $V \in C_{0,\Omega}(\mathbb{R}^n)$ satisfying $\|V - V_*\|_\infty \leq R_*$.

Proof. We search for a solution in the form of a perturbation around the constant state V_* . Let $B_R = \{\varphi \in C_{0,\Omega}(\mathbb{R}^n) : \|\varphi\|_\infty \leq R\}$ be a closed ball of radius R . We define the solution as

$$V(x) = V_* \mathbf{1}_\Omega(x) + \varphi(x), \quad (3.2.5)$$

where $\varphi \in B_R$. Using the Taylor expansion for the non-linearity f around V_* , we write

$$f(V) = f(V_*) + f'(V_*)\varphi + \mathcal{N}(\varphi), \quad (3.2.6)$$

where $\mathcal{N}(\varphi)$ is the higher-order remainder. Since the second derivative is bounded by M , the remainder is estimated by $\|\mathcal{N}(\varphi)\|_\infty \leq \frac{M}{2} \|\varphi\|_\infty^2$. We denote $C_2 := \frac{M}{2}$.

Substituting (3.2.5) and (3.2.6) into equation (3.2.4), we obtain

$$d_v V_* \mathcal{L}_D \mathbf{1}_\Omega + d_v \mathcal{L}_D \varphi + f(V_*) + f'(V_*)\varphi + \mathcal{N}(\varphi) = 0. \quad (3.2.7)$$

Since $f'(V_*) \leq -m < 0$ and the spectrum of $\mathcal{L}_D$ is bounded from above by $\beta_1 < 0$, the linear operator $(-d_v \mathcal{L}_D - f'(V_*)\text{Id})$ is strictly positive and invertible. Its inverse is bounded in the operator norm by $(m - \beta_1 d_v)^{-1}$. Rearranging (3.2.7), we reformulate the problem as a fixed-point equation $\varphi = \mathcal{F}(\varphi)$, where

$$\mathcal{F}(\varphi) := (-d_v \mathcal{L}_D - f'(V_*)\text{Id})^{-1} (d_v V_* \mathcal{L}_D \mathbf{1}_\Omega + f(V_*) + \mathcal{N}(\varphi)). \quad (3.2.8)$$

We first determine the conditions under which $\mathcal{F}$ maps B_R into itself. Utilizing the bounds from (3.2.3), we estimate the supremum norm of the image

$$\|\mathcal{F}(\varphi)\|_\infty \leq \frac{d_v V_* \|\mathcal{L}_D \mathbf{1}_\Omega\|_\infty + \epsilon + C_2 R^2}{m - \beta_1 d_v}. \quad (3.2.9)$$

We require $\|\mathcal{F}(\varphi)\|_\infty \leq R$. The minimal radius R_* satisfying this quadratic inequality corresponds to its smaller root

$$R_* = \frac{m - \beta_1 d_v}{2C_2} \left(1 - \sqrt{1 - \frac{4C_2(d_v V_* \|\mathcal{L}_D \mathbf{1}_\Omega\|_\infty + \epsilon)}{(m - \beta_1 d_v)^2}} \right). \quad (3.2.10)$$

For sufficiently small d_v and ϵ , the term under the square root is positive, ensuring R_* is well-defined, real, and positive.

Secondly, we verify that $\mathcal{F}$ acts as a contraction on B_{R_*} . For any $\varphi, \psi \in B_{R_*}$, the difference of their images is governed solely by the non-linear remainder

$$\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\|_\infty \leq \frac{1}{m - \beta_1 d_v} \|\mathcal{N}(\varphi) - \mathcal{N}(\psi)\|_\infty. \quad (3.2.11)$$

Applying the Fundamental Theorem of Calculus to the remainder difference, we have:

$$\mathcal{N}(\varphi) - \mathcal{N}(\psi) = \int_0^1 (f'(V_* + \psi + t(\varphi - \psi)) - f'(V_*)) (\varphi - \psi) dt. \quad (3.2.12)$$

Given $\|f''\|_\infty \leq M$, the integrand is bounded by $M\|\psi + t(\varphi - \psi)\|_\infty$. Using the triangle inequality for the convex combination, we obtain:

$$\begin{aligned} \|\mathcal{N}(\varphi) - \mathcal{N}(\psi)\|_\infty &\leq M\|\varphi - \psi\|_\infty \int_0^1 ((1-t)\|\psi\|_\infty + t\|\varphi\|_\infty) dt \\ &= \frac{M}{2} (\|\varphi\|_\infty + \|\psi\|_\infty) \|\varphi - \psi\|_\infty. \end{aligned} \quad (3.2.13)$$

Since $C_2 = M/2$ and $\varphi, \psi \in B_{R_*}$, equation (3.2.13) simplifies to:

$$\|\mathcal{N}(\varphi) - \mathcal{N}(\psi)\|_\infty \leq 2C_2 R_* \|\varphi - \psi\|_\infty. \quad (3.2.14)$$

Substituting (3.2.14) back into (3.2.11), the Lipschitz constant k for the operator $\mathcal{F}$ is:

$$k = \frac{2C_2 R_*}{m - \beta_1 d_v}. \quad (3.2.15)$$

Plugging the explicit formula for R_* from (3.2.10) into (3.2.15) yields:

$$k = 1 - \sqrt{1 - \frac{4C_2(d_v V_* \|\mathcal{L}_D \mathbf{1}_\Omega\|_\infty + \epsilon)}{(m - \beta_1 d_v)^2}}. \quad (3.2.16)$$

Since the discriminant is strictly positive, the square root is real and strictly positive, guaranteeing that $k < 1$. Thus, $\mathcal{F}$ is a strict contraction on B_{R_*} . By the Banach Fixed Point Theorem, there exists a unique solution $\varphi \in B_{R_*}$, completing the proof. $\square$

3.2.2 Continuous patterns

We continue by analyzing an abstract equation with a linear operator $\mathcal{L}$ and a nonlinear operator $\mathcal{N}$ given below

$$\mathcal{L}V + \mathcal{N}(V) = \lambda V. \quad (3.2.17)$$

We assume $\mathcal{N}$ is such that $\mathcal{N}(V) = o(\|V\|)$ when $\|V\| \rightarrow 0$. Obviously, there is a trivial solution $(0, \lambda)$ for all $\lambda \in \mathbb{R}$. If equation (3.2.17) has a variational formulation, then bifurcation techniques allow us to construct a family of its non-trivial solutions.

Definition 3.2.1 (Bifurcation point). *We call $(0, \lambda)$ a bifurcation point for equation (3.2.17) if every neighbourhood of $(0, \lambda)$ contains a non-zero solution.*

Now, for convenience of the reader we supply this subsection with the classical result by Rabinowitz [44].

Theorem 3.2.2 (Rabinowitz Bifurcation Theorem). *Let $\mathbb{H}$ be a Hilbert space, V a neighbourhood of 0 in $\mathbb{H}$, and $I \in C^2(V, \mathbb{R})$ with $DI(u) = \mathcal{L}u + \mathcal{N}(u)$ where $\mathcal{L} : \mathbb{H} \rightarrow \mathbb{H}$ is linear and the potential $\mathcal{N}(u) = o(\|u\|)$ when $\|u\| \rightarrow 0$. If λ is an isolated eigenvalue of $\mathcal{L}$ of finite multiplicity, then $(0, \lambda)$ is a bifurcation point of the equation (3.2.17). Moreover, at least one of the following alternatives occur.*

- (i) $(0, \lambda)$ is not an isolated solution in $\mathbb{H} \times \{\lambda\}$,
- (ii) There is a one-sided neighbourhood, Λ , of λ such that for all $\mu \in \Lambda \setminus \{\lambda\}$ there are at least two distinct nontrivial solutions,
- (iii) There is a neighbourhood, Λ , of λ such that for all $\mu \in \Lambda \setminus \{\lambda\}$ there is at least one nontrivial solution.

To apply this theorem in a Sobolev space setting on a bounded domain Ω , we must carefully distinguish between the hostile environment (Dirichlet) and the closed environment (Neumann), as their algebraic structure differs significantly due to boundary effects.

We first establish the commutation of the Dirichlet dispersal operator $\mathcal{L}_D$ with the weak derivative.

Lemma 3.2.1. *Let J satisfy Assumption 1.2.1 and assume J is sufficiently smooth. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Denoting by D^α the weak derivative with multi-index $\alpha \in \mathbb{Z}_{\geq 0}^n$, the following equation holds true in the weak sense for functions $u \in W_0^{|\alpha|, 2}(\Omega)$:*

$$D^\alpha \mathcal{L}_D u = \mathcal{L}_D D^\alpha u. \quad (3.2.18)$$

Proof. We present the proof for the first order weak derivative ∂_{x_i} . The crucial property of the Dirichlet case is that $u \in W_0^{1,2}(\Omega)$ can be extended by zero to the whole space $\mathbb{R}^n$, resulting in a valid Sobolev function $u \in W^{1,2}(\mathbb{R}^n)$. Thus, we re-write the integral part of $\mathcal{L}_D u(x)$ as an integral over $\mathbb{R}^n$. For any test function $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} (\partial_{x_i} \mathcal{L}_D u)(x) \varphi(x) dx &= - \int_{\Omega} (\mathcal{L}_D u)(x) \partial_{x_i} \varphi(x) dx \\ &= - \int_{\Omega} \left(\int_{\mathbb{R}^n} J(x-y) u(y) dy - u(x) \right) \partial_{x_i} \varphi(x) dx. \end{aligned}$$

Focusing on the convolution term and utilizing the property $\partial_{x_i} J(x - y) = -\partial_{y_i} J(x - y)$, we obtain

$$\begin{aligned} - \int_{\Omega} \int_{\mathbb{R}^n} J(x - y) u(y) dy \partial_{x_i} \varphi(x) dx &= \int_{\Omega} \partial_{x_i} \left(\int_{\mathbb{R}^n} J(x - y) u(y) dy \right) \varphi(x) dx \\ &= \int_{\Omega} \left(\int_{\mathbb{R}^n} (-\partial_{y_i} J(x - y)) u(y) dy \right) \varphi(x) dx. \end{aligned}$$

Since u has compact support inside Ω , integrating by parts in $\mathbb{R}^n$ with respect to y_i generates no boundary terms

$$\int_{\mathbb{R}^n} (-\partial_{y_i} J(x - y)) u(y) dy = \int_{\mathbb{R}^n} J(x - y) \partial_{y_i} u(y) dy.$$

Because $\partial_{y_i} u(y) = 0$ outside Ω , we restrict the integral back to Ω . Handling the local $-u(x)$ term similarly yields exactly $\int_{\Omega} (\mathcal{L}_D \partial_{x_i} u)(x) \varphi(x) dx$. $\square$

Remark 3.2.1. For the Neumann operator $\mathcal{L}_N$, the commutation $D^\alpha \mathcal{L}_N = \mathcal{L}_N D^\alpha$ generally fails on a bounded domain Ω . Because the integration is strictly restricted to $\Omega \times \Omega$, integrating by parts necessarily generates boundary integrals on $\partial\Omega$. The strict commutation holds only on manifolds without boundaries, such as the periodic torus $\mathbb{T}^n$. However, the boundedness of the operator $\mathcal{L}_N$ on Sobolev spaces $W^{k,2}(\Omega)$ still holds true.

Next, we establish the correct variational formulations. The energy functionals must be defined differently depending on the boundary conditions.

Lemma 3.2.2. Let J satisfy Assumption 1.2.1 and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that for $f \in C^1(\mathbb{R})$ there exists a primitive $F' = f$. Define a nonlinear functional $I : \Theta \rightarrow \mathbb{R}$, where $\Theta \subset W^{k,2}(\Omega)$ with $k > \frac{n}{2}$ is an open neighbourhood of 0.

For the Neumann case ($\mathcal{L}_N$):

$$I_N(u) = -\frac{1}{4} \iint_{\Omega \times \Omega} J(x - y) (u(y) - u(x))^2 dy dx + \int_{\Omega} F(u(x)) dx. \quad (3.2.19)$$

For the Dirichlet case ($\mathcal{L}_D$): Extending u by zero on $\mathbb{R}^n \setminus \Omega$,

$$I_D(u) = -\frac{1}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y) (u(y) - u(x))^2 dy dx + \int_{\Omega} F(u(x)) dx. \quad (3.2.20)$$

Then $I \in C^2(\Theta, \mathbb{R})$. Moreover, the Frechet derivative for respective functionals is given exactly by:

$$DI(u) = \mathcal{L}u + f(u),$$

where $\mathcal{L}$ is either $\mathcal{L}_N$ or $\mathcal{L}_D$.

Proof. We compute the Gateaux derivative for the Dirichlet case $I_D(u)$ (the Neumann case $I_N(u)$ is strictly analogous, substituting $\mathbb{R}^n$ with Ω). We evaluate the directional derivative at u in the direction of a test function $\varphi \in C_c^\infty(\Omega)$, extended by zero

$$\begin{aligned} \langle DI_D(u), \varphi \rangle &= \left. \frac{d}{d\epsilon} I_D(u + \epsilon\varphi) \right|_{\epsilon=0} \\ &= -\frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y) (u(y) - u(x)) (\varphi(y) - \varphi(x)) dy dx + \int_{\Omega} f(u(x)) \varphi(x) dx. \end{aligned}$$

By the symmetry of the integration domain and the kernel $J(x - y) = J(y - x)$, we split the first integral

$$\begin{aligned} & \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(u(y) - u(x))\varphi(y) dy dx - \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(u(y) - u(x))\varphi(x) dy dx \\ &= \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(u(x) - u(y))\varphi(x) dx dy - \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(u(y) - u(x))\varphi(x) dy dx \\ &= - \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} J(x - y)u(y) dy - u(x) \int_{\mathbb{R}^n} J(x - y) dy \right) \varphi(x) dx. \end{aligned}$$

Because $\int_{\mathbb{R}^n} J(x - y)dy = 1$ and $\varphi(x) = 0$ outside Ω , this simplifies precisely to

$$- \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(u(y) - u(x))(\varphi(y) - \varphi(x)) dy dx = \int_{\Omega} (\mathcal{L}_D u)(x)\varphi(x) dx. \quad (3.2.21)$$

Including the reaction term yields $DI_D(u) = -\mathcal{L}_D u - f(u)$.

Finally, we compute the second directional derivative of I at u in the directions $\varphi, \psi \in W^{k,2}(\Omega)$

$$\begin{aligned} \langle D^2 I_D(u)\varphi, \psi \rangle &= \frac{d}{d\epsilon} \langle DI_D(u + \epsilon\psi), \varphi \rangle \Big|_{\epsilon=0} \\ &= \int_{\Omega} (\mathcal{L}_D \psi)(x)\varphi(x) dx + \int_{\Omega} f'(u(x))\psi(x)\varphi(x) dx \\ &= -\frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} J(x - y)(\psi(y) - \psi(x))(\varphi(y) - \varphi(x)) dy dx \\ &\quad + \int_{\Omega} f'(u(x))\psi(x)\varphi(x) dx. \end{aligned}$$

The regularity $I \in C^2(\Theta, \mathbb{R})$ follows directly from the Sobolev embedding $W^{k,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ since $k > \frac{n}{2}$, ensuring that the non-linear terms $F(u)$, $f(u)$ and $f'(u)$ are bounded. $\square$

We now leverage the bifurcation theory to obtain the following result.

Theorem 3.2.3 (Continuous equilibrium). *Let J satisfy Assumption 1.2.1 and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $f \in C^1(\mathbb{R})$ is such that there exist $v_k \in \mathbb{R}$ satisfying $f(v_k) = 0$, $f'(v_k) = \beta_k$, where β_k is the k -th eigenvalue of the respective dispersal operator $\mathcal{L}$. Consider the steady state equation*

$$\mathcal{L}V + f(V) = \lambda V. \quad (3.2.22)$$

Then for $\ell > \frac{n}{2}$ there exists a sequence of non-constant solutions $\{V_i\}_{i=0}^\infty \subset W^{\ell,2}(\Omega)$ and corresponding numbers $\{\lambda_i\}_{i=0}^\infty \subset \mathbb{R}$ to equation (3.2.22) such that

$$(V_i, \lambda_i) \xrightarrow{i \rightarrow \infty} (v_k, 0).$$

Proof. Applying Taylor's Theorem around the spatially uniform root v_k , we re-write equation (3.2.22) as

$$\mathcal{L}V + f'(v_k)V + R_1(V) = \lambda V.$$

Using $f'(v_k) = \beta_k$ and rearranging terms yields an equation driven by the linear operator $\mathcal{L}$ perturbed by a nonlinear component

$$\mathcal{L}V + R_1(V) = (\lambda - \beta_k)U. \quad (3.2.23)$$

Taylor's Theorem guarantees that $R_1(V) = o(\|V\|_\infty)$ as $\|V\|_\infty \rightarrow 0$. The Sobolev embedding $W^{\ell,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ assures that R_1 is a valid potential operator. By Lemma 3.2.2, the problem (3.2.23) admits a variational formulation defined by the functional $I(V)$.

Since, by Theorem 2.4.1 and Theorem 2.4.2, β_k is an isolated eigenvalue of finite multiplicity for the linear operator $\mathcal{L}$, the Rabinowitz Bifurcation Theorem 3.2.2 applies. Thus, there exists a sequence of parameters $\lambda_i \rightarrow 0$ and a non-constant functions $V_i \rightarrow v_k$ solving (3.2.22). $\square$

3.2.3 Discontinuous patterns

In this subsection, we investigate the existence of discontinuous, spatially heterogeneous stationary states to the non-local dispersal problem. We work in regime where the dispersal events are highly non-local, meaning the dispersal range is very large compared to the domain size.

To formalize this, we introduce a scaling parameter $\varepsilon > 0$ and consider the scaled dispersal kernel $J_\varepsilon(z) = \frac{1}{\varepsilon^n} J\left(\frac{z}{\varepsilon}\right)$. Note that this scaling preserves the unit mass, e.g. $\int_{\mathbb{R}^n} J_\varepsilon(z) dz = \int_{\mathbb{R}^n} J(z) dz = 1$. The scaled steady-state equation reads

$$\mathcal{L}_\varepsilon V(x) + f(V(x)) = 0, \quad x \in \Omega, \quad (3.2.24)$$

where

$$\mathcal{L}_\varepsilon V(x) = \int_{\Omega} J_\varepsilon(x-y)V(y) dy - b_\varepsilon(x)V(x), \quad (3.2.25)$$

with

$$b_\varepsilon(x) = \int_{\Omega} J_\varepsilon(x-y) dy \quad (3.2.26)$$

The main result of this subsection establishes that as the dispersal range grows ($\varepsilon \rightarrow \infty$), the probability of landing in the finite domain Ω vanishes, heavily suppressing the spatial coupling. In this highly non-local limit, the local reaction kinetics dominate, allowing the domain to partition into sub-regions defined by different stable roots of the reaction function.

Theorem 3.2.4 (Discontinuous equilibrium). *Consider the scaled integral kernel J_ε and domain Ω satisfying Assumption 1.2.1. Let the dispersal operator be of Neumann-type, e.g. we denote $\mathcal{L}_\varepsilon = \mathcal{L}_{\varepsilon,N}$. Assume that $f \in C^2(\mathbb{R})$ is such that there exist two distinct roots $v_1, v_2 \in \mathbb{R}$ satisfying $f(v_1) = f(v_2) = 0$ with $f'(v_1) \neq 0$ and $f'(v_2) \neq 0$.*

Let $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ be arbitrary open, non-empty sets such that:

$$\Omega_1 \cap \Omega_2 = \emptyset \quad \text{and} \quad \Omega = (\overline{\Omega_1 \cup \Omega_2})^o. \quad (3.2.27)$$

If the dispersal range parameter $\varepsilon > 0$ is sufficiently large, then the steady problem (3.2.24) admits a discontinuous solution of the form $V = \tilde{V} + \psi$. The base jump function $\tilde{V}$ is defined almost everywhere by:

$$\tilde{V}(x) = \begin{cases} v_1 & \text{for } x \in \Omega_1, \\ v_2 & \text{for } x \in \Omega_2, \end{cases} \quad (3.2.28)$$

and the perturbation satisfies $\psi \in \overline{B(0, R)} \subset L^\infty(\Omega)$ for some small radius $R > 0$.

Proof. First, we evaluate the impact of the scaling $\varepsilon \rightarrow \infty$ on the norm of the dispersal operator acting on bounded domains. For any $u \in L^\infty(\Omega)$:

$$\|\mathcal{L}_\varepsilon u\|_\infty \leq \sup_{x \in \Omega} \int_\Omega J_\varepsilon(x-y)|u(y)| dy + \sup_{x \in \Omega} b_\varepsilon(x)\|u\|_\infty \leq 2\|u\|_\infty \sup_{x \in \Omega} \int_\Omega J_\varepsilon(x-y) dy.$$

Because Ω is bounded and J is continuous, the following upper bound holds true

$$\int_\Omega J_\varepsilon(x-y) dy \leq |\Omega| \sup_{z \in \Omega-x} J_\varepsilon(z) \leq |\Omega| \frac{\|J\|_\infty}{\varepsilon^n}. \quad (3.2.29)$$

Thus, there exists a constant $C > 0$ independent of ε such that $\|\mathcal{L}_\varepsilon\|_\infty \leq \frac{C}{\varepsilon^n}$.

We substitute the ansatz $V = \tilde{V} + \psi$ into equation (3.2.24). Because $\tilde{V}(x) \in \{v_1, v_2\}$ almost everywhere in Ω , and $f(v_i) = 0$, it follows that $f(\tilde{V}) = 0$ in $L^\infty(\Omega)$. Adding and subtracting $f'(\tilde{V})\psi$, we obtain:

$$\mathcal{L}_\varepsilon[\tilde{V} + \psi] + f(\tilde{V} + \psi) - f(\tilde{V}) - f'(\tilde{V})\psi + f'(\tilde{V})\psi = 0. \quad (3.2.30)$$

Since $f'(v_1) \neq 0$ and $f'(v_2) \neq 0$, the function $f'(\tilde{V})$ is bounded strictly away from zero. Let $m := \min(|f'(v_1)|, |f'(v_2)|) > 0$. We rewrite the problem as a fixed-point equation $\psi = \mathcal{F}_\varepsilon(\psi)$, where the operator $\mathcal{F}_\varepsilon$ is defined as

$$\mathcal{F}_\varepsilon(\psi) := \frac{-1}{f'(\tilde{V})} \left(\mathcal{L}_\varepsilon[\tilde{V} + \psi] + f(\tilde{V} + \psi) - f(\tilde{V}) - f'(\tilde{V})\psi \right). \quad (3.2.31)$$

Take $\psi \in \overline{B(0, R)} \subset L^\infty(\Omega)$. Using Taylor's theorem for the non-linear terms and the decay property in equation (3.2.29), we estimate the L^∞ -norm

$$\begin{aligned} \|\mathcal{F}_\varepsilon(\psi)\|_\infty &\leq \left\| \frac{1}{f'(\tilde{V})} \right\|_\infty \left(\|\mathcal{L}_\varepsilon\|_\infty (\|\tilde{V}\|_\infty + \|\psi\|_\infty) + \frac{1}{2} \|f''(\xi)\|_\infty \|\psi\|_\infty^2 \right) \\ &\leq \frac{1}{m} \left(\frac{C}{\varepsilon^n} (\|\tilde{V}\|_\infty + R) + \frac{1}{2} \|f''(\xi)\|_\infty R^2 \right), \end{aligned} \quad (3.2.32)$$

where $\xi \in B(\tilde{V}, \max(R, v_1, v_2)) \subset L^\infty(\Omega)$. Since $f \in C^2(\mathbb{R})$, there exists $M > 0$ such that $\|f''(\xi)\|_\infty \leq M$. Thus, for $\mathcal{F}_\varepsilon$ to map $\overline{B(0, R)}$ into itself ($\|\mathcal{F}_\varepsilon(\psi)\|_\infty \leq R$), we require

$$\frac{C}{m\varepsilon^n} (\|\tilde{V}\|_\infty + R) + \frac{M}{2m} R^2 \leq R. \quad (3.2.33)$$

Next, we derive the contractivity criterion. For any $\varphi, \psi \in \overline{B(0, R)}$, the Mean Value Theorem yields

$$\begin{aligned} \|\mathcal{F}_\varepsilon(\varphi) - \mathcal{F}_\varepsilon(\psi)\|_\infty &\leq \frac{1}{m} \left\| \mathcal{L}_\varepsilon(\varphi - \psi) + f(\tilde{V} + \varphi) - f(\tilde{V} + \psi) - f'(\tilde{V})(\varphi - \psi) \right\|_\infty \\ &\leq \frac{1}{m} \left(\frac{C}{\varepsilon^n} + \sup_{\zeta \in B(\tilde{V}, R)} \|f'(\zeta) - f'(\tilde{V})\|_\infty \right) \|\varphi - \psi\|_\infty \\ &\leq \frac{1}{m} \left(\frac{C}{\varepsilon^n} + MR \right) \|\varphi - \psi\|_\infty. \end{aligned} \quad (3.2.34)$$

To have that $\mathcal{F}_\varepsilon$ is a contraction, we require the Lipschitz constant to be strictly less than 1

$$\frac{C}{m\varepsilon^n} + \frac{M}{m}R < 1. \quad (3.2.35)$$

By first choosing $R > 0$ sufficiently small such that $\frac{M}{m}R < \frac{1}{2}$, and subsequently choosing the dispersal range ε sufficiently large so that $\frac{C}{m\varepsilon^n}$ is arbitrarily small, both conditions (3.2.33) and (3.2.35) are simultaneously satisfied. The existence of a unique $\psi \in \overline{B(0, R)}$ follows from the Banach Fixed Point Theorem. This establishes the discontinuous pattern and proves Theorem 3.2.4. $\square$

3.3 Reaction-diffusion-dispersal models

In this section, we extend our analysis to systems coupling multiple interacting species or resources, where the transport mechanisms differ. Specifically, we investigate the existence of stationary states to reaction-diffusion-dispersal problems. The general steady-state system takes the form

$$\begin{cases} d_v \mathcal{L}V(x) + f(V(x), W(x)) = 0 & x \in \Omega, \\ d_w \Delta W(x) + g(V(x), W(x)) = 0 & x \in \Omega, \end{cases} \quad (3.3.1)$$

$$\quad (3.3.2)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain. The functions V and W represent the stationary densities of the interacting components (e.g., vegetation biomass and soil water). The parameters $d_v, d_w > 0$ denote the dispersal and diffusion rates, respectively. The scalar functions $f, g \in C^1(\mathbb{R}^2)$ describe the local non-linear reaction kinetics.

As in the single-equation case, we construct the solutions $(V, W) \in X_v \times X_w$. The choice of the function spaces follows from the boundary conditions

- **Dispersing component (V):** For a hostile environment (Dirichlet case) we take $X_v = L^2_{0,\Omega}(\mathbb{R}^n)$ or $X_v = C_{0,\Omega}(\mathbb{R}^n)$. For a closed environment (Neumann case) we take $X_v = L^2(\Omega)$ or $X_v = C(\overline{\Omega})$.
- **Diffusing component (W):** We assume a hostile boundary for the diffusing resource, imposing homogeneous Dirichlet boundary conditions $W = 0$ on $\partial\Omega$. Accordingly, we set $X_w = C_0(\Omega)$, where the boundary condition is enforced within the standard domain of the Dirichlet Laplacian Δ .

Having established this framework, our goal is to construct non-constant stationary solutions (V, W) to the coupled system (3.3.1)–(3.3.2).

3.3.1 Reduction of the diffusion part

Before analyzing the full coupled system, we reduce its dimensionality by exploiting the regularizing properties of the Laplacian operator. Assuming the dispersal component $V \in C(\Omega)$ is given, the second equation (3.3.2) becomes a semi-linear elliptic boundary value problem for W

$$\begin{cases} -d_w \Delta W = g(V(x), W) & x \in \Omega, \\ W = 0 & x \in \partial\Omega. \end{cases} \quad (3.3.3)$$

$$\quad (3.3.4)$$

We establish the existence, uniqueness, and Lipschitz continuity of the map $V \mapsto W(V)$ under appropriate structural assumptions on the reaction term g .

Lemma 3.3.1 (Existence of the resource profile). *Let $V \in C(\Omega)$ be a given non-negative function. Assume that the reaction function $g(V, W)$ is smooth, monotonically decreasing with respect to W (i.e., $\partial_W g(V, W) \leq -c < 0$ for some $c > 0$), and that $g(V, 0) \geq 0$.*

Then there exists a unique, non-negative strong solution $W \in W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega)$ (for any $1 < p < \infty$) to the boundary value problem (3.3.3)–(3.3.4). Consequently, we can define a well-posed solution map $V \mapsto W(V)$.

Proof. The existence and uniqueness follow from standard elliptic theory for semi-linear equations. Because $\partial_W g(V, W) \leq -c < 0$, the operator $W \mapsto -d_w \Delta W - g(V, W)$ is strongly monotone. The condition $g(V, 0) \geq 0$ ensures that $\underline{W} \equiv 0$ acts as a sub-solution:

$$-d_w \Delta \underline{W} - g(V, \underline{W}) = -g(V, 0) \leq 0.$$

Assuming $g(V, W)$ is bounded from above for bounded V , one can construct a large constant super-solution $\overline{W}$. The method of sub- and super-solutions then guarantees the existence of a unique, non-negative classical solution W . The $W^{2,p}$ regularity follows from standard L^p elliptic estimates [24]. $\square$

Next, we prove that the solution map $V \mapsto W(V)$ is Lipschitz continuous with respect to the supremum norm. This property is crucial for substituting $W(V)$ back into the non-local dispersal equation.

Lemma 3.3.2 (Lipschitz continuity of the resource map). *Let $B_R \subset C_{0,\Omega}(\mathbb{R}^n)$ be a closed ball of radius $R > 0$. Assume that the mapping $V \mapsto g(V, W)$ is uniformly Lipschitz continuous on B_R with constant L_g , meaning $|g(V_1, W) - g(V_2, W)| \leq L_g \|V_1 - V_2\|_\infty$. Further assume $\partial_W g(V, W) \leq -c < 0$.*

Then the solution map $W : B_R \rightarrow C(\overline{\Omega})$ defined by Lemma 3.3.1 is Lipschitz continuous:

$$\|W(V_1) - W(V_2)\|_\infty \leq \frac{L_g}{c} \|V_1 - V_2\|_\infty \quad \text{for all } V_1, V_2 \in B_R. \quad (3.3.5)$$

Proof. Take arbitrary $V_1, V_2 \in B_R \subset C_{0,\Omega}(\mathbb{R}^n)$. Let $W_1 = W(V_1)$ and $W_2 = W(V_2)$ be the corresponding unique solutions to (3.3.3). Define the difference $U = W_1 - W_2$. Subtracting the equations for W_1 and W_2 , we find that U satisfies the boundary value problem

$$\begin{cases} -d_w \Delta U = g(V_1, W_1) - g(V_2, W_2) & x \in \Omega, \\ U = 0 & x \in \partial\Omega. \end{cases} \quad (3.3.6)$$

$$(3.3.7)$$

We can rewrite the right-hand side of (3.3.6) by adding and subtracting $g(V_1, W_2)$

$$\begin{aligned} g(V_1, W_1) - g(V_2, W_2) &= (g(V_1, W_1) - g(V_1, W_2)) + (g(V_1, W_2) - g(V_2, W_2)) \\ &= \partial_W g(V_1, \xi)(W_1 - W_2) + (g(V_1, W_2) - g(V_2, W_2)) \\ &= \partial_W g(V_1, \xi)U + (g(V_1, W_2) - g(V_2, W_2)), \end{aligned} \quad (3.3.8)$$

where $\xi(x)$ is an intermediate value between $W_1(x)$ and $W_2(x)$ provided by the Mean Value Theorem. Substituting (3.3.8) back into (3.3.6) and rearranging yields

$$-d_w \Delta U - \partial_W g(V_1, \xi)U = g(V_1, W_2) - g(V_2, W_2). \quad (3.3.9)$$

Since $\partial_W g(V_1, \xi) \leq -c < 0$, the operator on the left-hand side satisfies the conditions of the strong maximum principle. Applying the maximum principle to equation (3.3.9), we obtain the bound

$$\|U\|_\infty \leq \frac{1}{c} \|g(V_1, W_2) - g(V_2, W_2)\|_\infty. \quad (3.3.10)$$

Using the uniform Lipschitz continuity of g with respect to V , we estimate the right-hand side

$$\|g(V_1, W_2) - g(V_2, W_2)\|_\infty \leq L_g \|V_1 - V_2\|_\infty. \quad (3.3.11)$$

Combining (3.3.10) and (3.3.11), and substituting $U = W(V_1) - W(V_2)$, we arrive at the desired result

$$\|W(V_1) - W(V_2)\|_\infty \leq \frac{L_g}{c} \|V_1 - V_2\|_\infty. \quad (3.3.12)$$

□

By utilizing the mapping $W(V)$ established above, we can substitute the variable W out of the first equation (3.3.1). This successfully reduces the coupled reaction-diffusion-dispersal system (3.3.1)–(3.3.2) to a single, non-local, non-linear integro-differential equation for the dispersal component V

$$d_v \mathcal{L}V(x) + F_{red}(V(x)) = 0, \quad x \in \Omega, \quad (3.3.13)$$

where the reduced reaction functional is defined as $F_{red}(V) := f(V, W(V))$. The properties of F_{red} depend strictly on the interplay between the local kinetics f and the elliptic solution map $W(V)$.

3.3.2 The Klausmeier-Gray-Scott model

In this subsection, we apply our general theoretical framework to the specific interaction kinetics of the Klausmeier vegetation model. The local reaction terms for vegetation biomass V and soil water W are given by

$$f(V, W) = V^2 W - BV, \quad \text{and} \quad g(V, W) = -V^2 W - W + A, \quad (3.3.14)$$

where $A > 0$ is the rainfall rate and $B > 0$ is the plant mortality. Throughout this section we shall assume additionally that domain Ω satisfies the following supplementary conditions

Assumption 3.3.1. *We assume that $\Omega \subset \mathbb{R}^n$ is bounded, open and satisfies the following properties:*

- (O1) **Uniform estimates:** *Let $\{\phi_j\}_{j \in \mathbb{N}}$ be the eigenfunctions of Δ with Dirichlet boundary conditions. There exists a constant $C := C(\Omega) > 0$ such that $\sup_{j \in \mathbb{N}} \|\phi_j\|_\infty \leq C$.*
- (O2) **Boundary regularity:** *The boundary of Ω is sufficiently regular, i.e. $\partial\Omega \in C^k$ with suitable $k \in \mathbb{N}$.*

Now, we study the spatially uniform steady states (V_*, W_*) of the purely kinetic system, e.g. $f(V_*, W_*) = 0$ and $g(V_*, W_*) = 0$.

Proposition 3.3.1. *Let f, g be defined as in the equations (3.3.14). Consider the following problem of finding steady states (V^*, W^*) in kinetic system*

$$\begin{cases} 0 = f(V^*, W^*), \\ 0 = g(V^*, W^*). \end{cases} \quad (3.3.15)$$

The following statements hold true regarding the number of solutions for the system (3.3.15)

- If $A > 2B$ then there are 3 constant stationary solutions.
- If $A = 2B$ then there are 2 constant stationary solutions.
- If $A < 2B$ then there is 1 constant stationary solution of.

Proof. To find solutions of (3.3.15) one needs to solve for (V, W) the following system of equations

$$\begin{cases} 0 = V^2W - BV, \\ 0 = A - W - V^2W. \end{cases} \quad (3.3.16)$$

From the second equation in (3.3.16) we get formula $W = \frac{A}{V^2+1}$. Thus the first equation becomes

$$-BV^3 + AV^2 - BV = 0 \quad (3.3.17)$$

For all parameters $A > 0, B > 0$ we can find the desert state solution of following form

$$(V_1^*, W_1^*) = (0, A), \quad (3.3.18)$$

Provided that $A > 2B$, the equation (3.3.17) admits two more solutions

$$(V_2^*, W_2^*) = \left(\frac{A - \sqrt{A^2 - 4B^2}}{2B}, \frac{2B^2}{A - \sqrt{A^2 - 4B^2}} \right), \quad (3.3.19)$$

$$(V_3^*, W_3^*) = \left(\frac{A + \sqrt{A^2 - 4B^2}}{2B}, \frac{2B^2}{A + \sqrt{A^2 - 4B^2}} \right). \quad (3.3.20)$$

Finally, if $A = 2B$, then the solutions (3.3.19) and (3.3.20) merge into

$$(V_4^*, W_4^*) = (1, B). \quad (3.3.21)$$

For $A < 2B$ there are no other, real solutions of (3.3.17) then (3.3.18). $\square$

Now, in order to apply Theorem 3.2.1 to the non-local Dirichlet problem, we analyze existence and estimates for the reduced variable $W(V_*)$.

Lemma 3.3.3 (Proximity of the reduced water profile). *Assume the domain Ω satisfies Assumption 3.3.1. Let (V_*, W_*) be the positive uniform steady state (3.3.15). Let $W(V_*)$ be the unique solution to the elliptic problem:*

$$-d_w \Delta W + (V_*^2 + 1)W = A \quad \text{in } \Omega, \quad W = 0 \quad \text{on } \partial\Omega. \quad (3.3.22)$$

Then for any $\epsilon > 0$, there exists a sufficiently small diffusion rate $d_w > 0$ such that:

$$\|W(V_*) - W_*\|_\infty < \epsilon. \quad (3.3.23)$$

Proof. We expand the constant function $A = \sum_{j=1}^{\infty} a_j \phi_j$ using the orthonormal eigenbasis of the Dirichlet Laplacian, e.g. where $-\Delta \phi_j = \lambda_j \phi_j$. Because $W(V_*) \in L^2(\Omega)$, we can expand it as $W(V_*) = \sum_{j=1}^{\infty} c_j \phi_j$. Substituting these into (3.3.22) and using orthogonality yields the algebraic relation for the coefficients:

$$d_w \lambda_j c_j + (V_*^2 + 1)c_j = a_j \implies c_j = \frac{a_j}{V_*^2 + 1 + d_w \lambda_j}.$$

Recalling that the exact uniform water state is $W_* = \frac{A}{V_*^2 + 1} = \sum_{j=1}^{\infty} \frac{a_j}{V_*^2 + 1} \phi_j$, we estimate the L^∞ norm of the difference:

$$\begin{aligned} \|W(V_*) - W_*\|_\infty &= \left\| \sum_{j=1}^{\infty} \left(\frac{1}{V_*^2 + 1 + d_w \lambda_j} - \frac{1}{V_*^2 + 1} \right) a_j \phi_j \right\|_\infty \\ &\leq C \sum_{j=1}^{\infty} \left| \frac{d_w \lambda_j a_j}{(V_*^2 + 1)(V_*^2 + 1 + d_w \lambda_j)} \right|. \end{aligned} \quad (3.3.24)$$

Since A is a constant, the sequence of coefficients $\{a_j\}$ decays rapidly enough. For any fixed $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that the tail of the series $\sum_{j=N+1}^{\infty} |a_j|$ is arbitrarily small. For the finite sum $j \in \{1, \dots, N\}$, the terms are proportional to d_w . Thus, choosing $d_w > 0$ sufficiently small makes the entire expression (3.3.24) strictly less than ϵ . $\square$

With this proximity result, we can show that the reduced Klausmeier system fulfills the requirements for the existence of almost constant solutions near the boundary.

Proposition 3.3.2 (Conditions for almost constant vegetation). *Assume $A > 2B$, and let $V_* > 1$ be the stable root of the system (3.3.15). Define the reduced reaction functional for the vegetation as $F_{red}(V) := V^2 W(V) - BV$, where $W(V)$ is the solution map from Lemma 3.3.1.*

Then for every $\epsilon > 0$, there exists $d_w > 0$ sufficiently small, and positive constants $m, M > 0$ such that:

$$\|F_{red}(V_*)\|_\infty \leq \epsilon, \quad F'_{red}(V_*) \leq -m < 0, \quad \text{and} \quad \|F''_{red}\|_\infty \leq M. \quad (3.3.25)$$

Proof. First, utilizing $V_*^2 W_* - BV_* = 0$, we bound the evaluation of the reduced functional at the uniform state:

$$\|F_{red}(V_*)\|_\infty = \|V_*^2 W(V_*) - BV_* - (V_*^2 W_* - BV_*)\|_\infty = V_*^2 \|W(V_*) - W_*\|_\infty.$$

By Lemma 3.3.3, choosing d_w small enough ensures $\|W(V_*) - W_*\|_\infty \leq \epsilon/V_*^2$, yielding $\|F_{red}(V_*)\|_\infty \leq \epsilon$.

Next, we compute the Fréchet derivative of F_{red} evaluated at the constant function V_* acting on a perturbation h :

$$F'_{red}(V_*)[h] = 2V_* W(V_*)h + V_*^2 W'(V_*)[h] - Bh.$$

The linear operator $W'(V_*)[h]$ is the derivative of the solution map. Differentiating (3.3.22) with respect to V in the direction h yields:

$$-d_w \Delta(W'(V_*)[h]) + (V_*^2 + 1)W'(V_*)[h] = -2V_* W(V_*)h. \quad (3.3.26)$$

In the limit as $d_w \rightarrow 0$, the Laplacian term vanishes, and $W(V_*) \rightarrow W_*$. Equation (3.3.26) degenerates into an algebraic relation, giving $W'(V_*)[h] \rightarrow -\frac{2V_*W_*}{V_*^2+1}h$. Substituting this into the Fréchet derivative limit:

$$\lim_{d_w \rightarrow 0} F'_{red}(V_*) = 2V_*W_* - V_*^2 \left(\frac{2V_*W_*}{V_*^2+1} \right) - B = B - B \left(\frac{2V_*^2}{V_*^2+1} \right) = B \left(\frac{1-V_*^2}{1+V_*^2} \right).$$

Because $V_* > 1$ (since $A > 2B$), this limit is strictly negative. Let $L = B \frac{1-V_*^2}{1+V_*^2} < 0$. By continuity with respect to d_w , we can choose d_w small enough such that $F'_{red}(V_*) \leq L/2 =: -m < 0$.

Finally, differentiating $F'_{red}(V)$ once more yields the second derivative bilinear form, which depends on $W(V)$, $W'(V)$, and $W''(V)$. Since the elliptic operator $-d_w\Delta + (V^2+1)\text{Id}$ has a bounded inverse, W and its derivatives are bounded in $L^\infty(\Omega)$, which guarantees the existence of a finite upper bound M such that $\|F''_{red}\|_\infty \leq M$. $\square$

Consequently, the non-local Dirichlet Klausmeier model satisfies all the assumptions of Theorem 3.2.1, guaranteeing the existence of almost constant vegetation profiles when the dispersal and diffusion rates are sufficiently small.

3.4 Numerical experiments

To validate our theoretical findings and explore the macroscopic consequences of non-local interactions, we conduct a series of numerical experiments on the Klausmeier model. We contrast the classical local reaction-diffusion system with its non-local reaction-diffusion-dispersal counterpart under Dirichlet boundary conditions (the hostile environment).

3.4.1 Simulation framework and kernel calibration.

Throughout these experiments, we consider either the symmetric, one-dimensional interval or a two-dimensional square-shaped domain, e.g. $\Omega = (-L, L)^n$ with $n \leq 2$. To systematically investigate the influence of the dispersal (specifically, the "heaviness" of the dispersal tails), we introduce the generalized Gaussian kernel family parameterized by the shape exponent $p > 0$

$$J_p(x) = C_p e^{-\lambda_p |x|^p}, \quad x \in \mathbb{R}^n. \quad (3.4.1)$$

The constants C_p and λ_p are uniquely calibrated such that the kernel fulfills two conditions

$$\int_{\mathbb{R}^n} J_p(z) dz = 1, \quad \int_{\mathbb{R}^n} J_p(z) |z|^2 dz = 1. \quad (3.4.2)$$

Next, solving equations (3.4.2), the parameters C_p and λ_p are given explicitly in terms of the Gamma function $\Gamma(\cdot)$, spatial dimension n and exponent p as follows

$$\lambda_p = \left(\frac{\Gamma\left(\frac{n+2}{p}\right)}{\Gamma\left(\frac{n}{p}\right)} \right)^{p/2}, \quad C_p = \frac{p \Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2} \Gamma\left(\frac{n}{p}\right)} \lambda_p^{n/p}. \quad (3.4.3)$$

This calibration is crucial for our further numerical studies. Namely, by examining the Taylor expansion of the non-local operator in hostile environment, we observe that

$$\begin{aligned}\mathcal{L}_{D,p}V &= \int_{\mathbb{R}^n} J_p(x-y) (V(y) - V(x)) \, dy \\ &\approx \frac{1}{2} \left(\int_{\mathbb{R}^n} J_p(z) z^2 dz \right) \Delta V + \text{h.o.t.} \\ &= \frac{1}{2} \Delta V + \text{h.o.t.}\end{aligned}\tag{3.4.4}$$

By matching the second moment of the non-local kernel to the local diffusion operator, the leading-order spatial dynamics are identical. This allows us to isolate impact of the higher order terms from equation (3.4.4). In particular, we focus on the following stationary models

Non-local models	Local counterpart	
$d_v \mathcal{L}_{D,p}V + V^2W - BV = 0$	$\frac{d_v}{2} \Delta V + V^2W - BV = 0$	$x \in \Omega,$
$d_w \Delta W - V^2W - W + A = 0$	$d_w \Delta W - V^2W - W + A = 0$	$x \in \Omega,$
$V = 0$	$V = 0$	$x \in \mathbb{R}^n \setminus \Omega \text{ or } \partial\Omega,$
$W = 0$	$W = 0$	$x \in \partial\Omega.$

(3.4.5)

3.4.2 Critical Patch Size and Boundary drop-off

Critical patch size We investigate the existence of a critical patch size for vegetation survival, a concept established in our theoretical analysis (see Remark 4.2.2 below). The goal is to compare the local model against two representative non-local cases with respect to the dispersal range (fat vs thin kernel).

Experiment 3.4.1 (Initial detection of the Critical Patch Size and Boundary Drop-off).

Numerical Method and Setup: To investigate the critical patch size L_{crit} required for population survival, we simulate the system on a spatial domain $[-L, L]$ using an Implicit-Explicit (IMEX) numerical scheme. The local diffusion and non-local dispersal operators are treated implicitly, while the nonlinear reaction terms are evaluated explicitly. Homogeneous Dirichlet boundary conditions are strictly enforced at every time step. The simulation is initialized with the analytically derived uniform state (V_*, W_*) (see equation (3.3.20)), perturbed by a small-amplitude cosine profile. The system evolves until the ℓ^2 -norm of the difference between consecutive time steps falls below a strict tolerance. The critical patch size L_{crit} is numerically identified as the minimum patch half-width for which the steady-state average vegetation biomass remains above a survival threshold (set equal to 0.01).

Experimental Parameters: The specific values and ranges for the numerical settings and ecological model parameters are summarized in Table 3.1.

Symbol	Value / Range	Description
A	1.8	Non-dimensionalized global parameter
B	0.45	Non-dimensionalized global parameter
d_v	2.0	Dispersal rate
d_w	0.1	Diffusion rate
p	$\{0.5, 1.0, 2.0\}$	Kernel's exponents
L	$[1.0, 100.0]$	Patch half-width range (50 log-spaced points)
h_t	10^{-4}	Time step size for the IMEX forward Euler scheme
N	$\lfloor 20 + 5L \rfloor$	Number of spatial grid points (domain-adaptive)
ϵ_{tol}	10^{-5}	Stopping criterion tolerance (ℓ^2 -norm difference)

Table 3.1: Parameters used in Experiment 3.4.1

In Figure 3.1 below, we present results of numerical simulations done according to the procedure described in Experiment 3.4.1.

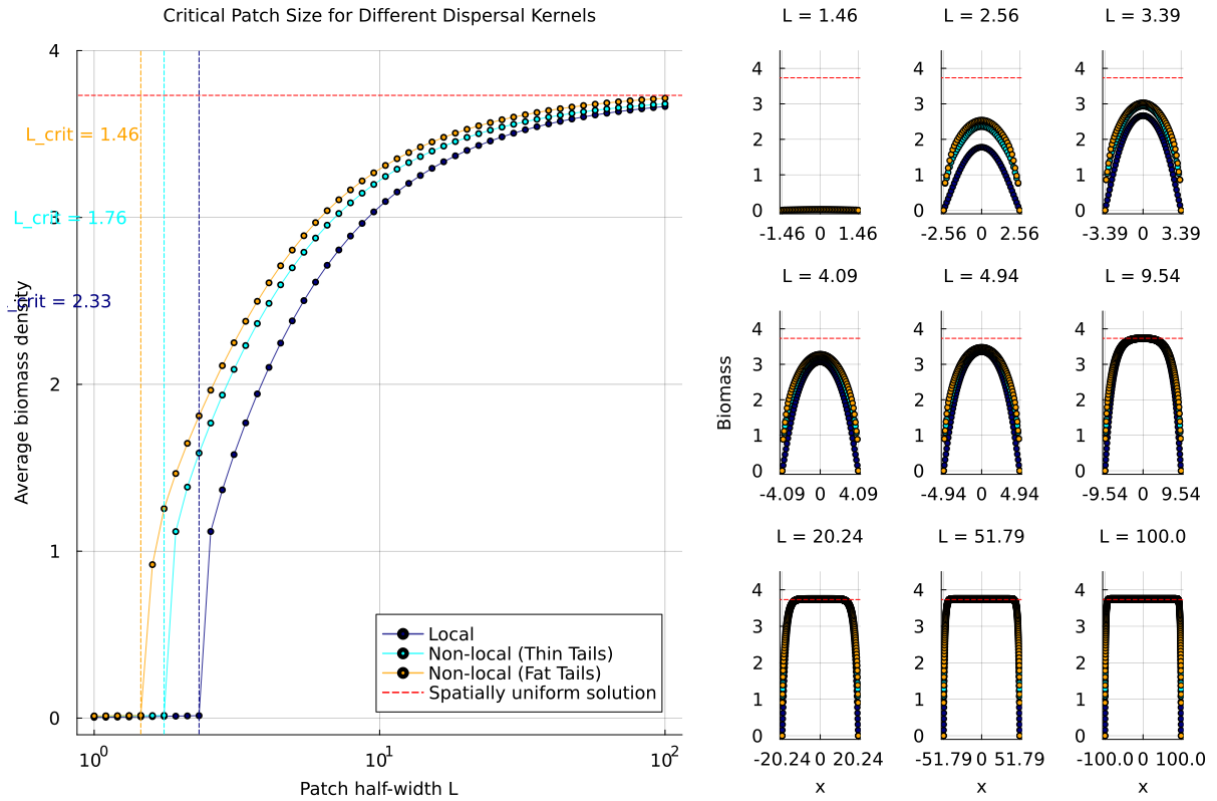


Figure 3.1: Left panel: The average stationary biomass for the non-local models with thin and fat tails (cyan and orange) and the local model (blue) as a function of the patch half-width L . Vertical dashed lines mark the numerically estimated critical patch sizes. Right panel: A gallery of stationary biomass density profiles for different patch half-widths L .

Conclusion 3.4.1. *As shown in the left panel of Figure 3.1, while all models eventually collapse to the desert state on small domains, the critical patch size required for survival differs significantly. The model with the fat-tailed kernel is the most resilient, exhibiting the smallest critical patch size ($L_{crit} \approx 1.46$). This is followed by the thin-tailed kernel ($L_{crit} \approx 1.76$), while the local model is the least resilient ($L_{crit} \approx 2.33$). This confirms that long-range dispersal events significantly enhance vegetation persistence in fragmented habitats.*

Next, we study the dependence of the critical patch size on exponent p of the dispersal kernel J_p , dispersal and diffusion rates d_v, d_w .

Experiment 3.4.2 (Critical Patch Size vs kernel's tail heaviness).

Numerical Method and Setup: We use the same procedure as presented in Experiment 3.4.1. This time we additionally iterate over the range of exponents p to analyze its impact on the critical patch size.

Experimental Parameters: The specific values for parameters $h_t, N, \epsilon_{tol}, L, A, B, d_v, d_w$ can be found in Table 3.1. Here we specifically consider

Symbol	Value / Range	Description
p	$[2^{-2}, 2^2]$	Kernel's exponents (40 log-spaced points)

Table 3.2: Parameters used in the Experiment 3.4.2.

In Figure 3.2 below, we present results of numerical simulations examining behaviour of Critical Patch Size with respect to changing kernel's tail heaviness.

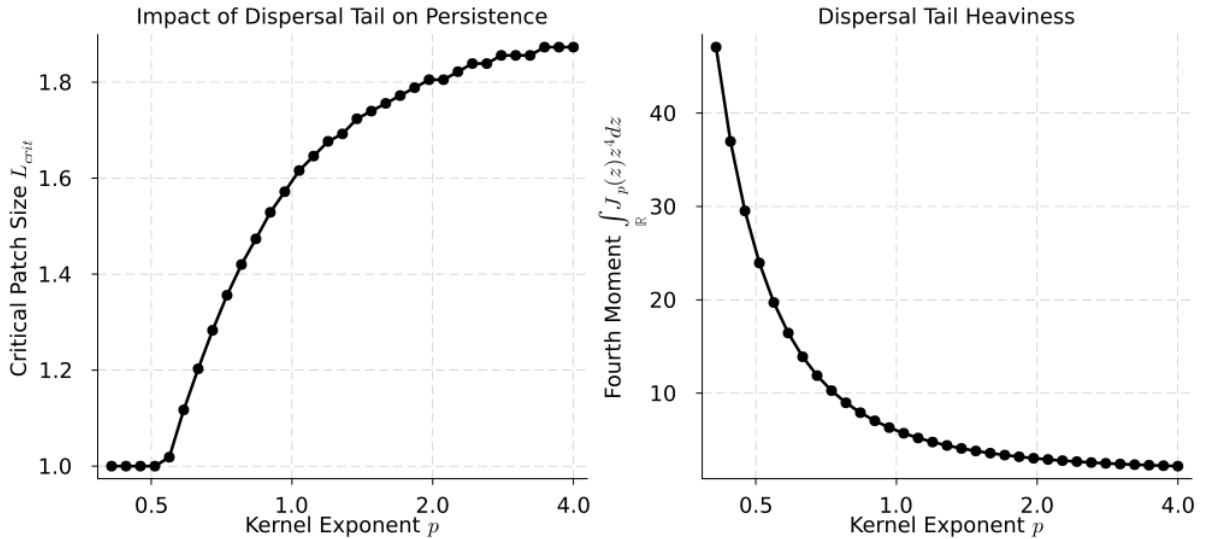


Figure 3.2: Left panel: The critical patch size L_{crit} required for population survival as a function of the kernel shape exponent p . Right panel: The analytically calculated fourth moment of the calibrated kernel J_p , demonstrating the rapid increase in dispersal tail heaviness as $p \rightarrow 0$.

Conclusion 3.4.2. *The results in Figure 3.2 establish a monotonic relationship between dispersal tail heaviness and critical patch size. As the exponent p decreases, the dispersal tails become*

increasingly “fat”, which is reflected by the sharp exponential growth in the kernel’s fourth moment (right panel). This enhanced capacity for long-range dispersal significantly lowers the critical patch size L_{crit} (left panel). Consequently, populations governed by fat-tailed dispersal can persist in much smaller and more fragmented habitats compared to those restricted by thin-tailed or purely local diffusion regimes.

Boundary drop-off As seen in Figure 3.1, the distinguished feature of non-local dispersal in bounded habitats is the emergence of a sharp boundary drop-off, where the stationary biomass density exhibits a discontinuous gap adjacent to the lethal exterior. We propose the following experiment to systematically quantify this behaviour and investigate its sensitivity to the dispersal kernel’s tail heaviness and the domain size.

Experiment 3.4.3 (Boundary Drop-off vs kernel’s tail heaviness).

Numerical Method and Setup: We use the same procedure as presented in Experiment 3.4.1. The boundary gap amplitude is then quantified by averaging the steady-state biomass at the first strictly interior nodes: $V(-L^+) \approx (V_2 + V_{N-1})/2$.

Experimental Parameters: The specific values for parameters $h_t, N, \epsilon_{\text{tol}}, L, A, B, d_v, d_w$ can be found in Table 3.1. Here we specifically consider

Symbol	Value / Range	Description
p	$\{0.5, 1.0, 2.0\}$ or $[2^{-2}, 2^2]$	Kernel’s exponents (fixed or 40 log-spaced points)

Table 3.3: Parameters used in the Experiment 3.4.3.

In the next figure, we show the visualizations obtained from Experiment 3.4.3.

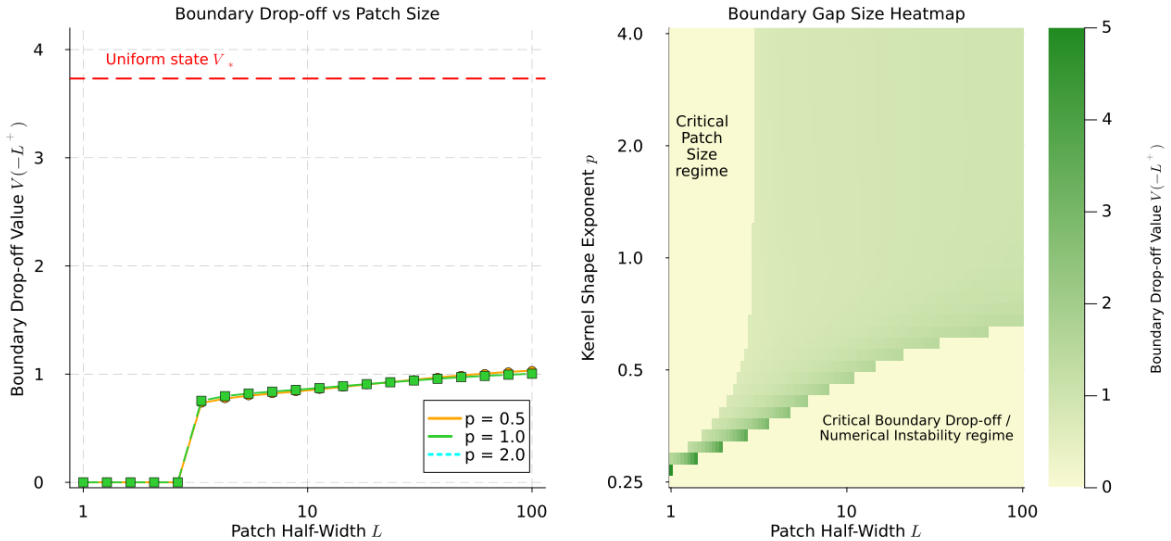


Figure 3.3: Left panel: The boundary drop-off value $V(-L^+)$ as a function of the patch half-width L for different kernel shape exponents p . Right panel: A heatmap illustrating the magnitude of the boundary drop-off across the (L, p) parameter space, revealing distinct dynamical regimes such as the critical patch size and areas of theoretical breakdown.

Conclusion 3.4.3. *As shown in the left panel of Figure 3.3, the magnitude of the boundary drop-off $V(-L^+)$ increases with the patch half-width L until it reaches a constant, non-zero level, which varies depending on the kernel exponent p .*

Conclusion 3.4.4. *The right panel of Figure 3.3 provides a broader view of the (L, p) parameter space, revealing two extreme behavioral regions. In the upper-left region (small L), we observe the critical patch size regime, where the habitat is too small to sustain a non-trivial state. Conversely, in the lower-right region (large L and small p , corresponding to heavy-tailed dispersal), the boundary drop-off becomes exceptionally large and the numerical scheme ultimately diverges. We hypothesize that in this heavy-tailed regime, the solution constructed in Theorem 3.2.1 ceases to exist. Specifically, this breakdown occurs because the boundary drop-off magnitude grows until it matches the uniform state, effectively making the radius defined in Equation 3.2.10 equal to the uniform state itself.*

3.4.3 Bifurcation analysis and pattern formation

Bifurcation analysis We now investigate the critical aggregation levels required for vegetation persistence and explore the emergence of spatially inhomogeneous solutions.

Experiment 3.4.4 (Bifurcation diagram via continuation method).

Numerical Method and Setup: We compute bifurcation diagrams for the local and non-local Klausmeier models using *BifurcationKit.jl* (see [63]). The domain $[-L, L]$ is discretized into N nodes, applying second-order central finite differences for the Laplacian and trapezoidal quadrature for the non-local operator, under strictly enforced homogeneous Dirichlet boundaries. Steady-state solution branches are tracked with respect to the rainfall parameter A via Pseudo-Arc Length Continuation (PALC) and a Newton solver. To trigger pattern formation, the continuation initializes at $A = 3.0$ from the uniform state (V_*, W_*) , perturbed by a cosine profile and random noise.

Experimental Parameters:

The numerical solver settings, continuation parameters, and ecological constants used in this analysis are summarized in Table 3.4.

Symbol / Parameter	Value / Range	Description
A	$[0.05, 3.0]$	Rainfall level (continuation parameter)
B	0.45	Non-dimensionalized mortality rate
d_v	2.0	Vegetation dispersal rate
d_w	80.0	Water diffusion rate (fast diffusion regime)
L	25.0	Patch half-width
N	$6L$	Number of spatial grid points
ϵ_{tol}	10^{-10}	Tolerance for the Newton solver

Table 3.4: Parameters used in the Experiment 3.4.4.

Next, we present results of Experiment 3.4.4 conducted in the slow diffusion regime.

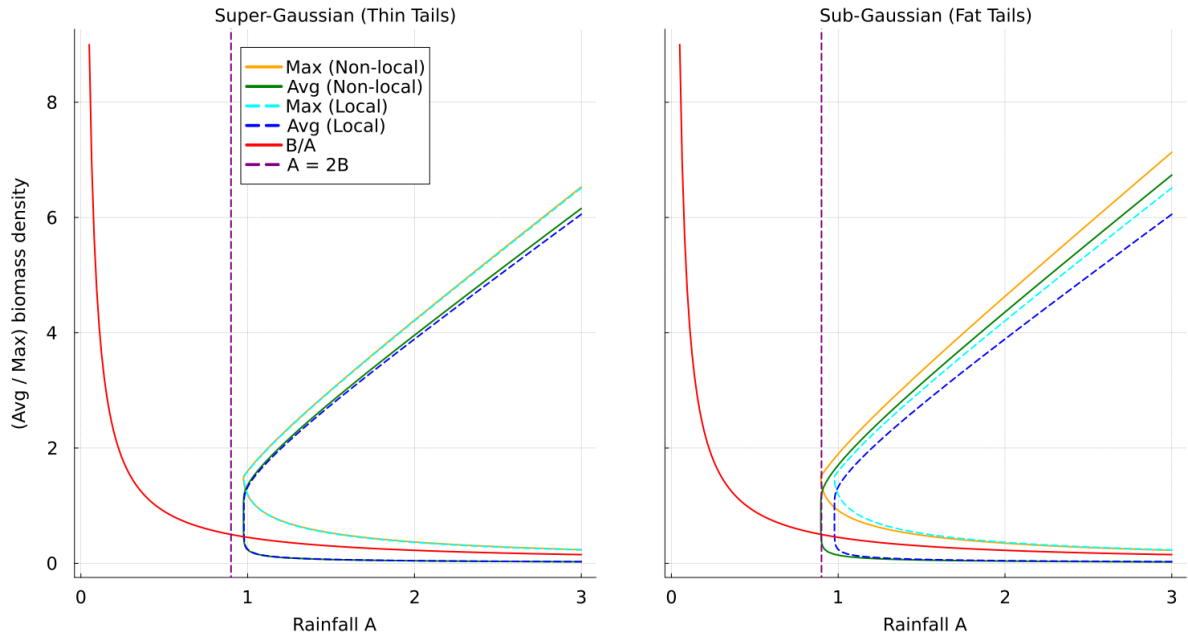


Figure 3.4: Bifurcation diagrams for the slow diffusion regime ($d_w = 0.1$). Solid orange and green lines represent the maximum and average biomass for non-local models; dashed lines correspond to the local model. The critical biomass threshold $\sim B/A$ is marked with a red line.

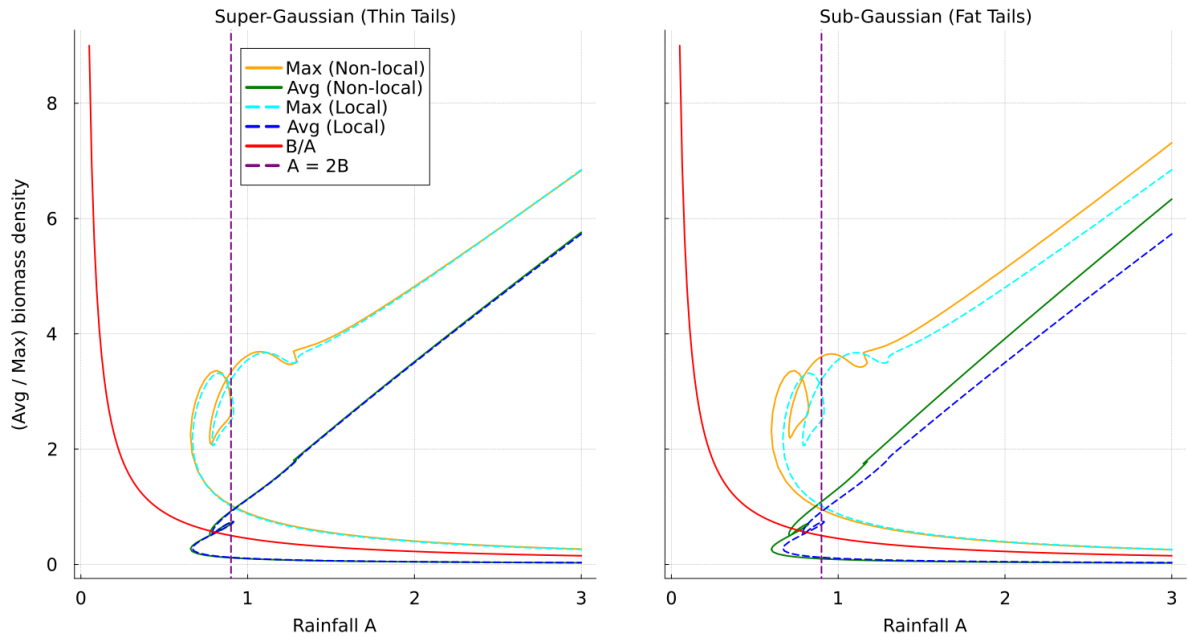


Figure 3.5: Bifurcation diagrams for the fast diffusion regime ($d_w = 80.0$). The non-local models exhibit complex branch structures extending into sub-critical rainfall regimes ($A < 2B$).

Conclusion 3.4.5. For slow diffusion ($d_w = 0.1$), the results in Figure 3.4 confirm the existence

of a stable steady state that ceases to exist near the theoretical kinetic threshold $A = 2B$. Furthermore, every stationary solution strictly respects the theoretical lower biomass bound $\sim B/A$.

Conclusion 3.4.6. *In the fast diffusion regime (Figure 3.5), the dispersal kernel's shape impacts the dynamics. The sub-Gaussian (fat-tailed, $p = 1$) kernel supports non-deserted, patterned solutions over a wider range of the rainfall parameter A , extending well below the kinetic threshold $A = 2B$.*

For sake of the exposition, we provide a gallery of spatial visualizations of the patterned states emerging along the bifurcating branches in Figure 3.5.

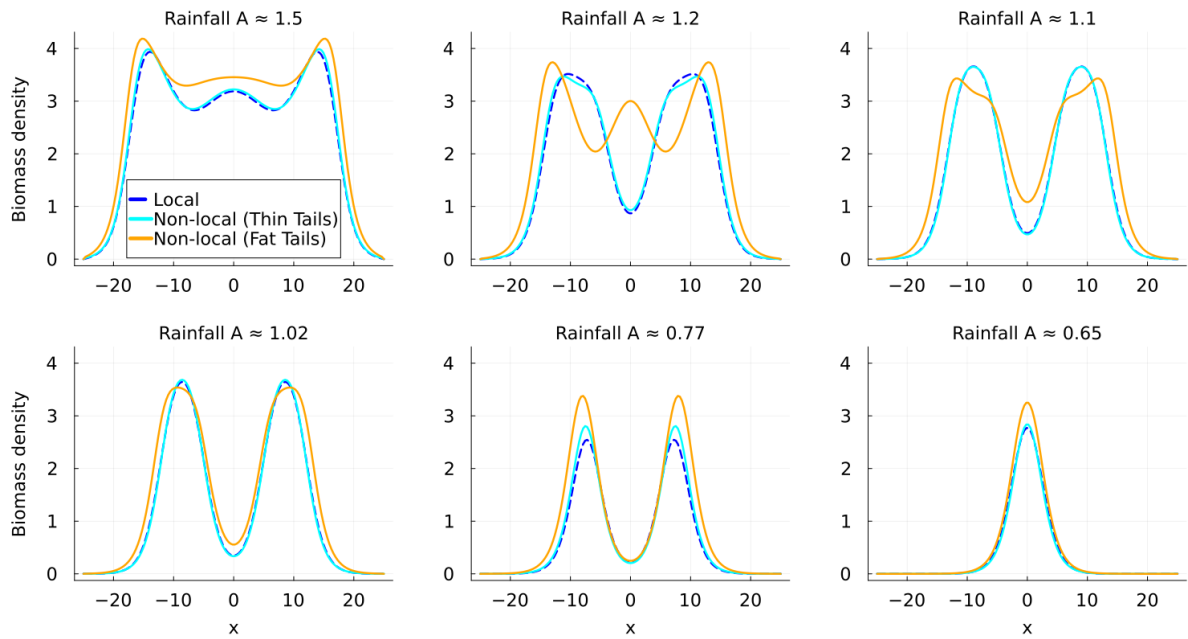


Figure 3.6: A gallery of one-dimensional stationary biomass densities for non-local (cyan and orange) and local (blue) models. Profiles are sampled from the upper stable branches of the bifurcation diagram in Figure 3.5.

Conclusion 3.4.7. *The extracted 1D vegetation profiles (Figure 3.6) reveal a morphological difference compared to the Experiment 3.4.1, where diffusion was slow ($d_w = 0.1$). Here, in the high diffusion regime ($d_w = 80.0$), the sharp boundary drop-off phenomenon entirely disappears. While the bifurcation diagram for the fat-tailed kernel exhibits similar loopy structures to the local model, the resulting morphology may differ as environmental stress varies (e.g., at $A \approx 1.2$).*

To complement these 1D findings, we further extended the numerical continuation to two-dimensional domains $\Omega = (-L, L) \times (-L, L)$ using an iterative Gauss–Seidel method. The spatial domain was discretized into a 50×50 grid, tracking the steady states sequentially from $A = 1.8$ down to 0.1. We analyzed the morphological differences induced by standard local diffusion versus the sub-Gaussian (fat-tailed) and super-Gaussian (thin-tailed) kernels.

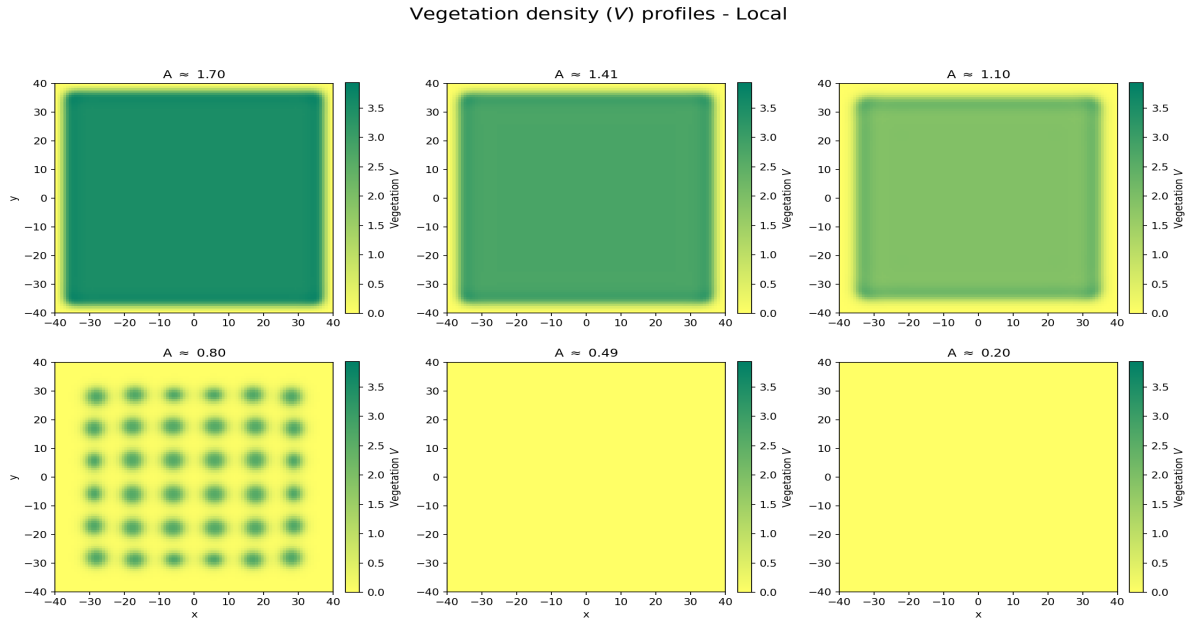


Figure 3.7: Two-dimensional stationary vegetation biomass density (V) profiles for the classical **local** reaction-diffusion model at decreasing precipitation levels A . Parameters: $d_v = 1.0$, $d_w = 10.0$, $B = 0.45$, $L = 40.0$.

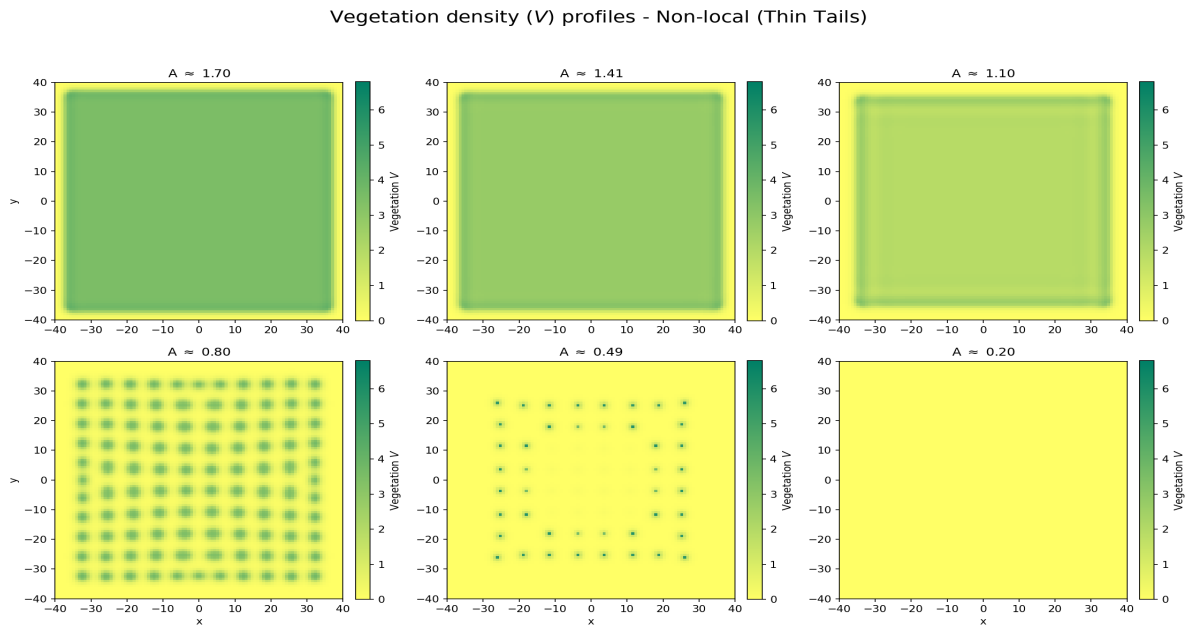


Figure 3.8: Two-dimensional stationary vegetation biomass density (V) profiles for the **non-local** model with a **super-Gaussian (thin-tailed)** dispersal kernel. Parameters: $d_v = 1.0$, $d_w = 10.0$, $B = 0.45$, $L = 40.0$.

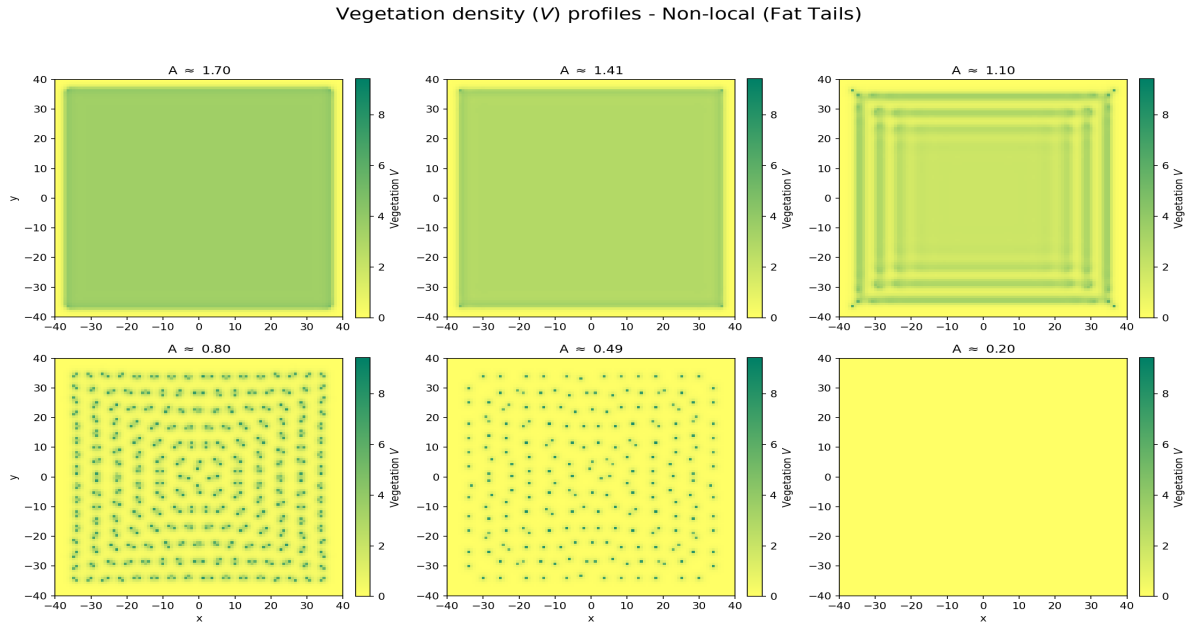


Figure 3.9: Two-dimensional stationary vegetation biomass density (V) profiles for the **non-local** model with a **sub-Gaussian (fat-tailed)** dispersal kernel. Parameters: $d_v = 1.0, d_w = 10.0, B = 0.45, L = 40.0$.

Conclusion 3.4.8. *The two-dimensional results mirror the resilience trends observed in 1D but expose rich spatial dynamics.*

- The **local model** (Figure 3.7) exhibits a classic pattern formation sequence: transitioning from a uniform interior to an array of isolated spots ($A \approx 0.80$), before completely collapsing to a bare desert already at $A \approx 0.49$.
- The **thin-tailed non-local model** (Figure 3.8) preserves the transition to spotted patterns but demonstrates enhanced resilience, maintaining a sparse array of vegetation spots at $A \approx 0.49$, long after the local model has collapsed.
- The **fat-tailed non-local model** (Figure 3.9) reveals a fundamentally different morphology. The impact of higher-order moments manifests as a highly pronounced, continuous boundary ridge of high vegetation density forming along the edges of the habitat. This prominent boundary aggregation provides a clear 2D visualization of the sharp drop-off phenomenon, confirming that long-range dispersal induces profound structural changes in fragmented ecosystems.

Chapter 4

Non-local evolutionary problems

In the previous chapter, we investigated the existence and spatial structure of stationary states, which represent the long-term stable equilibria of ecosystems. However, to fully understand the dynamics of population we must study evolutionary problems as well. From an ecological perspective, time-dependent problems are crucial for answering fundamental questions, such as whether a newly introduced population will establish itself and spread, or whether it will inevitably face extinction, and how exactly the system approaches its steady-state configuration.

Mathematically, as in the stationary case, we distinguish between two main classes of problems in this chapter. The first class consists of scalar reaction-dispersal models, which describe the spatio-temporal evolution of a single species. The governing Cauchy problem is formulated as the following non-local integro-differential equation

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) = d_v \mathcal{L}v(x, t) + f(v(x, t)), & x \in \Omega, \ t > 0, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (4.0.1)$$

where $v := v(x, t)$ denotes the population density, $d_v > 0$ is the dispersal rate, $\mathcal{L}$ is the non-local dispersal operator (incorporating either closed Neumann-type or hostile Dirichlet-type boundary conditions), f represents the local reaction kinetics, and $v_0(x)$ is the given initial population distribution.

The second class extends this evolutionary framework to multi-component systems, capturing complex biological interactions where distinct components exhibit fundamentally different transport mechanisms. These hybrid reaction-diffusion-dispersal systems are governed by the coupled evolution equations

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) = d_v \mathcal{L}v(x, t) + f(v(x, t), w(x, t)), & x \in \Omega, \ t > 0, \\ \frac{\partial w}{\partial t}(x, t) = d_w \Delta w(x, t) + g(v(x, t), w(x, t)), & x \in \Omega, \ t > 0, \end{cases} \quad (4.0.2)$$

subject to appropriate initial data $v(x, 0) = v_0(x)$ and $w(x, 0) = w_0(x)$, as well as standard Dirichlet boundary conditions for the locally diffusing resource $w(x, t)$ on $\partial\Omega$.

In this chapter, we rigorously investigate the well-posedness, global existence, and fundamental properties of solutions to these evolutionary systems, including the establishment of invariant regions. We place a particular emphasis on the long-time behavior of the solutions, establishing rigorous criteria for population extinction and persistence, and demonstrating the global asymptotic stability of the stationary states derived in Chapter 3.

4.1 Literature review

Foundational work in understanding the global dynamics of non-local evolution equations was established by Bates and Zhao [8], who analyzed a non-local dispersal equation arising in population dynamics. By employing super- and sub-solution methods along with monotone iteration techniques, they rigorously proved the existence, uniqueness, and global asymptotic stability of positive steady states. This framework was further extended by Zhang, Li, and Sun [68], who investigated the asymptotic behavior of a general non-local dispersal equation subject to hostile boundary conditions (Dirichlet type), explicitly demonstrating how the convergence to a unique stationary solution or the decay to zero depends strictly on the principal eigenvalue of the corresponding spatial operator.

The lack of compactness in non-local semi-flows introduces significant difficulties, particularly for multi-stable systems. Hutson and Grinfeld [27] tackled this issue in a non-local equation with a bistable nonlinearity. They developed a regular perturbation technique to prove pointwise convergence of positive semi-orbits to an equilibrium for small dispersal rates, a result that highlights the contrast with local diffusion models where such convergence is often guaranteed unconditionally. Furthermore, Sun, Li, and Wang [57] explored the long-time behavior of a non-local logistic equation with spatial degeneracy. They revealed that in regions lacking self-limitation, the time-dependent solutions can blow up as $t \rightarrow \infty$ if the intrinsic growth rate exceeds the principal eigenvalue, contrasting sharply with standard reaction-diffusion dynamics.

A substantial body of literature has been dedicated to connecting spectral theory with the long-time persistence and extinction of species. Berestycki, Coville, and Vo [9] established sharp persistence criteria for populations described by non-local Fisher-KPP equations, proving that the long-time convergence of the evolving population to a bounded positive stationary state is perfectly characterized by the sign of a generalized principal eigenvalue. This connection between spectral bounds and evolution of solutions was also thoroughly analyzed by Sun [55] in heterogeneous environments. The study comprehensively characterized the limits of the time-dependent solutions, demonstrating how extreme dispersal rates (both large and small) govern the ultimate survival or extinction of the species over time. Su, Li, and Yang [54] later utilized the global asymptotic stability of positive steady states in non-local evolution equations to analyze the total biomass of a species, establishing that non-local dispersal can allow dynamically evolving populations to achieve a total biomass strictly greater than the environment's carrying capacity.

Recent advancements have successfully extended these evolutionary frameworks to complex multi-stage biological processes and coupled systems. Wu and Zhao [67] investigated the spatial invasion of a species with distinct reproductive and non-local dispersal stages using an impulsive integro-differential model, proving threshold dynamics for the initial-value problem and establishing the existence of invasion speeds and traveling waves. In the context of epidemiology, the long-time dynamics of infectious diseases spreading via non-local pathways have been extensively modeled. Djilali, Sarmad, and Tridane [15] explored a hybrid local-nonlocal SIR epidemic model, rigorously proving the uniform persistence of the disease and global asymptotic stability of the evolving equilibria based on the basic reproduction number. Similarly, Zhou, Xu, and Zhang [69] addressed a non-local SIS epidemic model incorporating the flow of air pollutants, utilizing Lyapunov functionals and semigroup theory to determine the global dynamics and the eventual spatial stabilization of the disease over time.

4.2 Reaction-dispersal models

We study properties of solutions $v := v(x, t)$ of the following non-linear, non-local model describing the evolution of a density of single species.

$$\begin{cases} v_t(x, t) = \mathcal{L}v(x, t) + f(v(x, t)) & (x, t) \in \Omega \times (0, \infty), \\ v(x, 0) = v_0(x) & x \in \Omega, \end{cases} \quad (4.2.1)$$

$$(4.2.2)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain. We assume that initial datum $v_0 \in X$, where the choice of the space depends on the boundary effect. In particular for

- **Hostile environment:** we take either $X = L^2_{0,\Omega}(\mathbb{R}^n)$ or $X = C_{0,\Omega}(\mathbb{R}^n)$.
- **Closed environment:** we take either $X = L^2(\Omega)$ or $X = C(\bar{\Omega})$.

Next, we analyze local and global in time behaviour of the solutions to the system (4.2.1)-(4.2.2).

4.2.1 Well-posedness and properties

We begin with results on solvability and properties of solutions to the problem (4.2.1)-(4.2.2).

Theorem 4.2.1 (Existence of solutions in closed environment). *Consider the dispersal operator in the closed environment, e.g. $\mathcal{L} = \mathcal{L}_N$. Let the integral kernel J satisfy Assumptions 1.2.1. Assume that $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\Omega)$. Then, the following statements hold true*

- For every $v_0 \in L^2(\Omega)$ there exists a unique solution $v \in C([0, \infty), L^2(\Omega))$ to the Cauchy problem (4.2.1)-(4.2.2).
- Solutions v, w to the Cauchy problem (4.2.1)-(4.2.2) depend continuously on the initial data v_0, w_0 respectively. In particular for all $T > 0$ there is a constant $C(T) > 0$ such that

$$\|v - w\|_2 \leq C(T)\|v_0 - w_0\|_2. \quad (4.2.3)$$

- If $v_0 \in C(\bar{\Omega})$ then both $v \in C(\bar{\Omega} \times [0, T])$ and $v_t \in C(\bar{\Omega} \times [0, T])$ for each $T > 0$

Proof. Observe that (i) is a consequence of a more general result by Sun [43], who constructed a unique solution $v \in C([0, \infty), L^1(\Omega))$. This solution is obtained as a fixed point of the following equation

$$v(x, t) = v_0(x) + \int_0^t \mathcal{L}_\varepsilon v(x, s) ds + \int_0^t f(v(x, s)) ds. \quad (4.2.4)$$

Obviously $L^2(\Omega) \subset L^1(\Omega)$ for a bounded domain Ω and hence (i) follows.

Denote by $u = v - w$ the difference of solutions v, w to initial value problem (4.2.1)-(4.2.2) and proceed for $t \in [0, T]$ as follows

$$\begin{aligned} \|u(t)\|_2 &\leq \|u_0\|_2 + \int_0^t \|\mathcal{L}w(s)\|_2 + \|f(v(s)) - f(w(s))\|_2 ds \\ &= \|u_0\|_2 + \int_0^t \left(\int_\Omega \int_\Omega J^2(x - y) (u(y, s) - u(x, s))^2 dy dx \right)^{1/2} ds \\ &\quad + \int_0^t \left(\int_\Omega (f(v(x, s)) - f(w(x, s)))^2 dx \right)^{1/2} ds \\ &\leq \|u_0\|_2 + \left(\sqrt{2}\|J\|_\infty |\Omega|^{1/2} + \|f'\|_\infty \right) \int_0^t \|u(s)\|_2 ds. \end{aligned}$$

The Grönwall inequality yields the following upper bound

$$\|u(t)\|_2 \leq e^{T(\sqrt{2}\|J\|_\infty|\Omega|^{1/2}+\|f'\|_\infty)}\|u_0\|_2.$$

This proves (ii) with the explicit constant $C(T) = e^{T(\sqrt{2}\|J_\varepsilon\|_\infty|\Omega|^{1/2}+\|f'\|_\infty)}$.

Next, let $v_0 \in C(\overline{\Omega})$. Denote the Banach space $C(\overline{\Omega} \times [0, T_0])$ equipped with the norm

$$|||u||| = \max_{t \in [0, T_0]} \max_{x \in \overline{\Omega}} |u(x, t)|.$$

Now apply the Banach Fixed Point Theorem and the usual procedure. We deduce that there exists unique solution $v \in C(\overline{\Omega} \times [0, T])$. But now observe that second and third terms in formula (4.2.4) become continuously differentiable in time, and so in fact $v_t \in C(\overline{\Omega} \times [0, \infty))$ which is (iii). $\square$

Remark 4.2.1 (Existence of solutions in hostile environnement). *Consider the dispersal operator in the hostile environment, e.g. $\mathcal{L} = \mathcal{L}_D$. Let the integral kernel J satisfy Assumptions 1.2.1. Assume that $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\Omega)$. Then, the following statements hold true*

- (i) *For every $v_0 \in L^2_{0,\Omega}(\mathbb{R}^n)$ there exists a unique solution $v \in C([0, \infty), L^2_{0,\Omega}(\mathbb{R}^n))$ to the Cauchy problem (4.2.1)-(4.2.2).*
- (ii) *Solutions v, w to the Cauchy problem (4.2.1)-(4.2.2) depend continuously on the initial data v_0, w_0 . In particular for all $T > 0$ there is a constant $C(T) > 0$ such that*

$$\|v - w\|_2 \leq C(T)\|v_0 - w_0\|_2. \quad (4.2.5)$$

- (iii) *If $v_0 \in C_{0,\overline{\Omega}}(\mathbb{R}^n)$ then both $v \in C_{0,\overline{\Omega}}(\mathbb{R}^n \times [0, T])$ and $v_t \in C_{0,\overline{\Omega}}(\mathbb{R}^n \times [0, T])$ for each $T > 0$.*

Proof. This proof is completely analogous to the proof of Theorem 4.2.1, hence we skip it. $\square$

Now, we aim to prove existence of the invariant region. We do that by differentiating maximum and minimum of the solution to (4.2.1)-(4.2.2). The first step is to solve the problem when $v_0 \in C(\overline{\Omega})$ or $v_0 \in C_{0,\overline{\Omega}}(\mathbb{R}^n)$. Indeed we have

Lemma 4.2.1. *For $T > 0$ and bounded domain $\Omega \subset \mathbb{R}^n$ let us denote the solution u of the system (4.2.1)-(4.2.2). For the Neumann type case $\mathcal{L} = \mathcal{L}_N$ we assume*

$$v \in C(\overline{\Omega} \times [0, T]), \quad v_t \in C(\overline{\Omega} \times [0, T])$$

while in the Dirichlet case $\mathcal{L} = \mathcal{L}_D$, we consider

$$v \in C_{0,\overline{\Omega}}(\mathbb{R}^n, [0, T]), \quad v_t \in C_{0,\overline{\Omega}}(\mathbb{R}^n, [0, T])$$

Denote the functions $\xi : [0, T] \rightarrow \overline{\Omega}$, $\zeta : [0, T] \rightarrow \overline{\Omega}$ and $\mu : [0, T] \rightarrow \mathbb{R}$, $\nu : [0, T] \rightarrow \mathbb{R}$ given by the following formulas

$$\mu(t) := u(\xi(t), t) = \max_{x \in \Omega} u(x, t), \quad \nu(t) := u(\zeta(t), t) = \min_{x \in \Omega} u(x, t) \quad (4.2.6)$$

Then both μ and ν are differentiable almost everywhere and their derivatives are given by

$$\mu'(t) = u_t(\xi(t), t), \quad \nu'(t) = u_t(\zeta(t), t). \quad (4.2.7)$$

Proof. Let $t > 0, s > 0$. Assume that $\mu(t) > \mu(s)$. formula (4.2.6) yields

$$0 < \mu(t) - \mu(s) \leq v(\xi(t), t) - v(\xi(t), s) \leq \max_{(x,t) \in \bar{\Omega} \times [0,T]} |v_t(x, t)| |t - s|$$

which yields Lipschitz continuity of μ . By the Radamacher theorem [17] we conclude that μ is differentiable almost everywhere. Now, we estimate the difference quotient as follows

$$\begin{aligned} \frac{\mu(t+h) - \mu(t)}{h} &= \frac{v(\xi(t+h), t+h) - v(\xi(t), t)}{h} \\ &\leq \frac{v(\xi(t+h), t+h) - v(\xi(t+h), t)}{h} \\ &= v_t(\xi(t+h), c(t, h)), \end{aligned}$$

where by the Mean Value Theorem $c(t, h) \in [t-h, t+h]$. Passing to the limit as $h \rightarrow 0$ we get

$$\mu'(t) \leq v_t(\xi(t), t). \quad (4.2.8)$$

The reverse estimate can be written down as follows

$$\begin{aligned} \frac{\mu(t+h) - \mu(t)}{h} &\geq \frac{u(\xi(t), t+h) - u(\xi(t), t)}{h} \\ &= v_t(\xi(t), c(t, h)). \end{aligned}$$

Similarly as before, we pass to the limit as $h \rightarrow 0$ and obtain

$$\mu'(t) \geq v_t(\xi(t), t). \quad (4.2.9)$$

Combining inequalities (4.2.8) and (4.2.9) we get formula (4.2.7) for μ' . The proof for ν' is analogous. $\square$

Now we show that μ and ν are bounded by the minimum and maximum of initial datum v_0 .

Lemma 4.2.2. *Let the assumptions of the Lemma 4.2.2 hold true. Denote the functions ξ, ζ, μ, ν as in the Lemma 4.2.2. Then for all $t \geq 0$ the following estimates hold true*

$$\nu(0) \leq \nu(t) \leq \mu(t) \leq \mu(0).$$

Proof. Note, that both $\mathcal{L}_N$ and $\mathcal{L}_D$, satisfy the following inequality in the maximum point ξ

$$\mathcal{L}v(\xi(t), t) = \int_{\text{domain}} J(x-y) (v(y, t) - v(\xi(t), t)) \, dy \leq 0.$$

Analogously, in the minimum point ζ we estimate from below

$$\mathcal{L}v(\zeta(t), t) = \int_{\text{domain}} J(x-y) (v(y, t) - v(\zeta(t), t)) \, dy \geq 0.$$

By Lemma 4.2.1 both μ and ν are differentiable almost everywhere in time $t > 0$. Formula (4.2.7) yields

$$\mu'(t) = v_t(\xi(t), t) = \mathcal{L}v(\xi(t), t) + f(\mu(t)) \leq f(\mu(t)).$$

Similarly we have

$$\nu'(t) = v_t(\zeta(t), t) = \mathcal{L}v(\zeta(t), t) + f(\nu(t)) \geq f(\nu(t)).$$

The proof is completed by the usual comparison principle for the differential inequalities [40]. $\square$

Now, we are ready to prove the following Theorem

Theorem 4.2.2 (Invariant region). *Let the assumptions of the Theorem 4.2.1 or Remark 4.2.1 hold true. Additionally assume there exist number $v_1 < v_2$ such that $f(v_1) = f(v_2) = 0$. Denote by $v := v(x, t)$ the solution of initial value problem (4.2.1)-(4.2.2) with an initial datum $u_0 \in L^2(\Omega)$ in the Neumann case or $u_0 \in L_{0,\Omega}(\mathbb{R}^n)$ in the Dirichlet case. Define the set $\Gamma = [v_1, v_2]$. If $u_0(x) \in \Gamma$ a.e. then $v(x, t) \in \Gamma$ a.e. for all $t > 0$.*

Proof. observe that every $v_0 \in L^2(\Omega)$ can be approximated by the sequence $v_0^n \in C(\bar{\Omega})$. Now let the functions $v \in C([0, T], L^2(\Omega))$ and $v^n \in C(\bar{\Omega}, [0, T])$ be the solutions of the Cauchy problem (4.2.1)-(4.2.2) equipped respectively with the initial data v_0, v_0^n . Assume that $v_0(x) \in \Gamma$ a.e., then $v_0^n \in \Gamma$ a.e.. From Theorem 4.2.1 (or Remark 4.2.1 in the Dirichlet case), we have the following estimate

$$\|v - v^n\|_2 \leq C(T)\|v_0 - v_0^n\|_2.$$

Passing to the limit as $n \rightarrow \infty$ we get that $v^n \rightarrow v$ in the sense of L^2 norm. But now one can take the subsequence v^{n_k} such that $v^{n_k} \rightarrow v$ a.e. as $k \rightarrow \infty$. By the Lemma 4.2.2 $v^{n_k}(x, t) \in \Gamma$ a.e. for all $t \in [0, T]$. Hence $v(x, t) \in \Gamma$ a.e. for all $t \in [0, T]$. Thus the proof of Theorem 4.2.2 is completed. $\square$

4.2.2 Long time behavior of solutions

Now, we study the behaviour of solutions to the problem (4.2.1)-(4.2.2) in the long time range.

Hostile environment case: We first formalize the extinction regime, which mathematically underpins the concept of the Critical Patch Size (see Definition 1.2).

Theorem 4.2.3 (Extinction of vegetation). *Let the assumptions of Theorem 4.2.2 hold true, and let $v \in C([0, \infty), L_{0,\Omega}^2(\mathbb{R}^n))$ be a non-negative solution to the Cauchy problem (4.2.1)-(4.2.2) with non-negative initial datum $v_0 \in L_{0,\Omega}^2(\mathbb{R}^n)$. Assume $f(0) = 0$ and denote by $\beta_1 < 0$ the principle eigenvalue of $\mathcal{L}_D$. When the following inequality holds true*

$$\eta := \beta_1 + \sup_{v \in \Gamma \setminus \{0\}} \frac{f(v)}{v} < 0, \quad (4.2.10)$$

then the population goes extinct exponentially fast, and the solution satisfies:

$$\|v(t)\|_2 \leq \|v_0\|_2 e^{\eta t}. \quad (4.2.11)$$

Proof. Consider the energy of the solution v given by $E(t) = \frac{1}{2} \int_{\Omega} |v(x, t)|^2 dx$. Differentiating with respect to time yields

$$E'(t) = \int_{\Omega} v_t v dx = \int_{\Omega} (\mathcal{L}_D v + f(v)) v dx = \langle \mathcal{L}_D v, v \rangle + \int_{\Omega} f(v) v dx. \quad (4.2.12)$$

We bound the first term on the right hand side in equation (4.2.12) by the principle eigenvalue $\beta_1 < 0$ of dispersal operator $\mathcal{L}_D$

$$\langle \mathcal{L}_D v, v \rangle \leq \sup_{\|v\|_2=1} \langle \mathcal{L}_D v, v \rangle \|v\|_2^2 = \beta_1 \|v\|_2^2. \quad (4.2.13)$$

Second integral in (4.2.12) is estimated as follows

$$\int_{\Omega} f(v)v \, dx = \int_{\{x \in \Omega: u \neq 0\}} f(v)v \leq \sup_{v \in \Gamma \setminus \{0\}} \frac{f(v)}{v} \|v\|_2^2. \quad (4.2.14)$$

Combining estimates (4.2.13) and (4.2.14), we obtain the following differential inequality

$$E'(t) \leq \beta_1 \|v\|_2^2 + \sup_{v \in \Gamma \setminus \{0\}} \frac{f(v)}{v} \|v\|_2^2 = 2\eta E(t). \quad (4.2.15)$$

Solving inequality (4.2.15) and using assumption (4.2.10) we get upper bound for all $t \geq 0$

$$E(t) \leq E(0)e^{2\eta t}. \quad (4.2.16)$$

Note that inequality (4.2.16) directly implies

$$\|v(t)\|_2 \leq \|v_0\|_2 e^{\eta t}. \quad (4.2.17)$$

Thus, $\|v(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$, proving the exponential collapse to the bare-soil state. $\square$

Remark 4.2.2 (Critical Patch Size). *Theorem 4.2.3 provides a formulation of the Critical Patch Size. The extinction condition (4.2.10) represents a balance between the maximal per capita growth rate $\sup_{u \in \Gamma \setminus \{0\}} \frac{f(u)}{u}$ and the dispersal energy quantified by the principal eigenvalue $\beta_1 < 0$. By the spectral properties of the Dirichlet operator $\mathcal{L}_D$, the eigenvalue β_1 depends monotonically on the domain and decreases as $|\Omega|$ shrinks. Therefore, provided that the local reproduction rate is not excessively large, there exists a critical domain size threshold. For any habitat smaller than this threshold, boundary losses dominate, forcing $\eta < 0$ and leading to inevitable extinction.*

Next, we formulate the criterion for persistence of non-zero solutions.

Theorem 4.2.4 (Persistence of vegetation). *Assume there exists a strictly positive stationary solution $V \in L^\infty(\Omega)$ satisfying $\mathcal{L}_D V + f(V) = 0$. Let $v \in C([0, \infty), L_{0,\Omega}^2(\mathbb{R}^n))$ be a solution to the corresponding Cauchy problem (4.2.1)-(4.2.2). If V satisfies the following condition*

$$\gamma := \beta_1 + \sup_{x \in \Omega} f'(V(x)) < 0, \quad (4.2.18)$$

then V is locally exponentially stable in $L^2(\Omega)$. Specifically, for initial datum v_0 sufficiently close to V , the perturbation $u(x, t) = v(x, t) - V(x)$ satisfies

$$\|v(t) - V\|_2 \leq \|v_0 - V\|_2 e^{\gamma t}. \quad (4.2.19)$$

Proof. Let $u(x, t) = v(x, t) - V(x)$ denote the perturbation from the stationary state. The evolution of u is governed by the equation

$$u_t = \mathcal{L}_D(V + u) + f(V + u) - (\mathcal{L}_D V + f(V)) = \mathcal{L}_D u + f(V + u) - f(V). \quad (4.2.20)$$

We define the perturbation energy as $E_u(t) = \frac{1}{2} \int_{\Omega} |u(x, t)|^2 \, dx$. Differentiating with respect to time and using equation (4.2.20) yields

$$E'_u(t) = \int_{\Omega} u_t u \, dx = \langle \mathcal{L}_D u, u \rangle + \int_{\Omega} (f(V + u) - f(V)) u \, dx. \quad (4.2.21)$$

Utilizing the principal eigenvalue $\beta_1 < 0$ of $\mathcal{L}_D$, the dispersal term in (4.2.21) is bounded by

$$\langle \mathcal{L}_D u, u \rangle \leq \beta_1 \|u\|_2^2. \quad (4.2.22)$$

By the Mean Value Theorem, for each $x \in \Omega$ there exists an intermediate value $\xi(x, t)$ between $V(x)$ and $V(x) + u(x, t)$ such that $f(V + u) - f(V) = f'(\xi)u$. For perturbations remaining within the invariant region, we can bound the reaction term in (4.2.21) by

$$\int_{\Omega} f'(\xi) u^2 dx \leq \sup_{x \in \Omega} f'(V(x)) \int_{\Omega} u^2 dx = \sup_{x \in \Omega} f'(V(x)) \|u\|_2^2. \quad (4.2.23)$$

Plugging estimates (4.2.22) and (4.2.23) into equation (4.2.21), we obtain the following differential inequality

$$E'_u(t) \leq \beta_1 \|u\|_2^2 + \sup_{x \in \Omega} f'(V(x)) \|u\|_2^2 = \left(\beta_1 + \sup_{x \in \Omega} f'(V(x)) \right) \|u\|_2^2 = 2\gamma E_u(t). \quad (4.2.24)$$

Solving inequality (4.2.24) gives the energy bound

$$E_u(t) \leq E_u(0) e^{2\gamma t}. \quad (4.2.25)$$

Inequality (4.2.25) immediately yields the estimate for the L^2 norm

$$\|v(t) - V\|_2 \leq \|v_0 - V\|_2 e^{\gamma t}. \quad (4.2.26)$$

As we assumed $\gamma < 0$ in condition (4.2.18), the perturbation decays exponentially to zero as $t \rightarrow \infty$, confirming the stability of the stationary vegetation pattern. $\square$

Closed environment case: Here, we rigorously prove the sufficient condition for convergence of solutions to the average biomass, provided that it can not escape the domain Ω .

Theorem 4.2.5 (Convergence to the mean mass). *Let the assumptions of Theorem 4.2.2 hold true. Let $v \in C([0, \infty), L^2(\Omega))$ be a solution of the Cauchy problem (4.2.1) - (4.2.2) with an initial datum $v_0 \in L^2(\Omega)$. Denote the mean mass*

$$v_{\Omega}(t) = \frac{1}{|\Omega|} \int_{\Omega} v(x, t) dx.$$

There are positive numbers $a_1 > 0, a_2 > 0$ and a critical value given by the formula

$$\sigma = \beta_1 + \max_{v \in \Gamma} |f'(v)|, \quad (4.2.27)$$

where $\beta_1 < 0$ is the largest non-trivial eigenvalue of $\mathcal{L}_N$. If $\sigma < 0$, the following statements hold true for all $t > 0$:

(i) *The difference between the solution v and its mean mass v_{Ω} is estimated by*

$$\|v(t) - v_{\Omega}(t)\|_2 \leq a_1 e^{\sigma t}. \quad (4.2.28)$$

(ii) *The mean mass v_{Ω} satisfies the following ordinary differential equation*

$$\begin{cases} v'_{\Omega}(t) = f(v_{\Omega}(t)) + O(e^{\sigma t}), \\ v_{\Omega}(0) = \frac{1}{|\Omega|} \int_{\Omega} v_0(x) dx. \end{cases} \quad (4.2.29)$$

(iii) The Dirichlet form associated with $\mathcal{L}_N$ is estimated by

$$|\langle \mathcal{L}_N v(t), v(t) \rangle| \leq a_2 e^{2\sigma t}. \quad (4.2.30)$$

Proof. We introduce the notion of energy E_u for the distance function u by the formula

$$E_u(t) = \frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx, \quad u(x, t) = v(x, t) - v_{\Omega}(t),$$

where v is a solution to the Cauchy problem (4.2.1)-(4.2.2) and $v_{\Omega}(t)$ is the spatial mean mass. Since $\int_{\Omega} u(x, t) dx = 0$, the variable u belongs to the orthogonal complement of constants.

We differentiate the energy with respect to time to obtain

$$\begin{aligned} E'_u(t) &= \int_{\Omega} u_t u dx \\ &= \int_{\Omega} (v_t - v'_{\Omega}) u dx \\ &= \int_{\Omega} \left(\mathcal{L}_N v + f(v) - \frac{1}{|\Omega|} \int_{\Omega} f(v) dy \right) u dx \\ &= \langle \mathcal{L}_N v, u \rangle + \int_{\Omega} \left(f(v) - f(v_{\Omega}) - \frac{1}{|\Omega|} \int_{\Omega} (f(v) - f(v_{\Omega})) dy \right) u dx. \end{aligned}$$

By the mass-conserving property, $\mathcal{L}_N$ annihilates constants, yielding $\langle \mathcal{L}_N v, u \rangle = \langle \mathcal{L}_N(u + v_{\Omega}), u \rangle = \langle \mathcal{L}_N u, u \rangle$. Furthermore, the integral of u over Ω is zero. Using the Mean Value Theorem to bound the non-linear terms, we get:

$$\begin{aligned} E'_u(t) &\leq \langle \mathcal{L}_N u, u \rangle + \max_{v \in \Gamma} |f'(v)| \int_{\Omega} u^2 dx - \frac{1}{|\Omega|} \int_{\Omega} (f(v) - f(v_{\Omega})) dy \int_{\Omega} u dx \\ &= \langle \mathcal{L}_N u, u \rangle + \max_{v \in \Gamma} |f'(v)| \|u\|_2^2 \\ &= \langle \mathcal{L}_N u, u \rangle + 2 \max_{v \in \Gamma} |f'(v)| E_u(t). \end{aligned}$$

Since $\int_{\Omega} u = 0$, we can bound the dispersal quadratic form by the first non-trivial eigenvalue β_1 of $\mathcal{L}_N$:

$$\langle \mathcal{L}_N u, u \rangle \leq \beta_1 \|u\|_2^2 = 2\beta_1 E_u(t).$$

Substituting this bound, we obtain the final differential inequality:

$$\begin{aligned} E'_u(t) &\leq 2 \left(\beta_1 + \max_{v \in \Gamma} |f'(v)| \right) E_u(t) \\ &= 2\sigma E_u(t). \end{aligned}$$

Solving this differential inequality gives:

$$E_u(t) \leq E_u(0) e^{2\sigma t},$$

which directly implies:

$$\|v(t) - v_{\Omega}(t)\|_2 \leq \|v_0 - v_{\Omega}(0)\|_2 e^{\sigma t} =: a_1 e^{\sigma t}.$$

Next, we compute the derivative of the mean mass v_Ω and apply the Mean Value Theorem and the Cauchy-Schwarz inequality:

$$\begin{aligned}
v'_\Omega(t) &= \frac{1}{|\Omega|} \int_\Omega f(v) \, dx \\
&= f(v_\Omega) + \frac{1}{|\Omega|} \int_\Omega (f(v) - f(v_\Omega)) \, dx \\
&\leq f(v_\Omega) + \frac{\max_{v \in \Gamma} |f'(v)|}{|\Omega|} \int_\Omega |v - v_\Omega| \, dx \\
&\leq f(v_\Omega) + \max_{v \in \Gamma} |f'(v)| \frac{1}{\sqrt{|\Omega|}} \|u\|_2.
\end{aligned}$$

Using claim (i), we see that the difference is of order $O(e^{\sigma t})$, meaning the mean mass satisfies the following differential equation:

$$\begin{cases} v'_\Omega(t) = f(v_\Omega(t)) + O(e^{\sigma t}), \\ v_\Omega(0) = \frac{1}{|\Omega|} \int_\Omega v_0(x) \, dx. \end{cases}$$

Lastly, it suffices to use claim (i) once more to estimate the Dirichlet form. Since $\mathcal{L}_N$ is symmetric and $\mathcal{L}_N v_\Omega = 0$:

$$\begin{aligned}
|\langle \mathcal{L}_N v, v \rangle| &= |\langle \mathcal{L}_N u, v \rangle| \\
&= \left| \langle \mathcal{L}_N u, u \rangle + v_\Omega(t) \int_\Omega \mathcal{L}_N u \, dx \right| \\
&= |\langle \mathcal{L}_N u, u \rangle| \\
&\leq \|\mathcal{L}_N u\|_2 \|u\|_2 \\
&\leq \|\mathcal{L}_N\| \|u\|_2^2 \\
&\leq \|\mathcal{L}_N\| \|v_0 - v_\Omega(0)\|_2^2 e^{2\sigma t} =: a_2 e^{2\sigma t}.
\end{aligned}$$

With that, we conclude that the proof is completed. $\square$

4.3 Reaction-diffusion-dispersal models

We study the properties of solutions (v, w) of the following coupled non-linear, non-local model describing the interaction between vegetation biomass $v := v(x, t)$ and water density $w := w(x, t)$.

$$\begin{cases} v_t(x, t) = d_v \mathcal{L} v(x, t) + f(v(x, t), w(x, t)) & (x, t) \in \Omega \times (0, \infty), & (4.3.1) \\ w_t(x, t) = d_w \Delta w(x, t) + g(v(x, t), w(x, t)) & (x, t) \in \Omega \times (0, \infty), & (4.3.2) \\ w(x, t) = 0 & (x, t) \in \partial\Omega \times (0, \infty), & (4.3.3) \\ v(x, 0) = v_0(x) & x \in \Omega, & (4.3.4) \\ w(x, 0) = w_0(x) & x \in \Omega, & (4.3.5) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded and open domain, and $d_v, d_w > 0$ denote the dispersal and diffusion rates, respectively. We assume that the initial data $(v_0, w_0) \in X_v \times C_0(\Omega)$. For the non-local dispersal of vegetation v , the choice of the space dictates the boundary effect

- **Hostile environment:** we take $X_v = C_{0,\Omega}(\mathbb{R}^n)$.
- **Closed environment:** we take $X_v = C(\overline{\Omega})$.

Having established this framework, we will analyze the local and global-in-time behavior of the solutions to the system (4.3.1)-(4.3.5).

4.3.1 Well-posedness and properties

We begin by formalizing the properties of the linear transport operators. Denote the linear dispersal operator $\mathcal{A} : X_v \rightarrow X_v$ and the diffusion operator $\mathcal{B} : D(\mathcal{B}) \subset C_0(\Omega) \rightarrow C_0(\Omega)$ by

$$\mathcal{A}v = d_v \mathcal{L}v, \quad (4.3.6)$$

$$\mathcal{B}w = d_w \Delta w, \quad D(\mathcal{B}) = \{w \in C_0^2(\Omega) : \Delta w \in C_0(\Omega)\}. \quad (4.3.7)$$

Lemma 4.3.1. *The bounded linear operator $\mathcal{A}$ generates a uniformly continuous semigroup $\{\mathcal{T}_{\mathcal{A}}(t)\}_{t \geq 0}$ on X_v , given by $\mathcal{T}_{\mathcal{A}}(t) = e^{t\mathcal{A}}$.*

Proof. observe that by definition of $\mathcal{L}$, the operator $\mathcal{A}$ is bounded and linear. For each $v \in X_v$ the solution of the problem $v_t = \mathcal{A}v$ exists for all $t \geq 0$, it is unique and given explicitly by $u(t) = \mathcal{T}(t)u(0)$, with $\mathcal{T}(t) = e^{t\mathcal{A}} = \sum_{n=1}^{\infty} \frac{\mathcal{A}^n t^n}{n!}$. Hence, $\mathcal{T}(t)$ is well defined and $\{\mathcal{T}(t)\}_{t \geq 0}$ is a uniformly continuous semigroup, see [12, Proposition 1.2.2] and [41, Theorem 1.2]. Thanks to [13, Proposition 3.2] and [58, Theorem 2.1], the spectrum of the operator $\mathcal{L}$ is discrete and negative. Thus, for each $t \geq 0$, $\mathcal{T}(t) : X_v \rightarrow X_v$ is a contraction [16, Proposition V.1.7]. $\square$

Lemma 4.3.2. *The linear operator $\mathcal{B}$ generates a strongly continuous semigroup of contractions $\{\mathcal{T}_{\mathcal{B}}(t)\}_{t \geq 0}$ on $C_0(\Omega)$.*

Proof. The proof can be found in [4, Theorem 6.1.8]. $\square$

Next, let us define the product spaces $X = X_v \times D(\mathcal{B})$, $Y = X_v \times C_0(\Omega)$ and the block-diagonal operator $\mathcal{C} : X \rightarrow Y$ given by formula

$$\mathcal{C} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix}. \quad (4.3.8)$$

Note that $\mathcal{C}$ generates a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ on X . Denoting $u = (v, w)^T$, $v_0 = (v_0, w_0)^T$, and the non-linear reaction term $F(u) = (f(v, w), g(v, w))^T$, the system can be rewritten in the abstract form:

$$u'(t) = \mathcal{C}u(t) + F(u(t)), \quad u(0) = v_0. \quad (4.3.9)$$

Theorem 4.3.1 (Local-in-time existence and uniqueness). *Let the integral kernel J and domain Ω satisfy the assumptions of the non-local dispersal operator. Assume that the non-linearities $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are locally Lipschitz continuous. Then, for every initial datum $v_0 \in X$, there exists $T > 0$ and a unique mild solution $u \in C([0, T], X)$ satisfying the integral equation:*

$$u(t) = \mathcal{T}(t)v_0 + \int_0^t \mathcal{T}(t-s)F(u(s)) ds. \quad (4.3.10)$$

Furthermore, if $v_0 \in D(\mathcal{C})$, the solution is classical, meaning $u \in C([0, T], D(\mathcal{C})) \cap C^1([0, T], X)$.

Proof. Since $\mathcal{C}$ is the generator of a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ on the Banach space X , the mild solution is represented by the Duhamel formula (4.3.10). Because the non-linear mapping F is locally Lipschitz continuous, we construct the solution using the Banach fixed-point theorem.

For a given $T > 0$, we define the mapping $\Phi : C([0, T], X) \rightarrow C([0, T], X)$ by:

$$\Phi(u)(t) = \mathcal{T}(t)v_0 + \int_0^t \mathcal{T}(t-s)F(u(s)) ds.$$

Using the uniform boundedness of the semigroup $\mathcal{T}(t)$ on $[0, T]$ and the local Lipschitz property of F , one can show that for a sufficiently small time horizon $T > 0$, the mapping Φ is a strict contraction on a closed ball around the initial data in $C([0, T], X)$. Thus, the Banach fixed-point theorem yields a unique fixed point $u = \Phi(u)$, establishing the existence of a unique mild solution on $[0, T]$.

If the initial condition is sufficiently regular, $v_0 \in D(\mathcal{C})$, it follows from classical semigroup theory (see e.g., [41, Theorem 6.1.5]) that the mild solution is continuously differentiable in time and satisfies the differential equation (4.3.9) everywhere, yielding $u \in C([0, T], D(\mathcal{C})) \cap C^1([0, T], X)$. $\square$

While so far we have focused on local-in-time solutions, we next establish the non-negativity of solutions and the existence of an invariant region for the general system.

Theorem 4.3.2 (Invariant region). *Denote the initial datum $u_0 = (v_0, w_0) \in X_v \times C_0(\Omega)$. Assume there exists a rectangle $\Gamma = [v_1, v_2] \times [w_1, w_2] \subset \mathbb{R}^2$ such that $0 \in \Gamma$ and for all $(v, w) \in \Gamma$ the reaction terms satisfy*

$$f(v_1, w) \geq 0, \quad f(v_2, w) \leq 0, \quad (4.3.11)$$

$$g(v, w_1) \geq 0, \quad g(v, w_2) \leq 0. \quad (4.3.12)$$

If $(v_0(x), w_0(x)) \in \Gamma$ a.e. in Ω , then the solution $(v(x, t), w(x, t))$ ensured by Theorem 4.3.1 satisfies $(v(x, t), w(x, t)) \in \Gamma$ a.e. for all $t > 0$ within its maximal interval of existence.

Proof. First, assume that the initial data is sufficiently regular, $u_0 \in C(\overline{\Omega}) \times C_0(\Omega)$. The bounds for the local diffusion component w follow directly from the classical parabolic maximum principle. Because the boundary condition yields $0 \in [w_1, w_2]$, and the reaction term satisfies (4.3.12), the interval $[w_1, w_2]$ is an invariant region for $w(x, t)$.

For the non-locally dispersing vegetation density v , we utilize the differentiability of the spatial extrema established in Lemma 4.2.1. Let $\mu(t) = \min_{x \in \Omega} v(x, t)$ and $\nu(t) = \max_{x \in \Omega} v(x, t)$, attained at points $\xi(t)$ and $\zeta(t)$, respectively.

At the minimum point $\xi(t)$, the non-local dispersal operator satisfies $\mathcal{L}v(\xi(t), t) \geq 0$. Using the differential equation for v , we obtain the following differential inequality for μ :

$$\mu'(t) = d_v \mathcal{L}v(\xi(t), t) + f(\mu(t), w(\xi(t), t)) \geq f(\mu(t), w(\xi(t), t)). \quad (4.3.13)$$

Since $f(v_1, w) \geq 0$ for all $w \in [w_1, w_2]$ from (4.3.11), the standard comparison principle for differential inequalities [40] ensures that if $\mu(0) \geq v_1$, then $\mu(t) \geq v_1$ for all $t \geq 0$.

Similarly, at the maximum point $\zeta(t)$, we have $\mathcal{L}v(\zeta(t), t) \leq 0$. This yields the differential inequality for ν :

$$\nu'(t) = d_v \mathcal{L}v(\zeta(t), t) + f(\nu(t), w(\zeta(t), t)) \leq f(\nu(t), w(\zeta(t), t)). \quad (4.3.14)$$

Because $f(v_2, w) \leq 0$ for all $w \in [w_1, w_2]$, the comparison principle yields that if $\nu(0) \leq v_2$, then $\nu(t) \leq v_2$ for all $t \geq 0$.

For general initial datum $u_0 \in X_v \times C_0(\Omega)$, we can approximate each component of u_0 by a sequence of continuous functions $u_0^n \in C(\overline{\Omega}) \times C_0(\Omega)$ taking values in Γ . By the continuous dependence on initial data guaranteed by Theorem 4.3.1, the corresponding solutions satisfy $u^n \rightarrow u$. Passing to the limit for a pointwise converging subsequence ensures that $(v(x, t), w(x, t)) \in \Gamma$ almost everywhere. $\square$

4.3.2 The Klausmeier-Gray-Scott model

We now move our attention to the well known example of the water-vegetation dynamics model in semi-arid environments. We choose the following non-dimensionalised reaction terms

$$f(v, w) = \underbrace{v^2 w}_{\text{growth}} - \underbrace{Bv}_{\text{mortality}}, \quad g(v, w) = \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{v^2 w}_{\text{uptake}}. \quad (4.3.15)$$

We begin with a very simple classification of uniform solutions (V_*, W_*) of the kinetic system, obtained with Proposition 3.3.1

Proposition 4.3.1. *Let the reaction terms f, g be given by equations (4.3.15). Consider the kinetic system corresponding to the problem (4.3.1)-(4.3.5)*

$$\begin{cases} \frac{d}{dt} v = f(v, w) & t > 0, \\ \frac{d}{dt} w = g(v, w) & t > 0. \end{cases} \quad (4.3.16)$$

The following statements are true about number of stable steady states for kinetic system (4.3.16)

- If $A > 2B$ then there are two asymptotically stable steady states.
- If $A \leq 2B$ then there is one asymptotically stable steady state.

Proof. Observe that the Jacobian of the dynamical system (4.3.16) is given by

$$J(v, w) = \begin{pmatrix} 2vw - B & v^2 \\ -2vw & -1 - v^2 \end{pmatrix} \quad (4.3.17)$$

Substituting (V^*, W^*) , given by the formula (3.3.18), into (4.3.17) we obtain that

$$J(V_1^*, W_1^*) = \begin{pmatrix} -B & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with} \quad \det J(V_1^*, W_1^*) > 0, \quad \text{Tr } J(V_1^*, W_1^*) < 0.$$

This means that (V_1^*, W_1^*) is stable for all $A > 0, B > 0$. Next, we see that

$$V_i^* W_i^* = B \quad i \in \{2, 3, 4\} \quad (4.3.18)$$

Now, from equation (4.3.18) it follows that (4.3.17) becomes

$$J(V_i^*, W_i^*) = \begin{pmatrix} B & (V_i^*)^2 \\ -2B & -1 - (V_i^*)^2 \end{pmatrix} \quad (4.3.19)$$

The determinant is given by

$$\det J(V_i^*, W_i^*) = -B - B(V_i^*)^2 + 2B(V_i^*)^2 = -B(1 - (V_i^*)^2) \quad (4.3.20)$$

Observe that it is positive if and only if $V_i^* > 1$. For sake of exposition we list if this is satisfied with respect to $i \in \{2, 3, 4\}$.

- $V_2^* \leq 1$ for all positive B , thus determinant is negative. This already yields instability of V_2^* as one of the eigenvalues of $J(V_2^*, W_2^*)$ must have positive sign.
- $V_3^* > 1$ as we assumed $A > 2B$. Hence, we have that the determinant is positive.
- $V_4^* = 1$ and so the determinant vanishes. This means that one of the eigenvalues is zero.

Now, the trace of (4.3.19) is computed as follows

$$\text{Tr } J(V_i^*, W_i^*) = B - 1 - (v_i^*)^2, \quad (4.3.21)$$

which is negative if and only if $(V_i^*)^2 > B - 1$. As we have shown earlier, only V_3^* could be asymptotically stable solution. Observe that $V_3^* > 1$ and consequently if $B < 2$ then the trace is negative, and thus both eigenvalues must be strictly negative real numbers. This means that V_3^* is asymptotically stable. When $B > 2$ we need a stronger lower bound. This bound can be achieved for $A > 2$, however according to literature [30] this is not an ecologically feasible value, thus we do not tackle it here. $\square$

Remark 4.3.1 (Invariant region for the Klausmeier-Gray-Scott model). *By applying Theorem 4.3.2 to the reaction terms (4.3.15), we can identify a specific invariant rectangle for the Klausmeier-Gray-Scott model. Let the initial data (v_0, w_0) be non-negative and define $R_1 = \max\{\|w_0\|_\infty, A\}$. If the initial vegetation density is bounded such that $\|v_0\|_\infty \leq B/R_1$, then the region*

$$\Gamma = [0, B/R_1] \times [0, R_1]$$

is invariant for the system. Indeed, one can easily verify that $f(0, w) = 0 \geq 0$ and $f(B/R_1, w) \leq \frac{B}{R_1}(\frac{B}{R_1}R_1 - B) = 0$ for all $w \in [0, R_1]$. Simultaneously, $g(v, 0) = A \geq 0$ and $g(v, R_1) \leq A - R_1 \leq 0$, which perfectly satisfies the boundary conditions (4.3.11)–(4.3.12) of the general theorem.

Critical aggregation threshold: Finally, we derive an explicit criterion for vegetation to go extinct based on its initial concentration. We summarize this in the Theorem below.

Theorem 4.3.3. *Let $M = \max\{\|v_0\|_\infty, A\}$. Denote an interval $\Gamma = [0, \frac{B}{M}]$. Suppose that $v_0(x) \in \Gamma$ for all $x \in \Omega$. Then the solution (v, w) of the problem (4.3.1)–(4.3.5) with reaction terms f, g given by (4.3.15), satisfies*

$$v(x, t) \xrightarrow{t \rightarrow \infty} 0, \quad w(x, t) \xrightarrow{t \rightarrow \infty} W_0(x), \quad (4.3.22)$$

where we denoted the non-zero stationary water profile $W_0 := W(0)$ from Lemma 3.3.1.

Proof. The first limit is a consequence of the Lemma (4.2.1). Solving the differential inequality for $\nu(t) = \max u(x, t)$ and since $(B - M\nu(0)) \geq 0$ we get that for all $x \in \Omega$

$$v(x, t) \leq \frac{B\nu(0)}{M\nu(0) + (B - M\nu(0))e^{Bt}} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (4.3.23)$$

Now, recall the stationary problem for water density W with vegetation density set to $V = 0$

$$d_w \Delta W - W + A = 0, \quad x \in \Omega, \quad W = 0, \quad x \in \partial\Omega. \quad (4.3.24)$$

Lemma 3.3.1 yields existence of a non-constant solution W_0 of the problem (4.3.24). Denote $\psi(x, t) = w(x, t) - W_0(x)$. It's easy to verify that ψ satisfies the following differential equation

$$\begin{cases} \psi_t = d_w \Delta \psi - \psi - v^2 w & t > 0, x \in \Omega, \\ \psi = 0 & t > 0, x \in \partial\Omega, \\ \psi(0) = w_0 - W_0 & x \in \Omega. \end{cases} \quad (4.3.25)$$

Denote by $e^{t(d_w \Delta - \text{Id})}$ the heat semigroup, then apply the Duhamel principle to equation (4.3.25)

$$\psi(t) = e^{t(d_w \Delta - \text{Id})} \psi(0) - \int_0^t e^{(t-s)(d_w \Delta - \text{Id})} v^2(s) w(s) ds. \quad (4.3.26)$$

Equation (4.3.26) allows us to estimate the L^∞ norm of u .

$$\begin{aligned} \|\psi(t)\|_\infty &\leq e^{-t} \|\psi(0)\|_\infty + \int_0^t e^{-(t-s)} \|v^2(s)\|_\infty \|w(s)\|_\infty ds \\ &\leq e^{-t} \|\psi(0)\|_\infty + M \int_0^t e^{-(t-s)} \|v^2(s)\|_\infty ds. \end{aligned} \quad (4.3.27)$$

Now, we use inequality (4.3.23) to estimate the integral term as follows

$$\begin{aligned} M \int_0^t e^{-(t-s)} \|v^2(s)\|_\infty ds &\leq \frac{MB\nu(0)}{B - M\nu(0)} \int_0^t e^{-(t-s)} e^{-Bs} ds \\ &= \frac{MB\nu(0)}{(B - M\nu(0))(B - 1)} e^{-t} - \frac{MB\nu(0)}{(B - M\nu(0))(B - 1)} e^{-Bt}. \end{aligned} \quad (4.3.28)$$

Combining inequalities (4.3.27) and (4.3.28) yields

$$\begin{aligned} \|\psi(t)\|_\infty &\leq \left(\|\psi(0)\|_\infty + \frac{MB\nu(0)}{(B - M\nu(0))(B - 1)} \right) e^{-t} \\ &\quad - \frac{MB\nu(0)}{(B - M\nu(0))(B - 1)} e^{-Bt}. \end{aligned}$$

Thus, it follows that $\|\psi(t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. □

Roughly speaking Theorem 4.3.3 states that zero vegetation is a stable solution, while non-zero vegetation is unstable for too small initial vegetation. We continue by addressing the question under which conditions the vegetation can persist.

4.4 Numerical Experiments

To validate our theoretical findings and explore the macroscopic consequences of spatial homogenization near the tipping point, we conduct a series of numerical experiments. We investigate the transient dynamics of a population in a closed environment (governed by Neumann boundary conditions) subject to the bistable reaction kinetics of the Allee effect.

4.4.1 Simulation framework and kernel calibration

Throughout these experiments, we consider either the symmetric, one-dimensional interval $\Omega = (-L, L)$ or a two-dimensional square-shaped domain $\Omega = (-L, L)^n$. We introduce the generalized Gaussian kernel family parameterized by the shape exponent $p > 0$:

$$J_p(x) = C_p e^{-\lambda_p |x|^p}, \quad x \in \mathbb{R}^n. \quad (4.4.1)$$

Constants C_p and λ_p are calibrated such that the kernel integrates to one and has second moment equal to one. They are given explicitly as follows

$$\lambda_p = \left(\frac{\Gamma\left(\frac{n+2}{p}\right)}{\Gamma\left(\frac{n}{p}\right)} \right)^{p/2}, \quad C_p = \frac{p \Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2} \Gamma\left(\frac{n}{p}\right)} \lambda_p^{n/p}. \quad (4.4.2)$$

In all subsequent simulations, we study the non-local logistic model with the Allee effect:

$$\frac{\partial v}{\partial t} = d_v \mathcal{L}_{N,p} v + r v \left(1 - \frac{v}{K}\right) \left(\frac{v}{A} - 1\right), \quad x \in \Omega, \quad t > 0 \quad (4.4.3)$$

where we fix the intrinsic growth rate $r = 1.0$, the carrying capacity $K = 1.0$, and the critical Allee threshold $A = 0.6$.

4.4.2 Phase Transition

A critical ecological question arises when a highly fragmented population hovers exactly at the brink of survival. If the global spatial mean of the population strictly equals the unstable Allee threshold A , will the population survive by exploiting local high-density refuges, or will the spatial coupling homogenize the densities and drive the entire system to extinction?

Theorem 4.2.5 dictates that this fate is governed by the sign of $\sigma = d_v \beta_1(L) + \max |f'(v)|$. Because the principal eigenvalue $\beta_1(L) < 0$ depends monotonically on the domain size L , the survival of the ecosystem is determined by a phase transition in the (L, d_v) parameter space.

Experiment 4.4.1 (Initial detection of homogenization and fragmentation phenomena).

Numerical Method and Setup: To investigate this phase transition, we simulate the scalar integro-differential equation (1.3.2) on a 1D spatial domain $[-L, L]$ using an Implicit-Explicit (IMEX) numerical scheme with the mass-conserving Neumann operator $\mathcal{L}_N$.

The initial datum $v_0(x)$ is constructed as a symmetric cosine perturbation of the Allee threshold, e.g. $v_0(x) = A + \delta \cos(\pi x/L)$. Choice of $\delta > 0$ ensures that exactly 50% of the spatial locations start above the survival threshold A (local peaks), and 50% start below it (local valleys), while strictly maintaining the global spatial mean $v_\Omega(0) = A$.

We perform a highly resolved parameter sweep over the domain half-width L and the dispersal rate d_v . For each configuration, we evolve the system to a steady state and record the final spatial maximum, minimum, and mean biomass.

Experimental Parameters: The specific values and ranges for the numerical settings and ecological parameters are summarized in Table 4.1.

Symbol	Value / Range	Description
r, A, K, p	1.0, 0.6, 1.0, 2.0	Standard ecological parameters
d_v	$[10^{-2}, 10^0]$	Dispersal rate range (100 log-spaced points)
L	$[10^0, 10^2]$	Patch half-width range (100 log-spaced points)
h_t	0.01	Time step size for the IMEX scheme
N	200	Number of spatial grid points

Table 4.1: Parameters used in Experiment 4.4.1.

With that setup, we performed numerical experiments, which results are presented below

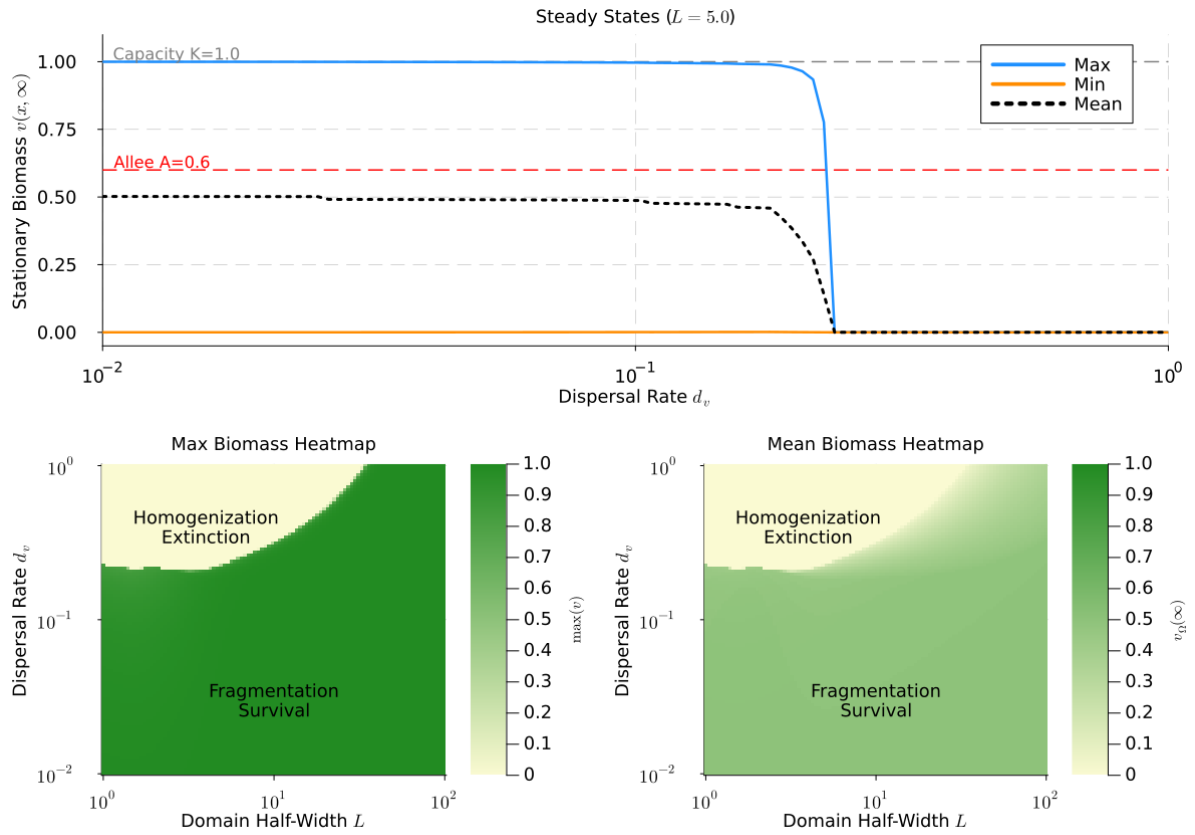


Figure 4.1: Top panel: The stationary spatial maximum, minimum, and mean of the biomass density as a function of the dispersal rate d_v for a fixed domain size $L = 5.0$. Bottom panels: Heatmaps of the final stationary maximum biomass (left) and mean biomass (right).

Conclusion 4.4.1. *The steady-state analysis (Figure 4.1, top panel) reveals a phase transition driven by the dispersal rate. When dispersal is limited ($d_v \lesssim 0.2$), the population successfully segregates into highly dense patches ($\max v \rightarrow K$) and deserted zones ($\min v \rightarrow 0$). As d_v exceeds a critical threshold, homogenization destroys the spatial patterns. Because $v_\Omega(0)$ was initialized exactly at the unstable threshold A , this forced spatial averaging to total collapse.*

Conclusion 4.4.2. *The heatmaps (Figure 4.1 bottom panels) validate our theoretical framework*

across varying habitat sizes. They show that smaller domains require less dispersal to trigger homogenization, establishing a strict, boundary between the survival and extinction regimes.

4.4.3 Convergence to Mean Mass (1D Homogenization)

Having established the phase boundary, we now explicitly validate the exponential convergence rates established in Theorem 4.2.5 within the homogenization regime ($\sigma < 0$).

Experiment 4.4.2 (Convergence to the average biomass).

Numerical Method and Setup: We simulate three distinct scenarios based on the initial mean mass $v_\Omega(0)$. In all cases, the initial profile is a symmetric cosine wave. We track the spatial

Mean above threshold	Mean exactly at threshold	Mean below threshold
$v_\Omega(0) = 0.7 > A$	$v_\Omega(0) = 0.6 = A$	$v_\Omega(0) = 0.3 < A$

mean, max and min, alongside the decay of the perturbation norm.

Experimental Parameters: We consider values of r, A, K, p, h_t, N the same as in Table 4.1. Values specific for this experiment are provided in Table 4.2 below.

Symbol	Value	Description
d_v	20.0	Dispersal rate (fast regime, $\sigma < 0$)
L	2.0	Domain half-width

Table 4.2: Parameters used in Experiment 4.4.2.

We present results of the Experiment 4.4.2 in the Figure 4.2 below.

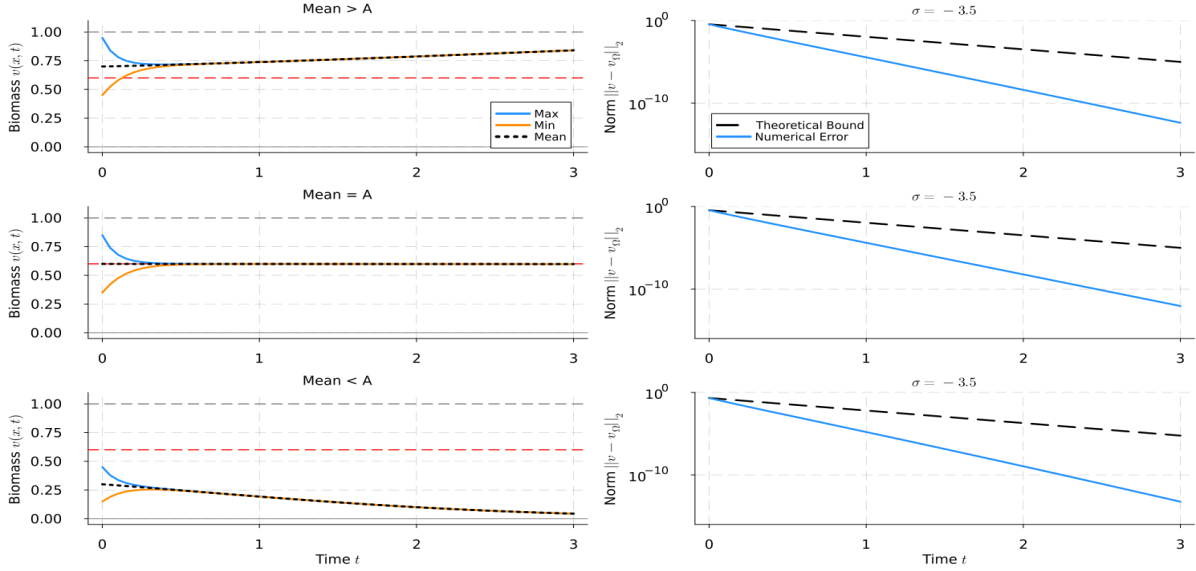


Figure 4.2: Rows correspond to different initial global means (Top: $> A$, Middle: $= A$, Bottom: $< A$). Left column: Transient time evolution of a global max, min, and the spatial mean under fast dispersal ($d_v = 20.0$). Right column: Log-linear plot of the perturbation norm $\|v(\cdot, t) - v_\Omega(t)\|_2$ over time, bounded by the theoretical exponential slope $\sigma < 0$ (dashed line).

Conclusion 4.4.3. *Figure 4.2 provides a numerical validation of Theorem 4.2.5. Because $\sigma < 0$, the spatial variance is exponentially damped in all three scenarios (right column), strictly bounded by the theoretical exponential decay rate. Consequently, the global maximum and minimum rapidly collapse toward the spatial mean (left column).*

Conclusion 4.4.4. *Figure 4.2 also shows that fate of the ecosystem depends entirely on where this homogenization occurs relative to the Allee threshold A . If homogenized above A (top row), the unified population survives and grows to K . If homogenized below A (bottom row), the local kinetics inevitably drive the newly uniform population to extinction.*

4.4.4 Discontinuous Patterns (2D Fragmentation)

Finally, we explore the dynamics in the survival regime ($\sigma > 0$), extending our framework to a two-dimensional habitat $\Omega = [-L, L] \times [-L, L]$. As highlighted in our theoretical analysis of stationary states, the lack of smoothing in non-local dispersal operators allows for the emergence of sharp, discontinuous patterns.

Experiment 4.4.3 (Pattern formation with arbitrary spatial morphology).

Numerical Method and Setup: *We simulate the 2D non-local logistic equation using a 2D mass-conserving Neumann operator. The spatial domain is discretized into a 50×50 grid. To emphasize the lack of spatial regularisation, we conduct simulations for two drastically different initial conditions $v_0(x, y)$:*

1. A classic, smooth **Checkerboard** (cosine) perturbation.
2. An arbitrary, highly discontinuous **"XD"** shape mask.

Both initial configurations are perfectly centered at the Allee threshold $A = 0.6$, ensuring that the global spatial mean strictly equals A . We apply a very slow dispersal rate $d_v = 0.05$ to force the system deep into the fragmentation regime.

Experimental Parameters: *The specific settings are summarized in Table 4.3.*

Symbol	Value	Description
r, A, K, p	1.0, 0.6, 1.0, 2.0	Standard ecological parameters
d_v	0.05	Dispersal rate (slow regime, $\sigma > 0$)
L	5.0	Domain half-width
h_t	0.05	Time step size
$N \times N$	50×50	Two-dimensional spatial grid

Table 4.3: Parameters used in Experiment 4.4.3.

Conclusion 4.4.5. *Under slow dispersal, the non-local coupling is too weak to homogenize the domain. As shown in the top panel of Figure 4.3, the local bistable kinetics dominate: the maximum density grows to the carrying capacity $K = 1.0$, while the minimum density perishes to 0.0.*

Conclusion 4.4.6. *The 2D stationary states (Figure 4.3middle and bottom right panels) exhibit remarkably sharp, discontinuous fragmentation. The fact that the arbitrary "XD" shape is perfectly preserved underscores the fundamental difference between non-local dispersal and classical local diffusion. This numerically demonstrates that in fragmented habitats with limited dispersal, persistent spatial heterogeneity acts as a critical mechanism for species survival, rescuing the population from an initial state that, if homogenized, would guarantee total extinction.*

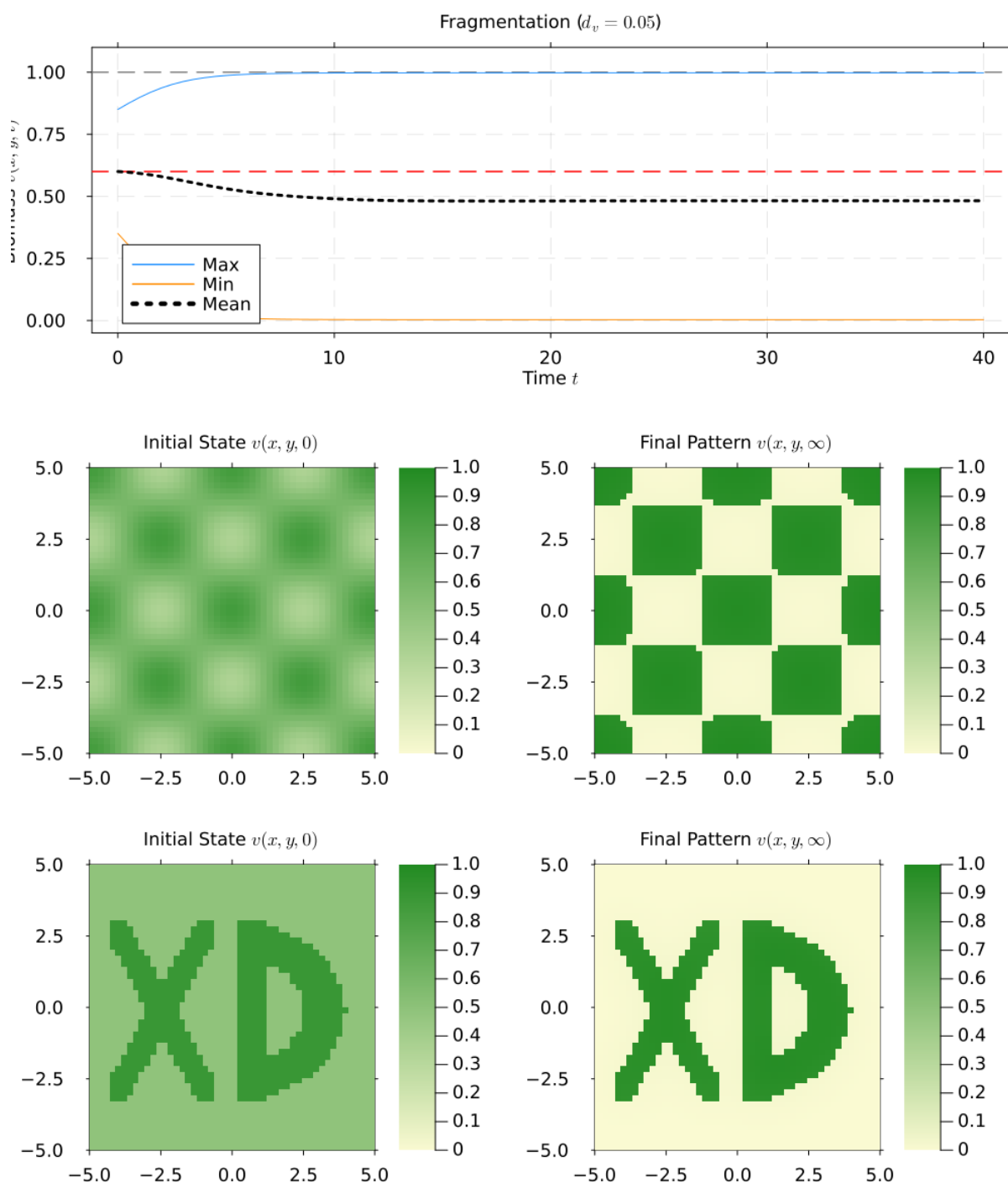


Figure 4.3: Top panel: The transient time evolution of the spatial maximum, minimum, and global mean under slow dispersal ($d_v = 0.05$). Middle panels: The initial spatial configuration $v(x, y, 0)$ and final stationary pattern $v(x, y, \infty)$ for the Checkerboard perturbation. Bottom panels: The initial and final states for the arbitrary "XD" mask perturbation. Both final states exhibit sharp fragmentation into distinct survival ($K = 1.0$) and extinction (0.0) zones.

Chapter 5

Numerical methods

In the preceding chapters, we established the rigorous theoretical foundation for both stationary and evolutionary non-local dispersal models. While analytical methods provide profound insights into well-posedness, spectral properties, and asymptotic behavior, the complex, non-linear, and inherently non-local nature of these systems often precludes the derivation of explicit closed-form solutions. To fully understand the transient dynamics, pattern formation, and bifurcation phenomena, we must resort to robust and efficient numerical simulations.

The primary computational challenge in simulating these integro-differential equations lies in the evaluation of the non-local dispersal operator $\mathcal{L}$. Unlike local diffusion governed by the classical Laplace operator Δ , which yields sparse, banded matrices upon standard discretization, the integral operator $\mathcal{K}v(x) = \int_{\Omega} J(x-y)v(y)dy$ globally couples all points within the spatial domain. Naive numerical integration techniques typically result in dense matrices, leading to a $\mathcal{O}(N^2)$ computational cost per time step for a spatial grid with N nodes. This bottleneck becomes especially severe when extending simulations to two-dimensional landscapes over long time horizons.

It is highly important to emphasize that we operate within a bounded spatial domain. This introduces a computational difficulty because the lack of periodic boundary conditions prevents the direct application of the standard Fast Fourier Transform to evaluate the convolution integrals. Consequently, we must approximate the integral operators using alternative strategies.

This chapter is dedicated to the development, implementation, and rigorous analysis of a numerical framework tailored specifically to overcome these hurdles. We begin by reviewing classical quadrature techniques to establish the basic matrix-vector representation of the discrete convolution operator. Subsequently, we introduce a highly efficient pseudo-spectral method based on orthogonal polynomial expansions, such as Chebyshev or Legendre polynomials. Finally, we provide a thorough convergence analysis and benchmark the proposed numerical scheme, ensuring its accuracy, stability, and reliability for the large-scale ecological simulations presented throughout this dissertation.

5.1 Literature review

The numerical approximation of integro-differential equations, particularly those involving non-local dispersal operators, inherently relies on the accurate and efficient discretization of integral terms. The foundational approach to handling such convolution integrals involves classical

quadrature rules. Comprehensive overviews of these standard techniques, including the Newton-Cotes formulas utilized in the preliminary sections of this chapter, can be found in classic numerical analysis textbooks. Most notably, the authoritative works by Kincaid and Cheney [29], as well as Burden and Faires [18], provide the rigorous error bounds and convergence properties essential for establishing baseline numerical schemes. For a more specialized and deeper theoretical treatment of integral equations of the second kind, which closely resemble the non-local operators analyzed in this dissertation, the monographs by Atkinson [5] and Hackbusch [26] serve as standard references.

While quadrature methods are robust and straightforward to implement, they generally exhibit algebraic convergence rates, which can become computationally expensive when high accuracy is required over large or multidimensional domains. To overcome this limitation and exploit the regularity of the solutions, spectral and pseudo-spectral methods are frequently employed. These methods offer exponential rates of convergence for sufficiently smooth problems. The highly influential textbooks by Trefethen [61, 62] provide a practical and highly accessible introduction to spectral collocation methods. It demonstrates their power in solving differential and integral equations efficiently using various basis functions, including Chebyshev polynomials which are perfectly suited for non-periodic bounded domains.

Building upon these polynomial based techniques, Chebyshev spectral collocation has been effectively applied to capture intricate spatial phenomena, such as the spike dynamics in the Gierer-Meinhardt reaction-diffusion model investigated by McCourt, Dovidio, and Gilbert [36]. Beyond standard reaction-diffusion systems, the accurate resolution of non-local models also relies heavily on carefully chosen spatial discretizations. Tian and Du [60] comprehensively analyzed various discretization schemes including quadrature based finite differences and finite elements for non-local diffusion and linear peridynamic equations, rigorously examining their convergence properties and behavior in the local limit. To achieve true spectral accuracy for non-local operators on non-Euclidean geometries, Slevinsky, Montanelli, and Du [53] developed a fast spectral method on the unit sphere, which leverages the diagonalizability of non-local Laplace-Beltrami operators in the spherical harmonics basis.

5.2 Preliminaries

In order to numerically simulate the non-local models discussed in the previous chapters, we must first discretize the spatial domain and appropriately approximate the non-local dispersal operator $\mathcal{L}v(x) = \int_{\Omega} J(x-y)v(y)dy - b(x)v(x)$. This consists of approximating the multiplication part $b(x)v(x)$ and the convolution integral $\mathcal{K}v(x) = \int_{\Omega} J(x-y)v(y)dy$.

While the methods, derivations, and error bounds presented in this section are well-known and standard in the numerical analysis literature, we recall them here to establish a consistent notation and to provide a self-contained mathematical foundation for the more advanced pseudo-spectral schemes developed later in this chapter.

For simplicity of presentation, we restrict our attention to the one-dimensional symmetric domain $\Omega = [-L, L]$, though the concepts extend naturally to higher dimensions via tensor products. We partition the domain Ω into N uniform subintervals using $N + 1$ equidistant nodes:

$$x_i = -L + ih, \quad \text{for } i = 0, 1, \dots, N, \quad (5.2.1)$$

where $h = \frac{2L}{N}$ is the spatial step size. We denote the discrete approximation of the function

$v(x_i)$ by v_i , and we gather these values into a column vector $\mathbf{v} = [v_0, v_1, \dots, v_N]^T \in \mathbb{R}^{N+1}$.

5.2.1 Second order quadrature

The composite trapezoidal rule is derived by approximating the integrand with a linear interpolant on each subinterval $[x_j, x_{j+1}]$. For a general sufficiently smooth function $f(x)$, the integral over a single subinterval is approximated as:

$$\int_{x_j}^{x_{j+1}} f(x) dx \approx \frac{h}{2} (f(x_j) + f(x_{j+1})). \quad (5.2.2)$$

Summing this approximation over all N subintervals yields the composite trapezoidal rule:

$$\int_{-L}^L f(x) dx \approx h \left(\frac{1}{2} f(x_0) + \sum_{j=1}^{N-1} f(x_j) + \frac{1}{2} f(x_N) \right). \quad (5.2.3)$$

Applying this rule to our convolution integral evaluated at a specific node x_i , we obtain:

$$\mathcal{K}v(x_i) = \int_{-L}^L J(x_i - y)v(y) dy \approx \sum_{j=0}^N w_j^{trap} J(x_i - x_j)v_j, \quad (5.2.4)$$

where the quadrature weights $\mathbf{w}^{trap} = [w_0^{trap}, \dots, w_N^{trap}]^T$ are given by:

$$w_j^{trap} = \begin{cases} h/2 & \text{for } j = 0 \text{ or } j = N, \\ h & \text{for } j = 1, \dots, N-1. \end{cases} \quad (5.2.5)$$

This operation can be elegantly expressed in matrix-vector form. Let $\mathbf{K}^{trap} \in \mathbb{R}^{(N+1) \times (N+1)}$ be the discrete integral operator matrix. Its elements are defined as $K_{i,j}^{trap} = J(x_i - x_j)w_j^{trap}$. Notice that $J(x_i - x_j) = J((i-j)h)$, which means that the underlying structure is a Toeplitz matrix. Thus, we can write the discrete operator as:

$$\mathbf{K}^{trap} = \mathbf{T}\mathbf{W}^{trap}, \quad (5.2.6)$$

where $\mathbf{W}^{trap} = \text{diag}(w_0^{trap}, \dots, w_N^{trap})$ is a diagonal weight matrix, and $\mathbf{T}$ is a symmetric Toeplitz matrix with entries $T_{i,j} = J((i-j)h)$. We recall the classical error bound for this approximation

Lemma 5.1 (Error of the Composite Trapezoidal Rule). *Let $f \in C^2([-L, L])$. The truncation error $E^{trap}(f)$ of the composite trapezoidal quadrature is bounded by*

$$|E^{trap}(f)| = \left| \int_{-L}^L f(x) dx - \sum_{j=0}^N w_j^{trap} f(x_j) \right| \leq \frac{L}{6} h^2 \|f''\|_{\infty}. \quad (5.2.7)$$

Proof. The proof is standard and follows from evaluating the error of linear interpolation. For a single subinterval $[x_j, x_{j+1}]$, the error is bounded by $\frac{h^3}{12} \max_{\xi \in [x_j, x_{j+1}]} |f''(\xi)|$. Summing over all $N = 2L/h$ subintervals yields $\sum_{j=0}^{N-1} \frac{h^3}{12} \|f''\|_{\infty} = \frac{Nh^3}{12} \|f''\|_{\infty} = \frac{2Lh^2}{12} \|f''\|_{\infty} = \frac{L}{6} h^2 \|f''\|_{\infty}$. $\square$

5.2.2 Fourth order quadratures

To achieve greater accuracy without refining the mesh, one can employ higher-order Newton-Cotes formulas, such as Simpson's 1/3 rule. This method approximates the integrand using quadratic polynomials over pairs of subintervals and requires the number of subintervals N to be even.

For a double subinterval $[x_{j-1}, x_{j+1}]$, the integral is approximated as:

$$\int_{x_{j-1}}^{x_{j+1}} f(x) dx \approx \frac{h}{3} (f(x_{j-1}) + 4f(x_j) + f(x_{j+1})). \quad (5.2.8)$$

By summing this over the domain (taking steps of 2), we obtain the composite Simpson's rule:

$$\int_{-L}^L f(x) dx \approx \frac{h}{3} \left(f(x_0) + 4 \sum_{j=1, j \text{ odd}}^{N-1} f(x_j) + 2 \sum_{j=2, j \text{ even}}^{N-2} f(x_j) + f(x_N) \right). \quad (5.2.9)$$

For the convolution operator, this yields the discrete matrix $\mathbf{K}^{simp} = \mathbf{T}\mathbf{W}^{simp}$, where $\mathbf{T}$ is the same Toeplitz matrix of kernel evaluations, and $\mathbf{W}^{simp}$ is a diagonal matrix containing the Simpson's weights $\mathbf{w}^{simp} = [w_0^{simp}, \dots, w_N^{simp}]^T$ defined by:

$$w_j^{simp} = \frac{h}{3} \times \begin{cases} 1 & \text{for } j = 0 \text{ or } j = N, \\ 4 & \text{for } j \text{ odd,} \\ 2 & \text{for } j \text{ even, } j \notin \{0, N\}. \end{cases} \quad (5.2.10)$$

The classical error estimate for this method demonstrates its superior convergence rate for smooth functions:

Lemma 5.2 (Error of the Composite Simpson's Rule). *Let $f \in C^4([-L, L])$. The truncation error $E^{simp}(f)$ of the composite Simpson's 1/3 quadrature is bounded by*

$$|E^{simp}(f)| = \left| \int_{-L}^L f(x) dx - \sum_{j=0}^N w_j^{simp} f(x_j) \right| \leq \frac{L}{90} h^4 \|f^{(4)}\|_{\infty}. \quad (5.2.11)$$

Proof. As with the trapezoidal rule, this is a standard result in numerical analysis. The error over a single double-interval $[x_{j-1}, x_{j+1}]$ is bounded by $\frac{h^5}{90} \max |f^{(4)}|$. Summing over the $N/2$ pairs of subintervals gives $\frac{N}{2} \frac{h^5}{90} \|f^{(4)}\|_{\infty} = \frac{2L/h}{2} \frac{h^5}{90} \|f^{(4)}\|_{\infty} = \frac{L}{90} h^4 \|f^{(4)}\|_{\infty}$. $\square$

While higher-order quadratures like Simpson's rule provide a theoretically faster convergence rate $\mathcal{O}(h^4)$ compared to the trapezoidal rule's $\mathcal{O}(h^2)$, their performance strongly depends on the smoothness of the integrand. In the context of non-local models, if the dispersal kernel $J(x)$ or the solution $v(x)$ exhibits steep gradients or discontinuities (which is common in pattern-forming biological systems), the theoretical higher-order accuracy might not be fully realized. Nevertheless, establishing these algebraic formulations in matrix form ($\mathbf{K} = \mathbf{T}\mathbf{W}$) is a vital preliminary step before transitioning to the pseudo-spectral approach.

5.2.3 The numerical scheme

To construct our numerical scheme based on classical quadrature methods, we consider a uniform spatial grid on $\Omega = [-L, L]$ with $N + 1$ nodes $x_0, \dots, x_N$. We look for semi-discrete solutions represented by the vector of function values at the spatial nodes $\mathbf{v}(t) = [v(x_0, t), \dots, v(x_N, t)]^T$.

Recall that the non-local dispersal operator decomposes into the sum of a purely integral convolution operator and a local multiplication operator

$$\mathcal{L}v(x, t) = \int_{\Omega} K(x, y)v(y, t) dy - b(x)v(x, t). \quad (5.2.12)$$

We first focus on the integral part. Let $\mathbf{W}$ be the diagonal matrix containing the chosen quadrature weights along its main diagonal, and let $\mathbf{T}$ be the symmetric Toeplitz matrix with entries $T_{i,j} = J(x_i - x_j)$. The discrete convolution operator $\mathbf{K}_{conv}$ acting on the state vector is explicitly given by

$$\mathbf{K}_{conv}\mathbf{v}(t) = \mathbf{T}\mathbf{W}\mathbf{v}(t). \quad (5.2.13)$$

A similar numerical treatment is applied to the multiplicative function $b(x)$, which is defined as an integral of the kernel over a specific domain. Since a closed-form analytical expression for $b(x)$ might not be available for arbitrary kernels, we compute it consistently within our quadrature framework. For the closed environment case where the integration domain is strictly Ω , we approximate the values of $b(x_i)$ using the same discrete weights. Let $\mathbf{w}$ be the column vector of the quadrature weights. The discrete vector $\mathbf{b}$ containing the approximated values of $b(x)$ at the grid points is given by $\mathbf{b} = \mathbf{T}\mathbf{w}$. We then construct the diagonal matrix $\mathbf{B} = \text{diag}(\mathbf{b})$ to represent the local multiplication operator in our numerical scheme. Note that in the non-local Dirichlet case the integral defining the vector $\mathbf{b}$ must be evaluated over the entire space $\mathbb{R}^n$, which yields that the diagonal matrix $\mathbf{B} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Scheme for a single population model We first formulate the numerical scheme for the scalar non-local equation. For the time integration, we apply an Implicit-Explicit scheme. We treat the integral operator matrix $\mathbf{T}\mathbf{W}$ and the local multiplicative boundary loss term $\mathbf{B}$ implicitly to maintain numerical stability without severe time-step restrictions. The local non-linear reaction kinetics $\mathbf{f}$ are handled explicitly. Let Δt be the time step size and $\mathbf{v}^n$ denote the solution vector at time $t_n = n\Delta t$. The fully discrete numerical scheme for the single variable v takes the form

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = d_v \mathbf{T}\mathbf{W}\mathbf{v}^{n+1} - d_v \mathbf{B}\mathbf{v}^{n+1} + \mathbf{f}(\mathbf{v}^n). \quad (5.2.14)$$

Rearranging the terms to solve for the unknown state $\mathbf{v}^{n+1}$ yields the linear system

$$\mathbf{M}_v \mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t \mathbf{f}(\mathbf{v}^n), \quad (5.2.15)$$

where

$$\mathbf{M}_v = \mathbf{I} - d_v \Delta t (\mathbf{T}\mathbf{W} - \mathbf{B}). \quad (5.2.16)$$

In the non-local Neumann case representing a closed environment, variable v demands a different conceptual approach. Here, the integral determining $b(x)$ is evaluated strictly over Ω . This configuration intrinsically guarantees mass conservation within the domain, and no manual modification of the matrix rows in the equation for v is required. On the other hand, in non-local Dirichlet case we have to impose the boundary condition, which we do by replacing rows of $\mathbf{M}_v$ corresponding to the boundary points with rows of the identity matrix $\mathbf{I}$.

Scheme for a coupled reaction-diffusion-dispersal model We extend this methodology to the hybrid system involving a non-locally dispersing variable v and a locally diffusing resource w , as seen in the modified Klausmeier-Gray-Scott model. The local diffusion of water is modeled by the classical Laplace operator. In the finite difference framework, the second partial derivative ∂_{xx} is approximated by a sparse banded differentiation matrix denoted as $\mathbf{D}_2$ and given by

$$(\mathbf{D}_2)_{i,j} = \begin{cases} +1, & i = j + 1, \\ -2, & i = j, \\ +1, & i = j - 1. \end{cases} \quad (5.2.17)$$

Consequently, the fourth order finite difference matrix approximation of the operator ∂_{xx} is given by

$$(\mathbf{D}_2)_{i,j} = \begin{cases} -1/12, & i = j + 2, \\ +4/3, & i = j + 1, \\ -5/2, & i = j, \\ +4/3, & i = j - 1, \\ -1/12, & i = j - 2. \end{cases} \quad (5.2.18)$$

Applying the same Implicit-Explicit strategy, we treat both the non-local and the local operators implicitly, while the interaction kinetics $\mathbf{f}$ and $\mathbf{g}$ are evaluated explicitly using the states from the previous time step. The coupled discrete system is formulated as two separate linear equations

$$\begin{cases} \mathbf{M}_v \mathbf{v}^{n+1} &= \mathbf{v}^n + \Delta t \mathbf{f}(\mathbf{v}^n, \mathbf{w}^n), \\ \mathbf{M}_w \mathbf{w}^{n+1} &= \mathbf{w}^n + \Delta t \mathbf{g}(\mathbf{v}^n, \mathbf{w}^n), \end{cases} \quad (5.2.19)$$

where

$$\mathbf{M}_w = \mathbf{I} - d_w \Delta t \mathbf{D}_2. \quad (5.2.20)$$

For the local variable w , the homogeneous Dirichlet boundary conditions require a direct modification of the algebraic system. To strongly enforce $w = 0$ at the domain edges, we replace the rows of the evolution matrix $\mathbf{M}_w$ corresponding to the boundary points with zeros and place a value of one on the main diagonal.

In the non-local Neumann case representing a closed environment, variable v demands a different conceptual approach. Here, the integral determining $b(x)$ is evaluated strictly over Ω . This configuration intrinsically guarantees mass conservation within the domain, and no manual modification of the matrix rows in the equation for v is required. On the other hand, in non-local Dirichlet case we have to impose the boundary condition, which we do by replacing rows of $\mathbf{M}_v$ corresponding to the boundary points with rows of the identity matrix $\mathbf{I}$.

Concurrently, to enforce the boundary conditions, we set the right hand side at the boundary indices equal to zero.

5.3 Pseudo-spectral method

As highlighted in the literature review, while classical local quadrature methods (such as the trapezoidal or Simpson's rule) are conceptually straightforward, they generally suffer from algebraic convergence rates and require dense matrix-vector multiplications that scale poorly,

typically as $\mathcal{O}(N^2)$. In contrast, spectral methods represent the solution as a global expansion in terms of smooth basis functions (e.g., trigonometric polynomials, Chebyshev polynomials, or Legendre polynomials). For problems with sufficiently smooth data, spectral methods exhibit "spectral accuracy" (i.e., exponential convergence rates), allowing one to achieve high precision with a remarkably small number of degrees of freedom.

The term *pseudo-spectral method* (or spectral collocation method) refers to the approach where the differential or integral equations are strictly enforced at a set of discrete grid points (collocation points), rather than minimizing a residual over the entire domain as in pure Galerkin methods. This approach relies on evaluating global approximations at discrete points, making it highly efficient when combined with fast transform algorithms like the Fast Fourier Transform (FFT). We refer the reader to the classic textbooks by Trefethen [61, 62] for a comprehensive and practical introduction to these concepts.

In the following subsections, we derive a numerical scheme based on approximating the integral kernel via a finite-dimensional basis (a Galerkin-type projection) and integrating the resulting coupled system in time using an Implicit-Explicit (IMEX) scheme.

5.3.1 Approximation with finite dimensional kernels

To proceed with the numerical approximation, we project the continuous and infinite dimensional non local problem onto a finite dimensional subspace. This procedure is essentially a Galerkin method applied to the integral kernel.

It is convenient to denote the integral kernel over the domain by the function $K(x, y) = J(x - y)$, where J is an integral kernel satisfying the assumptions defined in previous chapters. Observe that $K \in L^2(\Omega \times \Omega)$. Since the space $L^2(\Omega \times \Omega)$ is isomorphic to the tensor product space $L^2(\Omega) \otimes L^2(\Omega)$, we can expand the kernel K into a generalized Fourier series

$$K(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{k,l} \varphi_k(x) \varphi_l(y), \quad (5.3.1)$$

where $\{\varphi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$ and the real coefficients are given by the standard inner product $c_{k,l} = \iint_{\Omega \times \Omega} K(x, y) \varphi_k(x) \varphi_l(y) dx dy$.

To obtain a computationally tractable model, we consider a finite dimensional subspace $\mathcal{V}_N = \text{span}\{\varphi_0, \varphi_1, \dots, \varphi_N\}$. We define the truncated kernel K_N as the orthogonal projection of K onto $\mathcal{V}_N \otimes \mathcal{V}_N$

$$K_N(x, y) = \sum_{k=0}^N \sum_{l=0}^N c_{k,l} \varphi_k(x) \varphi_l(y). \quad (5.3.2)$$

Analogously, we denote the corresponding truncated non local dispersal operator $\mathcal{L}_N$ by the formula

$$\mathcal{L}_N u(x, t) = \int_{\Omega} K_N(x, y) u(y, t) dy - b(x) u(x, t). \quad (5.3.3)$$

Consequently, we wish to study the resulting truncated approximation of the scalar non local dispersal equation. We consider the single population model governed by a general local reaction kinetics function f . The truncated system is formulated as

$$\begin{cases} \partial_t u_N(x, t) = d_v \mathcal{L}_N u_N(x, t) + f(u_N(x, t)), & (x, t) \in \Omega \times (0, \infty), \\ u_N(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (5.3.4)$$

We now rigorously prove the convergence of the solutions of the truncated system (5.3.4) to the exact solutions of the full continuous system as the truncation order N tends to infinity.

Theorem 5.3.1. *Consider the initial data $u_0 \in C(\Omega)$ and assume that the nonlinear reaction term $f \in C^1(\mathbb{R})$. Let u, u_N denote the solutions of the problems (3.0.1) and (5.3.4) up to the arbitrary time of evolution $T > 0$. Then the approximate solution converges to the exact solution*

$$u_N \xrightarrow{N \rightarrow \infty} u,$$

where the convergence is understood in the space $C([0, T], L^2(\Omega))$ with respect to the supremum norm over time.

Proof. Let us denote the error function by $E_N(x, t) = u(x, t) - u_N(x, t)$. We subtract the governing equation for u_N from the equation for u to observe that E_N satisfies the following identity

$$\begin{aligned} \partial_t E_N &= d_v \mathcal{L}_N E_N + d_v (\mathcal{L} - \mathcal{L}_N) u + f(u) - f(u_N) \\ &= d_v \mathcal{L}_N E_N + d_v \int_{\Omega} \left(\sum_{k>N} \sum_{l>N} c_{k,l} \varphi_k(x) \varphi_l(y) \right) u(y, t) dy + f(u) - f(u_N). \end{aligned}$$

We multiply this equation by E_N and integrate over the spatial domain Ω to obtain the energy inequality

$$\frac{1}{2} \partial_t \|E_N(\cdot, t)\|_2^2 \leq d_v \langle \mathcal{L}_N E_N, E_N \rangle + d_v \langle \mathcal{R}_N u, E_N \rangle + \langle f(u) - f(u_N), E_N \rangle, \quad (5.3.5)$$

where $\mathcal{R}_N u$ represents the integral remainder term associated with the truncated tail of the kernel expansion.

We estimate the right hand side of (5.3.5) term by term. For the first term, we observe that $\mathcal{L}_N$ is a bounded linear operator on $L^2(\Omega)$. Therefore, the associated quadratic form is bounded from above by some finite real number $C_L > 0$ representing the spectral bound of the operator. This yields the estimate

$$d_v \langle \mathcal{L}_N E_N, E_N \rangle \leq d_v C_L \|E_N(\cdot, t)\|_2^2.$$

For the third term, we utilize the properties of the reaction function f . Because both u and u_N are strictly confined within the compact invariant region Γ for all $t \in [0, T_{max}]$, we only need to evaluate f on this bounded interval. Since f is continuously differentiable, its derivative is bounded on Γ . Thus, the local Lipschitz constant $L_f = \max_{\xi \in \Gamma} |f'(\xi)|$ is a well defined and finite positive number. This allows us to bound the reaction difference as

$$\langle f(u) - f(u_N), E_N \rangle \leq L_f \|E_N(\cdot, t)\|_2^2.$$

For the second term involving the remainder, we apply the Cauchy Schwarz inequality and Young's inequality

$$d_v \langle \mathcal{R}_N u, E_N \rangle \leq d_v \|\mathcal{R}_N u(\cdot, t)\|_2 \|E_N(\cdot, t)\|_2 \leq \frac{d_v}{2} \|\mathcal{R}_N u(\cdot, t)\|_2^2 + \frac{d_v}{2} \|E_N(\cdot, t)\|_2^2.$$

We must carefully estimate the norm of the remainder. Using the orthonormality of the basis functions, we can bound the remainder squared norm as

$$\|\mathcal{R}_N u(\cdot, t)\|_2^2 \leq |\Omega| \|u(\cdot, t)\|_{\infty}^2 \sum_{k>N} \sum_{l>N} c_{k,l}^2.$$

Because $u(x, t)$ is uniformly bounded in the supremum norm by the invariant region limits, the quantity $M_u = |\Omega| \max_{t \in [0, T_{max}]} \|u(\cdot, t)\|_\infty^2$ is a well defined and finite positive constant. Furthermore, because the original kernel K belongs to $L^2(\Omega \times \Omega)$, the sum of its squared expansion coefficients converges. This implies that the tail sum $\epsilon_N = \sum_{k>N} \sum_{l>N} c_{k,l}^2$ is finite and strictly approaches zero as N tends to infinity.

Plugging all these well defined estimates back into (5.3.5), we obtain the simplified differential inequality

$$\partial_t \|E_N(\cdot, t)\|_2^2 \leq (2d_v C_L + 2L_f + d_v) \|E_N(\cdot, t)\|_2^2 + d_v M_u \epsilon_N. \quad (5.3.6)$$

Let us define a composite positive constant $M = 2d_v C_L + 2L_f + d_v$. Because both systems start from the exact same initial data $u_0(x)$, we have the initial condition $\|E_N(\cdot, 0)\|_2^2 = 0$. We integrate the inequality (5.3.6) in time and apply Grönwall's inequality to deduce

$$\|E_N(\cdot, t)\|_2^2 \leq \int_0^t d_v M_u \epsilon_N e^{M(t-s)} ds \leq \frac{d_v M_u}{M} (e^{MT_{max}} - 1) \epsilon_N. \quad (5.3.7)$$

The right hand side of this final estimate consists entirely of finite constants multiplied by the tail sum ϵ_N . Since $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$, we conclude that $\|E_N(\cdot, t)\|_2 \rightarrow 0$ uniformly on the interval $[0, T_{max}]$. This completes the proof of strong convergence in $L^2(\Omega)$. $\square$

5.3.2 The numerical scheme

To construct our pseudo-spectral numerical scheme, we consider $\{P_k\}_{k \geq 0}$ to be the set of normalized Legendre polynomials on $\Omega = [-L, L]$. Note that this forms an orthonormal basis for $L^2(\Omega)$. Since they are dense in $C(\Omega)$, we can approximate the translation-invariant kernel $K(x, y) = J(x - y)$ as a tensorial product of Legendre polynomials. That is, there exists a sequence of symmetric coefficients $c_{k,l} = c_{l,k}$ such that

$$K(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{k,l} P_k(x) P_l(y). \quad (5.3.8)$$

We specifically look for semi-discrete solutions $v_N(x, t)$ expressed as a finite sum of Legendre polynomials with time-dependent coefficients

$$v_N(x, t) = \sum_{i=0}^N a_i(t) P_i(x). \quad (5.3.9)$$

Recall that the non-local dispersal operator decomposes into the sum of a purely integral convolution operator and a local multiplication operator

$$\mathcal{L}_N v(x, t) = \int_{\Omega} K_N(x, y) v(y, t) dy - b(x) v(x, t). \quad (5.3.10)$$

We first focus on the integral part. Let us plug the finite expansion (5.3.9) into the convolution term. By leveraging the orthonormality of the Legendre polynomials $\int_{\Omega} P_i(y) P_l(y) dy = \delta_{i,l}$,

we obtain

$$\begin{aligned}
 \int_{\Omega} K_N(x, y) v_N(y, t) dy &= \int_{\Omega} \left(\sum_{k=0}^N \sum_{l=0}^N c_{k,l} P_k(x) P_l(y) \right) \left(\sum_{i=0}^N a_i(t) P_i(y) \right) dy \\
 &= \sum_{k=0}^N \sum_{l=0}^N \sum_{i=0}^N c_{k,l} a_i(t) P_k(x) \left(\int_{\Omega} P_l(y) P_i(y) dy \right) \\
 &= \sum_{k=0}^N \left(\sum_{i=0}^N c_{k,i} a_i(t) \right) P_k(x).
 \end{aligned} \tag{5.3.11}$$

To build a practical pseudo-spectral method, we must transition between the spectral coefficients and the physical values evaluated at a discrete grid. We select $N + 1$ Gauss-Legendre quadrature nodes $x_0, \dots, x_N$ inside Ω . Let $\mathbf{P} \in \mathbb{R}^{(N+1) \times (N+1)}$ be the Vandermonde-type transformation matrix whose entries are defined as $P_{m,i} = P_i(x_m)$. The vector of spectral coefficients is denoted by $\mathbf{a}(t) = [a_0(t), \dots, a_N(t)]^T$, while the vector of function values at the spatial nodes is $\mathbf{v}(t) = [v_N(x_0, t), \dots, v_N(x_N, t)]^T$. The transformation between the spectral and physical domains is governed by the relation

$$\mathbf{v}(t) = \mathbf{P}\mathbf{a}(t) \iff \mathbf{a}(t) = \mathbf{P}^{-1}\mathbf{v}(t). \tag{5.3.12}$$

Let $\mathbf{C} \in \mathbb{R}^{(N+1) \times (N+1)}$ be the matrix of the kernel projection coefficients with entries $C_{k,i} = c_{k,i}$. Equation (5.3.11) demonstrates that the action of the integral operator corresponds exactly to the matrix-vector multiplication $\mathbf{P}\mathbf{C}\mathbf{a}(t)$. The discretized convolution operator $\mathbf{K}_{conv}$ is therefore explicitly given by

$$\mathbf{K}_{conv}\mathbf{v}(t) = \mathbf{P}\mathbf{C}\mathbf{a}(t) = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}\mathbf{v}(t). \tag{5.3.13}$$

A significant numerical challenge arises when dealing with the multiplicative function $b(x)$, which is defined as an integral of the kernel over a specific domain. Since a closed-form analytical expression for $b(x)$ might not be available for arbitrary kernels, we compute it consistently within our spectral framework. For the closed environment case where the integration domain is strictly Ω , we approximate $b(x)$ using the truncated kernel K_N . By integrating the expansion over y , we have

$$b_N(x) = \int_{\Omega} \sum_{k=0}^N \sum_{l=0}^N c_{k,l} P_k(x) P_l(y) dy = \sum_{k=0}^N P_k(x) \sum_{l=0}^N c_{k,l} \int_{\Omega} P_l(y) dy. \tag{5.3.14}$$

Because the basis functions are normalized on $[-L, L]$, the constant function 1 can be expressed as $\sqrt{2L}P_0(y)$. The orthogonality property then implies that $\int_{\Omega} P_l(y) dy = \sqrt{2L}\delta_{l,0}$. Consequently, the continuous function $b_N(x)$ simplifies exactly to

$$b_N(x) = \sqrt{2L} \sum_{k=0}^N c_{k,0} P_k(x). \tag{5.3.15}$$

Evaluating this at the collocation points yields the discrete vector $\mathbf{b} = \sqrt{2L}\mathbf{P}\mathbf{c}_0$, where $\mathbf{c}_0$ is the first column of the matrix $\mathbf{C}$. We then construct the diagonal matrix $\mathbf{B} = \text{diag}(\mathbf{b})$ to represent the local multiplication operator in our numerical scheme. Note, that in the non-local Dirichlet case the integral defining the vector $\mathbf{b}$ must be evaluated over the entire space $\mathbb{R}^n$ which yields that the diagonal matrix $\mathbf{B} = \mathbf{Id}$.

Scheme for a single population model We first formulate the numerical scheme for the scalar non-local equation. For the time integration, we apply an Implicit-Explicit scheme. We treat the integral operator matrix $\mathbf{PCP}^{-1}$ and the local multiplicative boundary loss term $\mathbf{B}$ implicitly to maintain numerical stability without severe time-step restrictions. The local non-linear reaction kinetics $\mathbf{f}$ are handled explicitly. Let Δt be the time step size and $\mathbf{v}^n$ denote the solution vector at time $t_n = n\Delta t$. The fully discrete numerical scheme for the single variable v takes the form

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = d_v \mathbf{PCP}^{-1} \mathbf{v}^{n+1} - d_v \mathbf{B} \mathbf{v}^{n+1} + \mathbf{f}(\mathbf{v}^n). \quad (5.3.16)$$

Rearranging the terms to solve for the unknown state $\mathbf{v}^{n+1}$ yields the linear system

$$\mathbf{M}_v \mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t \mathbf{f}(\mathbf{v}^n), \quad (5.3.17)$$

where

$$\mathbf{M}_v = \mathbf{I} - d_v \Delta t (\mathbf{PCP}^{-1} - \mathbf{B}). \quad (5.3.18)$$

In the non-local Neumann case representing a closed environment, variable v demands a different conceptual approach. Here, the integral determining $b(x)$ is evaluated strictly over Ω . This configuration intrinsically guarantees mass conservation within the domain, and no manual modification of the matrix rows in the equation for v is required. On the other hand, in non-local Dirichlet case we have to impose the boundary condition, which we do by replacing rows of $\mathbf{M}_v$ corresponding to the boundary points with rows of the identity matrix $\mathbf{I}$.

Scheme for a coupled reaction-diffusion-dispersal model We extend this methodology to the hybrid system involving a non-locally dispersing variable v and a locally diffusing resource w , as seen in the modified Klausmeier-Gray-Scott model. The local diffusion of water is modeled by the classical Laplace operator. In the pseudo-spectral framework, the second partial derivative is approximated by a dense pseudo-spectral differentiation matrix denoted as $\mathbf{D}_2$ and given by

$$(\mathbf{D}_2)_{i,j} = P'_i(x_j), \quad i, j \in \{1, \dots, N\}, \quad (5.3.19)$$

where $P'_i(x_j)$ denotes the derivative of i -th Legendre polynomial evaluated at the j -th node of Chebyshev-Lobatto grid.

Applying the same Implicit-Explicit strategy, we treat both the non-local and the local operators implicitly, while the interaction kinetics $\mathbf{f}$ and $\mathbf{g}$ are evaluated explicitly using the states from the previous time step. The coupled discrete system is formulated as two separate linear equations

$$\begin{cases} \mathbf{M}_v \mathbf{v}^{n+1} &= \mathbf{v}^n + \Delta t \mathbf{f}(\mathbf{v}^n, \mathbf{w}^n), \\ \mathbf{M}_w \mathbf{w}^{n+1} &= \mathbf{w}^n + \Delta t \mathbf{g}(\mathbf{v}^n, \mathbf{w}^n), \end{cases} \quad (5.3.20)$$

where

$$\mathbf{M}_w := \mathbf{I} - d_w \Delta t \mathbf{D}_2 \quad (5.3.21)$$

For the local variable w , the homogeneous Dirichlet boundary conditions require a direct modification of the algebraic system. To strongly enforce $w = 0$ at the domain edges, we replace the rows of the evolution matrix $\mathbf{M}_w$ corresponding to the boundary points with zeros and place a value of one on the main diagonal.

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Concurrently, to enforce the boundary conditions, we set the right hand side at the boundary indices equal to zero.

5.4 Convergence analysis and benchmarking

To ensure the reliability of the numerical framework developed in the previous sections, it is imperative to rigorously benchmark its performance. In this section we systematically analyze the convergence properties of the spatial discretization and the temporal integration schemes. First we investigate the spatial approximation errors of the non-local convolution operator and the multiplicative boundary loss function. We compare classical quadrature rules against the pseudo-spectral approach using kernels and test functions of varying regularity. Subsequently we assess the temporal accuracy of the Implicit-Explicit scheme by performing a time step refinement analysis for both the scalar non-local equation and the coupled reaction-diffusion-dispersal system.

5.4.1 Approximation of convolution operator $\mathcal{K}$

We begin by evaluating the accuracy of the spatial discretization methods applied to the pure integral operator $\mathcal{K}v(x) = \int_{-L}^L J(x-y)v(y)dy$. The error of the numerical approximation depends heavily on the smoothness of both the dispersal kernel J and the density function v .

Experiment 5.4.1 (Benchmarking the convolution approximation with respect to kernel's and integrand's regularity).

Numerical Method and Setup We compute the action of the operator $\mathcal{K}$ on a test function $v(x)$ using three different numerical methods evaluated at their respective grid points. We use the composite trapezoidal rule, the composite Simpson's 1/3 rule, and the pseudo-spectral method based on Legendre polynomials. We compute the discrete maximum error $E(N) = \|\mathcal{K}_{exact}v - \mathcal{K}_{num}v\|_\infty$ as a function of the number of spatial nodes N . The exact solution $\mathcal{K}_{exact}v$ is obtained either analytically or via a high-resolution numerical integration utilizing $N = 5000$ points.

Experimental Parameters We set the spatial domain to $\Omega = [-10.0, 10.0]$. We test two different kernels. The first is a smooth Gaussian kernel $J_{gauss}(z) = C_g e^{-z^2}$ representing a highly regular dispersal process. The second is a Laplace kernel $J_{laplace}(z) = C_l e^{-|z|}$ representing a process with a singularity in its first derivative at the origin. We apply these kernels to two test functions. The first test function is smooth $v_{smooth}(x) = \cos(\pi x/10)$. The second test function $v_{rough}(x) = (10^2 - x^2)^{1.5} \mathbb{1}_{[-10,10]}(x)$ possesses limited regularity at the boundaries. We vary the number of nodes N from 10 to 500.

Conclusion 5.4.1. From Figure 5.1, it is visible that the trapezoidal and Simpson's rules display their theoretical algebraic convergence rates of $\mathcal{O}(N^{-2})$ and $\mathcal{O}(N^{-4})$ respectively.

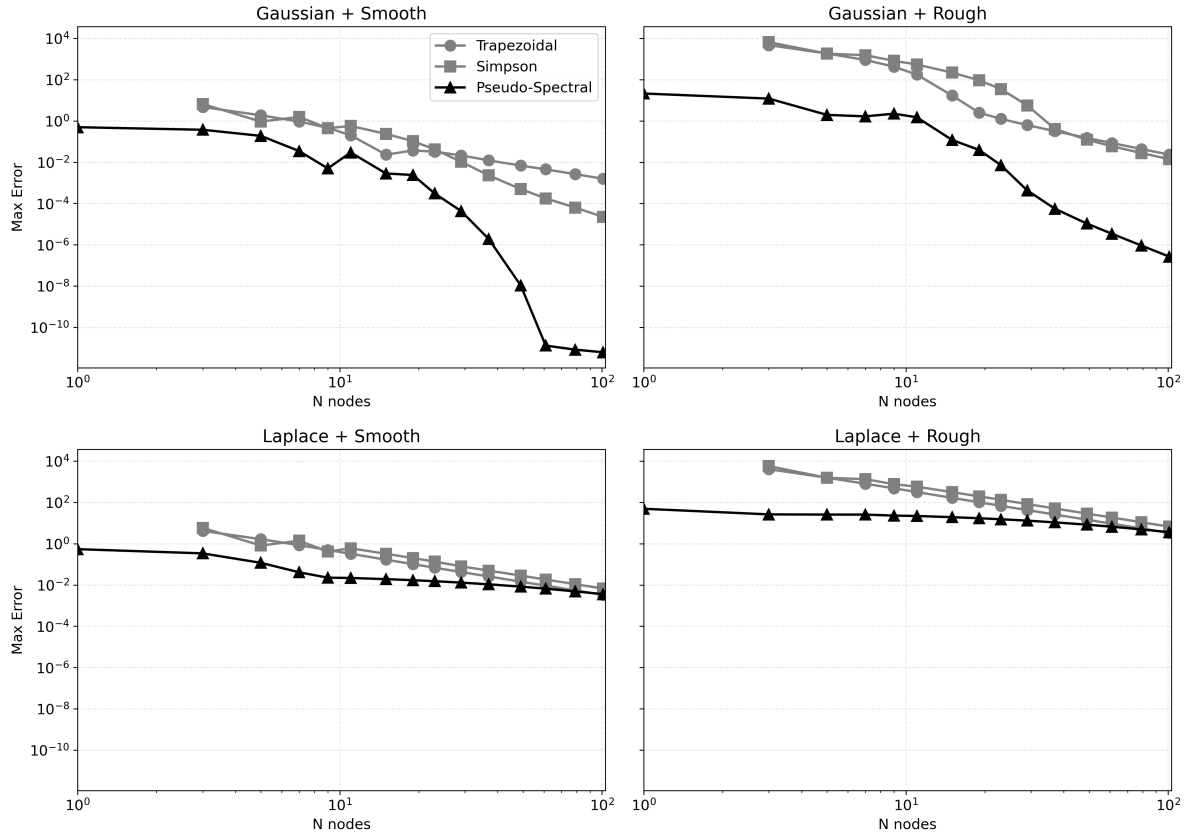


Figure 5.1: Maximum absolute error of the convolution operator approximation $\|\mathcal{K}v - \mathcal{K}_{num}v\|_\infty$ as a function of the number of spatial nodes N . The left panel shows the results for the smooth Gaussian kernel, while the right panel displays the results for the Laplace kernel. Different line styles correspond to the trapezoidal rule, Simpson's rule, and the pseudo-spectral method acting on smooth and rough test functions.

Conclusion 5.4.2. *The results presented in Figure 5.1 highlight the fundamental trade-offs between the numerical methods. For the smooth Gaussian kernel and the smooth test function, the pseudo-spectral method exhibits exponential convergence. Spectral method seems to work well also in the configuration with smooth kernel and rough test function. As the regularity reduces the pseudo-spectral method loses its exponential convergence and reverts to an algebraic rate, performing similarly to the classical quadrature rules. This confirms that spectral accuracy strictly requires high regularity of both the kernel and the state variable.*

5.4.2 Approximation of multiplicative function $b(x)$

The accurate computation of the multiplicative function $b(x) = \int_{\Omega} J(x-y)dy$ is crucial because it governs the boundary losses in the non-local Dirichlet problem and ensures mass conservation in the non-local Neumann problem.

Experiment 5.4.2 (Benchmarking the multiplicative function approximation with respect to kernel's regularity).

Numerical Method and Setup We approximate the function $b(x)$ over the domain Ω using the discrete weights of the trapezoidal rule, Simpson's rule, and the Legendre pseudo-spectral projection. We measure the maximum absolute error $E(N) = \|b_{\text{exact}} - \mathbf{b}_{\text{num}}\|_\infty$ across the grid points. The exact function $b_{\text{exact}}(x)$ is derived analytically utilizing the error function for the Gaussian kernel and direct integration for the Laplace kernel.

Experimental Parameters The domain is $\Omega = [-10.0, 10.0]$. We use the normalized Gaussian kernel and the normalized Laplace kernel as defined in Experiment 5.4.1. The number of nodes N ranges from 10 to 500.

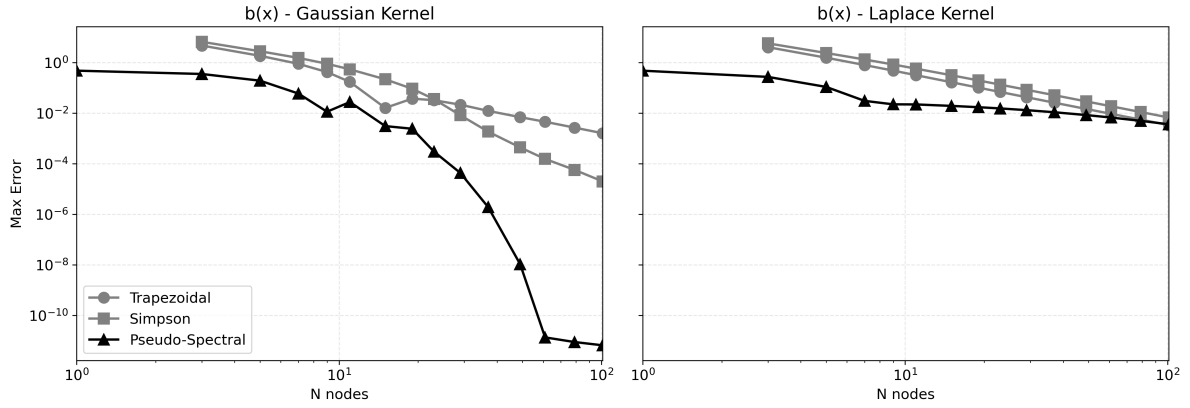


Figure 5.2: Maximum absolute error of the multiplicative function approximation $\|b_{\text{exact}} - \mathbf{b}_{\text{num}}\|_\infty$ as a function of the number of spatial nodes N . The left panel corresponds to the Gaussian kernel and the right panel corresponds to the Laplace kernel.

Conclusion 5.4.3. As observed in Figure 5.2, the computation of $b(x)$ for the Gaussian kernel yields exponential convergence under the pseudo-spectral method, drastically outperforming the standard quadratures. For the Laplace kernel, the derivative singularity at $z = 0$ propagates into the function $b(x)$, limiting its global smoothness. As a result the spectral method suffers from the Gibbs phenomenon and its error decays algebraically, closely matching the error slopes of the local quadrature methods.

5.4.3 Error of doubling the time step

To validate the temporal stability and the convergence order of our Implicit-Explicit integration scheme, we perform a time step refinement analysis. Since analytical solutions for the transient dynamics of these non-local systems are not available, we measure the convergence by comparing solutions obtained with successively halved time steps.

Experiment 5.4.3 (Benchmarking the numerical scheme for evolutionary problem).

Numerical Method and Setup We solve the evolutionary systems up to a fixed final time T . Let $u_{h_t}(t)$ denote the numerical solution obtained using a time step h_t . We compute a reference solution $u_{h_t/2}(t)$ using a halved time step $h_t/2$. The temporal error is estimated by the maximum difference between these two solutions at the final time $E(h_t) = \|u_{h_t}(T) - u_{h_t/2}(T)\|_\infty$. We perform this test for both the single scalar non-local Fisher-KPP equation and the coupled hybrid Klausmeier-Gray-Scott model.

Experimental Parameters We fix the spatial domain to $\Omega = [-20.0, 20.0]$ and use a highly resolved spatial grid with $N = 400$ nodes to isolate the temporal error. We integrate the systems up to $T = 5.0$. The initial conditions are smooth localized perturbations of the unstable trivial states. The time step h_t is systematically decreased from 10^{-1} down to 10^{-4} .

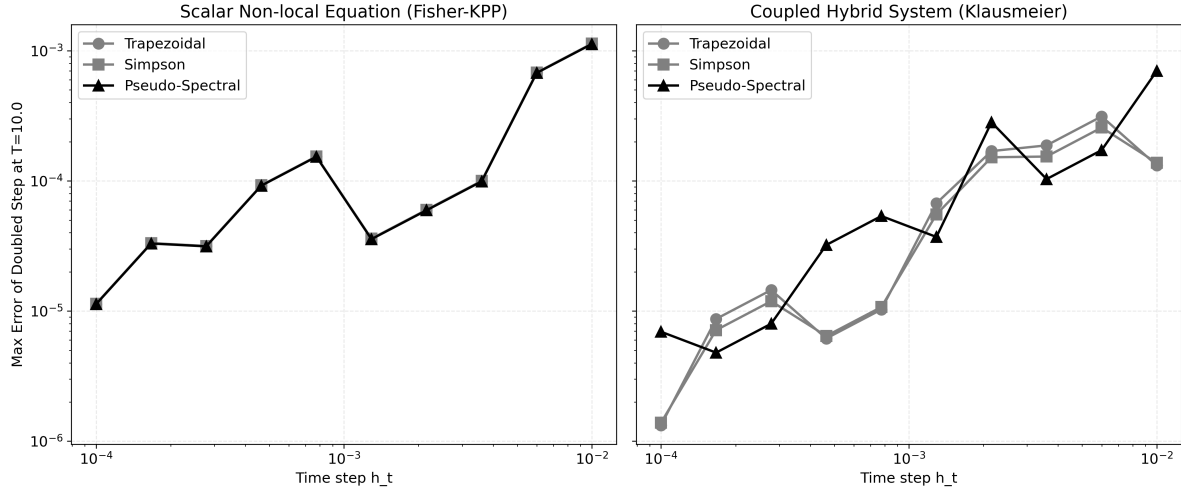


Figure 5.3: Temporal error estimate $\|u_{h_t}(T) - u_{h_t/2}(T)\|_\infty$ as a function of the time step size h_t plotted on a logarithmic scale. The left panel shows the temporal convergence for the scalar non-local Fisher-KPP model. The right panel shows the convergence for the coupled non-local Klausmeier-Gray-Scott model. The dashed reference line indicates a linear slope of $\mathcal{O}(h_t)$.

Conclusion 5.4.4. Figure 5.3 demonstrates that the temporal error decreases with the time step size h_t for both the scalar equation and the coupled system. The error curves run nearly parallel to the $\mathcal{O}(h_t)$ reference line. This confirms that our Implicit-Explicit temporal integration scheme correctly achieves the expected first-order convergence rate.

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