PhD Thesis

Ruin probability in multidimensional self-similar Gaussian risk models

Konrad Krystecki

under the supervision of Prof. Krzysztof Dębicki

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Chapter 1

Introduction

In recent years studies on the distributional properties of extrema of multidimensional stochastic processes gained significant interest. This is motivated by both new phenomena that appear in extensions of one-dimensional extreme value theory of random fields [7, 16, 21, 51] and applied probability problems [4, 5, 17, 23, 25, 50, 52, 53, 61]. In particular, multidimensional Brownian models have drawn a lot of attention due to their tractability and practical relevancy; see e.g. [17, 18, 20, 23, 25, 50, 53]. Among many models, Gaussian processes take the central role due to the significance of central limit theorems. An important methodological tools in the exploration of theory of extremes of Gaussian processes was developed in the papers of James Pickands III e.g. [64, 65], where the up-crossing probability was investigated for general Gaussian processes.

Let us recall that a stochastic process is called a Gaussian process if for any n, $a_1, \ldots, a_n \in \mathbb{R}, t_1, \ldots, t_n \in T$ random variable $\sum_{k=1}^{n} a_k X(t_k)$ has Gaussian distribution. Asymptotic theory for Gaussian processes finds many applications in physics, finance, queuing theory and risk theory. In this thesis we are interested in problems that are strongly motivated by open questions in the risk theory. The idea of diffusion approximation of risk process was introduced in [47], where the author considers a series of risk processes defined as

$$R_n(t) = u + c_n t - \frac{1}{\sqrt{n}} \sum_{k=1}^{N(nt)} X_k,$$

where $N(t)$ is a Poisson process with parameter λ, X_k have finite mean μ and standard deviation σ and $c_n = c + \lambda\mu\sqrt{n}$. One of those model interpretations is that X_k represent claims, c_n represent contributions by n clients and u represents initial capital. Then, for large enough n, representing number of clients in the system, such a risk process can be approximated by

$$R(t) := u + ct - \sqrt{\lambda}\sigma B(t),$$

where $B(t)$ is a standard Brownian motion. Standard Brownian motion $B(t), t \geq 0$ is a Gaussian process which satisfies

1. $B_0 = 0$ a.s.

2. B has independent increments

3. $B(t)$ has $N(0, t)$ distribution

4. B has almost surely continuous paths.

A natural extention of standard Brownian motion is the fractional Brownian motion $B_H(t), t \geq 0$, which is a Gaussian process such that

1. $B_H(0) = 0$ and $E[B_H(t)] = 0, t \geq 0$,

2. $E[B_H^2(t)] = t^{2H}, t \geq 0$,

3. $B_H(t)$ has a Gaussian distribution for $t > 0$,

4. $B_H(t)$ has stationary increments.

Naturally, Brownian motion is a specific case of fractional Brownian motion with $H = \frac{1}{2}$. Asymptotics of Brownian motion and fractional Brownian motion are the main focuses of this dissertation.

More generally one can define the risk process as $R(t) := u + ct - X(t)$, where $X(t)$ is any centered Gaussian process, u is the initial capital and c is a constant drift. For such risk process, the probability of ruin is defined as

$$\mathbb{P}\{\inf_{t \in \mathscr{T}} : R(t) < 0\} = \mathbb{P}\{\exists t \in \mathscr{T} : X(t) - ct > u\},$$

where $\mathscr{T}$ is some set. Since obtaining exact results for such problems is not practically feasible, various techniques and methodologies were developed to obtain the asymptotics behaviour of such probability as $u \to \infty$. Theory of Gaussian extremes has been explored in various directions. One of the crucial choices is the selection of $\mathscr{T}$. Usually it is given as a compact subset of $\mathbb{R}$ (e.g. $[0,T]$ interval for some constant $T > 0$), e.g. [15, 40], however infinite time interval is also investigated ($\mathscr{T} = [0,\infty)$), see e.g. [36]. $\mathscr{T}$ can also be a discrete set (see e.g. [48, 55]) or a random set as in [3].

Other interesting aspect is the dimensionality of the problem. Early research focused on one-dimensional problems, however recently most focus is laying on the multidimensional problems (see e.g. [8, 50, 51]). Additionally other types of functional, e.g. Parisian type ruin, defined as

$$\mathbb{P}\{\exists t \in \mathscr{T} \forall s \in \mathscr{S}_t : X(s) - cs > u\}, \tag{1.1}$$

gained interest, see e.g. contributions [14, 46, 63]. In this dissertation we aim at extending those results in various directions as described below.

In the one-dimension models, it has been observed that the asymptotics of the ruin concentrates around the point that maximizes the variance of the Gaussian process. Similar idea is being replicated with the definition of the so-called *generalized variance* in higher dimensions. As can be observed through the thesis we prove that with larger values of the generalized variance function, the process has larger chance to cross high barriers and hence the asymptotics is driven by the behaviour of such function. In this dissertation the generalized variance function is denoted as function q. It appears that the shape of q in the neighbourhood of the maximum has a crucial impact on the asymptotical results of the probability of ruin. Its exact form depends on the problem and will be defined separately in each chapter of the thesis.

This dissertation consists of the analysis of several problems that are motivated by Gaussian risk models. In Chapter 2 we study the probability of simultaneous ruin of two dimensional Brownian motion with drifts dependent on initial capital, i.e.

$$\mathbb{P}\big\{\exists_{t\in[0,T]} W_1(t) - c_1 u^\alpha t > a_1 u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\big\} \tag{1.2}$$

with W_i correlated Brownian motions, as $u \to \infty$. The motivation for this chapter comes from the so-called *many source exceedance probability* (see e.g. [27]) and leads the drift to be a polynomial function of initial capital. The problems studied in this chapter are related to both finite time and infinite time models, which in 1-dimensional setting were studied in e.g. [16].

Chapter 3 is devoted to studying the ruin problem modelled by fractional Brownian motion. This process is an interesting generalization of the Brownian motion that introduces correlation between the increments of the process. Finite time ruin probability of fractional Brownian motion is defined as

$$\widetilde{\pi}_{[0,T],H}(u) = \mathbb{P}\left\{ \sup_{t\in[0,T]} : B_H(t) - ct > u \right\} \tag{1.3}$$

with $c, T > 0, H \in (0,1]$. The probability in (1.3) has also been studied in the queuing models setting for fluid systems as in e.g. [35]. Papers [3, 30] extended the model (1.3) by supposing that T is a random variable with appropriately chosen tail distribution. Ruin models with fractional Brownian motion were also studied in higher dimensional setting, see e.g. [52]. Various other contexts and applications to the extreme value theory can be found in e.g. [9, 33, 34]. However, in majority of those papers, authors consider continuous time framework and are interested in behaviour of the supremum in the interval $t \in [0, \infty)$. Recently, an analog of (1.3) was also considered for deterministic grid (see e.g. [22, 48]), where the authors study the ruin probability only in certain equidistant time points. In Chapter 3 we aim to extend this direction of research and investigate the asymptotics of ruin probability in random inspection times, i.e.

$$\mathbb{P}\left\{ \sup_{i\geq 0, X_i\in[0,T]} : B_H(X_i) - cX_i > u \right\},$$

where B_H is a fractional Brownian motion and X_i are the inspection times represented by a counting process. In Chapter 3 we derive exact asymptotics of the probability above as $u \to \infty$.

In [57, 58] the authors studied the *Parisian ruin* probability of non-simultaneous two-dimensional Brownian motion, that is

$$\mathbb{P}\left\{ \exists_{s'\in[0,T],t'\in[0,T]} \forall_{s\in[s',s'+H_1(u)]} \forall_{t\in[t',t'+H_2(u)]} W_1(s) - c_1 s > u, W_2(t) - c_2 t > au \right\}$$

with W_i correlated Brownian motions, $H_1(u), H_2(u) \sim \frac{1}{u^2}$. Parisian type ruin has been studied in many contributions in 1-dimensional setting (e.g. [15], [13]) or in multi-dimensional setting (e.g.

[56, 57]). The choice of $H_1(u), H_2(u)$ in [57, 58] was connected to variance-covariance structure of the problem, which gave the ability to obtain the exact asymptotics for this particular case. In Chapter 4 we extend those results to other orders of $H_1(u), H_2(u)$. The idea of choosing such functions is closely connected to the so-called persistence problem (see e.g. [38, 71]). For the case of $H_i(u)$ of constant length, we restrict the model to positive correlation, which in light of [10, 37] can be considered as a natural path for understanding the behaviour of large companies from the same sector with macroeconomic factors overtaking the influence of competition. We calculate logarithmic asymptotics in the case of $H_1(u), H_2(u)$ converging to zero as $u \to \infty$. For the case of $H_1(u), H_2(u)$ not converging to zero, we show that the length of $H_1(u), H_2(u)$ can impact the asymptotics in a significant way. To provide practical application we present a study of simulations of multivariate Brownian motion.

For two-dimensional set parameter $\mathscr{T} = [0,1]^2$ the *non-simultaneous ruin* probability can be defined as

$$\mathbb{P}\left\{\exists_{s,t\in[0,1]} W_1(s) - c_1 s > u, W_2(t) - c_2 t > au\right\}$$

with W_1, W_2 correlated standard Brownian motions. Exact results for this model were given in [45], but as [59] points out, they are computationally ineffective and are not translatable to higher dimensions. Additionally, in [69] bounds can be found for two-dimensional model with no drifts. Asymptotic results for the two-dimensional non-simultaneous model were given in [24] for infinite time interval and in [18] for finite time interval. Chapter 5 aims at extending the known results to higher dimension by introducing ruin probability

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in [0,1]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{c}\cdot\boldsymbol{t} > \boldsymbol{\alpha} u\right\},$$

where $\boldsymbol{W}(\boldsymbol{t})$ is a centered multi-dimensional Brownian motion with correlated components as $u \to \infty$ and $\boldsymbol{c}\cdot\boldsymbol{t}$ is a component-wise multiplication. We specify conditions which are sufficient to observe no dimension reduction and present exact asymptotics under restriction -

$$A \gneqq \mathbf{0}, \boldsymbol{\alpha}\boldsymbol{\Sigma}_{\boldsymbol{t}}^{-1} > \mathbf{0}, \quad \boldsymbol{\alpha} > \mathbf{0}, \boldsymbol{t} \in [\mathbf{0}, \mathbf{1}]^{\mathbf{d}}, \tag{1.4}$$

where the studied process W_t is defined as

$$\boldsymbol{W}(\boldsymbol{t}) = A\boldsymbol{B}(\boldsymbol{t})$$

with $\boldsymbol{B}(\boldsymbol{t})$ $d-$dimensional Brownian motion with independent coordinates and $\Sigma_{\boldsymbol{t}}$ is a correlation matrix of $\boldsymbol{W}(\boldsymbol{t})$. The conditions above enforce positive correlations between components. The above assumptions go in line with observations of real financial market, e.g. in [37] it has been noticed that creating homogeneous groups is a viable strategy for designing the risk models for larger financial portfolios. As mentioned in [10] large companies often show a positive correlation, since their performance is more dependant on the state of the economy as a whole than on the cross-company competition. Additionally, in many sectors a positive correlation between companies occurs because of high dependence of those sectors on external factors and hence the need to model positively correlated portfolios. Similarly, claims for specific kinds of insurance (i.e. weather insurance) can have high positive correlation. We additionally find what is the most likely time of ruin for $\boldsymbol{W}(\boldsymbol{t})$ and provide upper bounds for (1).

In order to get a better understanding of the studied problems and to supplement theory with practice, in Chapter 4 we prepared a subsection dedicated to simulations of the two-dimensional non-simultaneous processes. These simulations are purely illustrative examples and should not be treated as a rigorous proof of any sorts. The simulation process, while in this dissertation used mainly as a presentation tool, can be used in order to yield interesting results to supplement theoretical findings. We can see the potential applications in e.g.

1. Testing the speed of convergence of the ruin probability with u.

   Such application would give a better understanding in quality of the asymptotics ruin of Gaussian processes with small values of u, which are crucial for practical applications of asymptotics results.

2. Finding the value of constants that cannot be obtained explicitly.

   Many of the constants calculated in the studies of asymptotics of Gaussian processes are shown to be finite and positive, however no exact value can be calculated for those. Such attempts have already been made in [12, 28, 49] and can be further enhanced with updated technology.

1.1 Notation

We introduce some basic mathematical notation that is used consistently in the thesis. By $X \stackrel{d}{=} Y$ we denote equality in distribution of random variables X and Y. Let $\varphi_t(\cdot,\cdot)$ denote the probability density function of $(W_1(t), W_2(t))$ with $W_1(t), W_2(t)$ being correlated standard Brownian motions. Similarly define $\varphi_t(\cdot)$ to be the probability density function of $W_1(t)$. Additionally let $\Psi(\cdot)$ be the tail distribution of a standard Normal random variable.

For two given positive functions $f(\cdot), g(\cdot)$ we write $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Additionally we define

$$f(x) \stackrel{\log}{\sim} g(x) \iff \lim_{x\to\infty} \frac{\log(f(x))}{\log(g(x))} = 1.$$

For the multidimensional problems we write

$$\mathbf{0} = (0, 0, \dots, 0), \mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R} \cup \{\infty\}^d.$$

Furthermore, for any matrices $X, Y \in \mathbb{R}^{n\times m}$ we write $X > Y$ if and only if

$$\forall_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} > Y_{ij},$$

and $X \gneqq Y$ if and only if

$$\forall_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} \geq Y_{ij}, \quad \exists_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} > Y_{ij}$$

and we write $X > 0$ if and only if

$$\forall_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} > 0,$$

and $X \gneqq 0$ if and only if

$$\forall_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} \geq 0, \quad \exists_{i\in 1,2,\dots,n, j\in 1,2,\dots,m} X_{ij} > 0.$$

Finally, for X, Y defined as above let

$$X \cdot Y = (X_{i,j} Y_{i,j})_{i\in 1,\dots,n, j\in 1,\dots,m}.$$

Chapter 2

Ruin probability of two-dimensional Brownian risk model with drift dependent on initial capital

2.1 Introduction

Let $(W_1(t), W_2(t)), t \geq 0$ be a standard bivariate Brownian motion with constant correlation $\rho \in (-1, 1)$, that is we can represent $W_1(t) = B_1(t), W_2(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)$ with $B_1(t), B_2(t)$ standard independent Brownian motions. Consider

$$\mathbb{P}\{\exists_{t\in[0,T]} W_1(t) - C_1 t > A_1, W_2(t) - C_2 t > A_2\}.$$

The asymptotic properties of the above probability, for $A_i = a_i u$, $a_i, C_i \in \mathbb{R}$ as $u \to \infty$ were recently analyzed in e.g. [16, 20, 25, 50, 52].

The model considered in this chapter goes in line with [43], where the extrema of one-dimensional Gaussian processes dependent on u were studied. We extend these findings to the two-dimensional case and allow for a specific structure of drift dependence on u. To be more precise, we take $C_i = c_i u^\alpha$ and $A_i = a_i u^\beta$ for $\boldsymbol{a} = (a_1, a_2), \boldsymbol{c} = (c_1, c_2) > (0, 0), \alpha, \beta \geq 0$ and consider

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) = \mathbb{P}\big\{\exists_{t\in[0,T]} W_1(t) - c_1 u^\alpha t > a_1 u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\big\}. \quad (2.1)$$

Probability (2.1) plays an important role in many areas of applied probability problems, including e.g. ruin theory, where the event under $p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u)$ is called *simultaneous ruin* with $W_i(t)$ representing accumulated claims for two dependent companies, $a_1u^\beta, a_2u^\beta > 0$ are the initial capitals and c_iu^α are the premium rates; see e.g. [16, 20, 50, 52].

An important motivation to analyze (2.1) comes from simultaneous ruin problem under the *many source* setup. To be more precise, let W_i, W_i' be standard Brownian motions with $Cov(W_i(t), W_i'(t)) = \rho t$ for $i = 1, \dots, N$ and $Cov(W_i(t), W_j'(t)) = 0$ for $i, j = 1, \dots, N, i \neq j$. Then we consider

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \sum_{i=1}^N (W_i(t) - c_1 t) > a_1 N, \sum_{i=1}^N (W_i'(t) - c_2 t) > a_2 N\right\}. \tag{2.2}$$

One can think of (2.2) as a model of two portfolios consisting of N i.i.d. sub-risk processes, representing independent businesses, that only share a common initial capital, which is proportional to the number of companies in the portfolios. Suppose that $N \to \infty$. Notice that for W, W' standard Brownian motions with $Cov(W(t), W'(t)) = \rho t$ we have

$$\begin{aligned}
&\mathbb{P}\left\{\exists_{t\in[0,T]} \sum_{i=1}^N (W_i(t) - c_1 t) > a_1 N, \sum_{i=1}^N (W_i'(t) - c_2 t) > a_2 N\right\}\\
&= \mathbb{P}\left\{\exists_{t\in[0,T]} \sqrt{N} W(t) - c_1 N t > a_1 N, \sqrt{N} W'(t) - c_2 N t > a_2 N\right\}\\
&= \mathbb{P}\left\{\exists_{t\in[0,2T]} W(t) - \frac{c_1\sqrt{N}t}{2} > a_1\sqrt{N}, W'(t) - \frac{c_2\sqrt{N}t}{2} > a_2\sqrt{N}\right\}.
\end{aligned}$$

By taking $u := \sqrt{N}$ we have that (2.2) equals to $p_{1,1,\rho,2T}(\boldsymbol{c},\boldsymbol{a},u)$. Related 1−dimensional many-source models were considered in the context of fluid queues, see e.g. [27, 53, 72].

Another problem covered by (2.1) is the model of junctions of three independent Brownian motions B_1, B_2, B_3. Let $Y(t) = B_1(t) + ct, X_1(t) = B_2(t) - a_2, X_2(t) = B_3(t) - a_3$. Suppose that $Y(0) > \max(X_1(0), X_2(0))$, i.e. $a_2, a_3 > 0$. We are interested in

$$\mathbb{P}\left\{\exists_{t\in[0,T]} Y(t) < X_1(t), Y(t) < X_2(t)\right\} \tag{2.3}$$

as $c \to \infty$. Suppose that $c = uC$ for $C > 0$. Using self-similarity of Brownian motion, we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\exists_{t\in[0,T]} B_2(t) - B_1(t) - Cut > a_2, B_3(t) - B_1(t) - Cut > a_3\right\}\\
&= \mathbb{P}\left\{\exists_{t\in[0,T]} W_1(t) - \frac{C}{\sqrt{2}}ut > \frac{a_2}{\sqrt{2}}, W_2(t) - \frac{C}{\sqrt{2}}ut > \frac{a_3}{\sqrt{2}}\right\},
\end{aligned}$$

where $(W_1(t), W_2(t)), t \geq 0$ is a two-dimensional Brownian motion with $Corr(W_1(t), W_2(t)) = \frac{1}{2}$. Therefore (2.3) corresponds to $p_{1,0,\rho,T}(\boldsymbol{c},\boldsymbol{a},u)$ with $\boldsymbol{c} = (C, C), \boldsymbol{a} = (a_2, a_3)$. Similar problem, but for the junction of two independent Gaussian processes has been studied for two-dimensional integrated Gaussian processes in e.g. [29][Sect. 4.2] or for Brownian motion in e.g. [62, 67].

The main results of this chapter are given in Theorems 2.2.1, 2.3.2, 2.3.3 and 2.4.1. It appears that the relation between α and β leads to three scenarios that require separate approaches which is also reflected in the form of the derived asymptotics. If $\alpha < \beta$, the asymptotics is dominated by u^β, while $\alpha \geq \beta$ leads to the case where both u^β and u^α have a vast impact on the asymptotics behavior of (2.1). The model analyzed in this chapter extends findings obtained in [16, 20] to the case $T \in (0, \infty)$ and the drift dependent on u. We note however that the proofs of the main results required modifications of the methodology that exists in the literature, which is reflected in new types of *generalized Pickands* constants that appear in the derived asymptotics (see Theorems 2.2.1 and 2.3.3).

The chapter is organized as follows. In the next section we tackle the case $\alpha < \beta$, for which we are able to obtain explicit formula for the cases in which one of the dimensions does not contribute to the asymptotics. Then we study case $\alpha \geq \beta$, which require more complex approach to the optimization problem and hence the differentiation between full dimensional case and dimension reduction case is not given explicitly.

We note that by self-similarity of Brownian motion

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) = p_{\alpha,\beta,\rho,\frac{T}{a_1^2}}(a_1\boldsymbol{c}, \frac{1}{a_1}\boldsymbol{a}, u) = p_{\alpha,\beta,\rho,\frac{T}{a_2^2}}(a_2\boldsymbol{c}, \frac{1}{a_2}\boldsymbol{a}, u).$$

Thus, without loss of generality, in the rest of the chapter we assume that $0 < a_2 \leq a_1 = 1$.

2.2 Case $\alpha < \beta$.

Suppose that $\alpha < \beta$ in (2.1). In this case the first component in (2.1) always influences the asymptotics, while the effect of the second one depends on the play between a_2 and ρ. The asymptotics behavior of (2.1) is dominated by u^β, and drifts appear in the results as lower order factors.

Theorem 2.2.1 *Let $\alpha < \beta$.*

(i) If $a_2 < \rho$, then as $u \to \infty$

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \sim 2\Psi\left(\frac{u^\beta + c_1 u^\alpha T}{\sqrt{T}}\right).$$

(ii) If $a_2 = \rho$, then as $u \to \infty$

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \sim \begin{cases} 2\Psi(\frac{u^\beta + c_1 u^\alpha T}{\sqrt{T}}) & \rho c_1 > c_2 \\ \Psi(\frac{u^\beta + c_1 u^\alpha T}{\sqrt{T}}) & \rho c_1 = c_2 \\ 2\Psi(-\frac{\rho c_1 - c_2}{\sqrt{(1-\rho^2)T}} T u^\alpha)\Psi(\frac{u^\beta + c_1 u^\alpha T}{\sqrt{T}}) & \rho c_1 < c_2 \end{cases}.$$

(iii) If $a_2 > \rho$, then as $u \to \infty$

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \sim \mathcal{P}_{\tilde{\boldsymbol{\lambda}}} u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T),$$

where

$$\mathcal{P}_{\boldsymbol{\lambda}} = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{t\in[0,\infty)} \begin{array}{c} W_1(t) - \frac{t}{T} > x \\ W_2(t) - \frac{a_2 t}{T} > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dx dy \in (0,\infty),$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) := \frac{1}{T}(\frac{1-a_2\rho}{1-\rho^2}, \frac{a_2-\rho}{1-\rho^2})$.

2.2.1 Proof of Theorem 2.2.1

We begin with the observation that a bivariate Brownian motion $(W_1(t), W_2(t)), t \geq 0$ can be represented in the following two ways

$$(W_1(t), W_2(t)) \stackrel{d}{=} (B_1(t), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)) \stackrel{d}{=} (\rho B_2(t) + \sqrt{1-\rho^2} B_1(t), B_2(t)), \tag{2.4}$$

where $\stackrel{d}{=}$ denotes equality in distribution and $B_i(t), t \geq 0$ are mutually independent standard Brownian motions.

Recall model (2.1) and let $\gamma = \alpha - \beta$. Let $\boldsymbol{a}_{u^\gamma}(t) = (1 + c_1 t u^\gamma, a_2 + c_2 t u^\gamma), \boldsymbol{a} = (1, a_2)$ and further

$$q_{\boldsymbol{a}_{u^\gamma}}(t) := \boldsymbol{a}_{u^\gamma}(t) \Sigma_t^{-1} \boldsymbol{a}_{u^\gamma}(t)^\top, q_{\boldsymbol{a}}(t) := \boldsymbol{a} \Sigma_t^{-1} \boldsymbol{a}^\top$$

where Σ_t denotes the covariance matrix of $(W_1(t), W_2(t))$.

Additionally let

$$q^*_{\boldsymbol{a}_{u^\gamma}}(t) = \min_{\boldsymbol{x} \geq \boldsymbol{a}_{u^\gamma}} q_{\boldsymbol{x}}(t), \quad q^*_{\boldsymbol{a}_{u^\gamma}} = \min_{t\in[0,T]} q^*_{\boldsymbol{a}_{u^\gamma}}(t) \tag{2.5}$$

and

$$q^*_{\boldsymbol{a}}(t) = \min_{\boldsymbol{x} \geq \boldsymbol{a}} q_{\boldsymbol{x}}(t), \quad q^*_{\boldsymbol{a}} = \min_{t\in[0,T]} q^*_{\boldsymbol{a}}(t). \tag{2.6}$$

Using properties of the tail of normal distribution (see [44]) we have that for $t \in [0,T]$

$$\log \mathbb{P}\{W_1(t) - c_1 u^\alpha t > u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\} = -u^{2\beta} \frac{q^*_{\boldsymbol{a}_{u^\gamma}}(t)}{2} + O(1) \tag{2.7}$$

as $u \to \infty$. Hence studying the behavior of $q^*_{\boldsymbol{a}_{u^\gamma}}(t)$ is crucial in understanding the asymptotics of $p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u)$.

Further, define $\boldsymbol{b}_{u^\gamma}(t) := \boldsymbol{a}_{u^\gamma}(t)\Sigma_t^{-1}, \boldsymbol{b}(t) := \boldsymbol{a}\Sigma_t^{-1}$ and denote

$$t_{u^\gamma} := u^{-\gamma} \sqrt{\frac{a_2^2 - 2a_2\rho + 1}{c_1^2 - 2c_1c_2\rho + c_2^2}} \text{ and } t^*_{u^\gamma} := \min\left(t_{u^\gamma}, T\right),$$

which we shall prove to be the solution to the quadratic programming problem (2.6).

Finally define $\boldsymbol{\lambda}_{u^\gamma} := \boldsymbol{b}_{u^\gamma}(t^*_{u^\gamma}) = \left(\frac{1-a_2\rho+(c_1-\rho c_2)Tu^\gamma}{(1-\rho^2)T}, \frac{a_2-\rho+(c_2-\rho c_1)Tu^\gamma}{(1-\rho^2)T}\right)$, which we prove in the following lemma to be the function that determines whether one of the coordinates is dominated by the other. Note that since $\gamma < 0$, it can be proven that for large enough u quantity $\boldsymbol{\lambda}_{u^\gamma}$ can be replaced with $\boldsymbol{\lambda} = \frac{1}{T}(\frac{1-a_2\rho}{1-\rho^2}, \frac{a_2-\rho}{1-\rho^2})$ (see proof of Lemma 2.2.4) and investigations of $q^*_{\boldsymbol{a}_{u^\gamma}}(t)$ can be simplified to investigations of $q^*_{\boldsymbol{a}}(t)$.

Lemma 2.2.2 *Let* $\boldsymbol{\lambda} > (0,0)$. *Then* $t^*_{u^\gamma}$ *is the unique point minimizing function* $q^*_{\boldsymbol{a}_{u^\gamma}}(t)$ *in the interval* $[0,T]$.

Proof of Lemma 2.2.2 It follows straightforwardly from [16][Remark A.1] that

$$q^*_{\boldsymbol{a}_{u^\gamma}}(t) = q_{\boldsymbol{a}_{u^\gamma}}(t).$$

With direct calculations we obtain that

$$\frac{d}{dt} q^*_{\boldsymbol{a}_{u^\gamma}}(t) = \frac{t^2 u^{2\gamma}(c_1^2 - 2c_1c_2\rho + c_2^2) - (a_2^2 - 2a_2\rho + 1)}{t^2(1-\rho^2)}.$$

Hence we get that either

$$t = u^{-\gamma} \sqrt{\frac{a_2^2 - 2a_2\rho + 1}{c_1^2 - 2c_1c_2\rho + c_2^2}}$$

or

$$t = -u^{-\gamma} \sqrt{\frac{a_2^2 - 2a_2\rho + 1}{c_1^2 - 2c_1c_2\rho + c_2^2}}.$$

Since $a_2 \in (0,1]$, hence $t_{u^\gamma} = u^{-\gamma} \sqrt{\frac{a_2^2-2a_2\rho+1}{c_1^2-2c_1c_2\rho+c_2^2}} > 0$ is the only possible critical point. Again direct calculations gives

$$\frac{d^2}{dt^2} q^*_{\boldsymbol{a}}(t_{u^\gamma}) = \frac{2(a_2^2 - 2a_2\rho + 1)}{t^3_{u^\gamma}(1-\rho^2)} > 0,$$

hence $t^*_{u^\gamma} = \min(t_{u^\gamma}, T)$ is the local minima. □

Notice that in the above we do not use any assumptions for γ. However, for $\gamma < 0$ we have clearly that $t^*_{u^\gamma} = T$ for large enough u. We need one more technical lemma that we use in several parts of the proof.

Lemma 2.2.3 *Let $\alpha < \beta, \boldsymbol{\lambda} > (0,0)$, $f(u)$ be a function that satisfies $\lim_{u\to\infty} f(u) = 0$. Then for large enough u*

$$\mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} W_1(t) - c_1 u^\alpha t > u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\right\} \le Ce^{-\frac{\tau}{2}f(u)u^{2\beta}} e^{-\frac{q^*_{\boldsymbol{a}}(T)}{2}u^{2\beta}},$$

for some $\tau, C > 0$.

Proof of Lemma 2.2.3 We have that for large enough u

$$\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} \begin{array}{c} W_1(t) - c_1 u^\alpha t > u^\beta \\ W_2(t) - c_2 u^\alpha t > a_2 u^\beta \end{array}\right\} &= \mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} \frac{W_1(t)}{1 + c_1 u^{\alpha-\beta} t} > u^\beta, \frac{W_2(t)}{a_2 + c_2 u^{\alpha-\beta} t} > u^\beta\right\} \\
&\le \mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} \frac{W_1(t)}{1 + c_1 u^{\alpha-\beta} t} + \frac{W_2(t)}{a_2 + c_2 u^{\alpha-\beta} t} > u^\beta\right\} \\
&\le \mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} Z_{u^\gamma}(t) > u^\beta\right\},
\end{aligned}$$

with $Z_{u^\gamma}(t) := \frac{\boldsymbol{b}(t)_{u^\gamma}(W_1(t), W_2(t))^\top}{\boldsymbol{b}(t)_{u^\gamma}\boldsymbol{a}^\top_{u^\gamma}}$, $\boldsymbol{b}_{u^\gamma}(t) := \boldsymbol{a}_{u^\gamma}\Sigma_t^{-1} > \mathbf{0}$. Straightforward calculations give $Var(Z_{u^\gamma}(t)) = \frac{1}{q_{\boldsymbol{a}_{u^\gamma}}(t)}$.

Using Lemma 2.2.2 we obtain that $\max_{t\in[0,T]} Var(Z_{u^\gamma}(t)) = Var(Z_{u^\gamma}(T))$ and further from Taylor expansion we have for $\tau := (q^*_{\boldsymbol{a}_{u^\gamma}}(T))' > 0$

$$\max_{t\in[0,T-f(u)]} Var(Z_{u^\gamma}(t)) = Var(Z_{u^\gamma}(T - f(u))) = \frac{1}{q^*_{\boldsymbol{a}_{u^\gamma}}(T) - \tau f(u) + o(1)}.$$

Hence with Borell-TIS inequality (see, e.g., [Thm 2.6.1] [2]) applied to the process $Z_{u^\gamma}(t)$ we have that for some $C > 0$

$$\mathbb{P}\left\{\exists_{t\in[0,T-f(u)]} W_1(t) - c_1 u^\alpha t > u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\right\} \le Ce^{-\frac{\tau}{2}f(u)u^{2\beta}} e^{-\frac{q^*_{\boldsymbol{a}_{u^\gamma}}(T)}{2}u^{2\beta}}.$$

This completes the proof. □

Further we use the idea of splitting the interval into smaller intervals of an appropriate size. The following two lemmas build up on that idea.

For $\Delta > 0$ let $k_u = T - \frac{(k-1)\Delta}{u^{2\beta}}$ and $E_{u,k} = [(k+1)_u, k_u]$.

Lemma 2.2.4 *Let* $\rho \in (-1,1), \boldsymbol{\lambda} > (0,0), \alpha < \beta, k \le \frac{u^\beta \log(u^\beta)}{\Delta}, \alpha \le \beta$ *and* $\Delta > 0$ *be given constants. Then, as* $u \to \infty$,

$$\begin{aligned} &\mathbb{P}\left\{\exists_{t\in E_{u,k}} W_1(t) - c_1 u^\alpha t > u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\right\} \\ &\sim \quad I(\Delta) u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T) e^{-\frac{u^{2\beta}}{2}(q_{\boldsymbol{a}}(k_u) - q_{\boldsymbol{a}}(T))}, \end{aligned} \tag{2.8}$$

where $I(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\Delta]} \begin{array}{c} W_1(t) - \frac{t}{T} > x \\ W_2(t) - \frac{a_2 t}{T} > y \end{array}\right\} e^{\lambda_1 x + \lambda_2 y} dxdy.$

Proof of Lemma 2.2.4 Let $A_u(x,y) := \left\{W_1(k_u) = u^\beta + c_1 u^\alpha k_u - \frac{x}{u^\beta}, W_2(k_u) = a_2 u^\beta + c_2 u^\alpha k_u - \frac{y}{u^\beta}\right\}$. Then for

$$\mathbb{P}_k := \mathbb{P}\left\{\exists_{t\in E_{u,k}} W_1(t) - c_1 u^\alpha t > u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\right\}$$

and $E = [-\Delta, 0]$, by using the total probability formula, we have

$$\begin{aligned} \mathbb{P}_k &= \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in E} \begin{array}{c} W_1(\frac{t}{u^{2\beta}} + k_u) - c_1 u^\alpha (\frac{t}{u^{2\beta}} + k_u) > u^\beta \\ W_2(\frac{t}{u^{2\beta}} + k_u) - c_2 u^\alpha (\frac{t}{u^{2\beta}} + k_u) > a_2 u^\beta \end{array} \middle| A_u(x,y)\right\} \\ &\quad \times u^{-2\beta} \varphi_{k_u}(u^\beta + c_1 u^\alpha k_u - \frac{x}{u^\beta}, a_2 u^\beta + c_2 u^\alpha k_u - \frac{y}{u^\beta}) dxdy \\ &= \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in E} \begin{array}{c} W_1(\frac{t}{u^{2\beta}} + k_u) - W_1(k_u) - \frac{c_1 u^{\alpha-\beta} t}{u^\beta} > \frac{x}{u^\beta} \\ W_2(\frac{t}{u^{2\beta}} + k_u) - W_2(k_u) - \frac{c_2 u^{\alpha-\beta} t}{u^\beta} > \frac{y}{u^\beta} \end{array} \middle| A_u(x,y)\right\} \\ &\quad \times u^{-2\beta} \varphi_{k_u}(u^\beta + c_1 u^\alpha k_u - \frac{x}{u^\beta}, a_2 u^\beta + c_2 u^\alpha k_u - \frac{y}{u^\beta}) dxdy. \end{aligned} \tag{2.9}$$

Using that

$$\begin{aligned} &\varphi_{k_u}(u^\beta + c_1 u^\alpha k_u - \frac{x}{u^\beta}, a_2 u^\beta + c_2 u^\alpha k_u - \frac{y}{u^\beta}) \\ &= \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T) e^{-\frac{u^{2\beta}}{2}(q_{\boldsymbol{a}}(k_u) - q_{\boldsymbol{a}}(T))} e^{\lambda_1 x + \lambda_2 y + O(\frac{x^2+y^2+xy}{u^{2\beta}})} \end{aligned}$$

in order to prove (2.8), it remains to show finiteness of

$$I_u(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in E} \begin{array}{c} W_1(\frac{t}{u^{2\beta}} + k_u) - W_1(k_u) - \frac{c_1 u^{\alpha-\beta} t}{u^\beta} > \frac{x}{u^\beta} \\ W_2(\frac{t}{u^{2\beta}} + k_u) - W_2(k_u) - \frac{c_2 u^{\alpha-\beta} t}{u^\beta} > \frac{y}{u^\beta} \end{array} \middle| A_u(x,y)\right\} e^{\lambda_1 x + \lambda_2 y} dxdy.$$

For this purpose, define $\chi_{k_u}(t) := \begin{pmatrix} W_1(\frac{t}{u^{2\beta}} + k_u) - W_1(k_u) - \frac{c_1 u^{\alpha-\beta} t}{u^\beta} \\ W_2(\frac{t}{u^{2\beta}} + k_u) - W_2(k_u) - \frac{c_2 u^{\alpha-\beta} t}{u^\beta} \end{pmatrix}$. Then we have that

$$\mathbb{E}[\chi_{k_u}(t) | A_u(x,y)] = \left(-\frac{c_1 u^{\alpha-\beta} t}{u^\beta} + \frac{t + c_1 u^{\alpha-\beta} k_u t}{u^\beta k_u}, -\frac{c_2 u^{\alpha-\beta} t}{u^\beta} + \frac{a_2 t + c_2 u^{\alpha-\beta} k_u t}{u^\beta k_u}\right)$$

$$\begin{aligned}
&+ \left(O\left(\frac{x}{u^{3\beta}}\right), O\left(\frac{y}{u^{3\beta}}\right)\right) \\
&= \left(\frac{t}{u^\beta k_u} + O\left(\frac{x}{u^{3\beta}}\right), \frac{a_2 t}{u^\beta k_u} + O\left(\frac{y}{u^{3\beta}}\right)\right)
\end{aligned}$$

and with $\Sigma_{-\frac{t}{u^{2\beta}}}$ being a covariance matrix of $\chi_{k_u}(t)$ we have

$$Var(\chi_{k_u}(t)|A_u(x,y)) = \Sigma_{-\frac{t}{u^{2\beta}}} - \Sigma_{-\frac{t}{u^{2\beta}}}\Sigma_{k_u}^{-1}\Sigma_{-\frac{t}{u^{2\beta}}} = \Sigma_{-\frac{t}{u^{2\beta}}}\left(1 + O\left(\frac{1}{u^{2\beta}}\right)\right).$$

Let

$$\begin{pmatrix} \chi^*_{1,u,k_u,x}(t) \\ \chi^*_{2,u,k_u,y}(t) \end{pmatrix} := \begin{pmatrix} u^\beta(W_1(\frac{t}{u^{2\beta}} + k_u) - W_1(k_u) - \frac{c_1 u^{\alpha-\beta} t}{u^\beta}) \\ u^\beta(W_2(\frac{t}{u^{2\beta}} + k_u) - W_2(k_u) - \frac{c_2 u^{\alpha-\beta} t}{u^\beta}) \end{pmatrix}.$$

Note that since $\alpha < \beta$, then as $u \to \infty$ $\chi^*_{1,u,k_u,x}(t)$ weakly converges to $W_1(t) - \frac{t}{T}$ and $\chi^*_{2,u,k_u,y}(t)$ weakly converges to $W_2(t) - \frac{a_2 t}{T}$ and with weak convergence what remains to be proven is the ability to use the dominated convergence theorem. Indeed, in the asymptotics of (2.9) for sufficiently large u we get

$$\begin{aligned}
I_u(\Delta) &= \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in E} : \begin{array}{c} \chi^*_{1,u,k_u,x}(t) > x \\ \chi^*_{2,u,k_u,y}(t) > y \end{array} \middle| A_u \right\} e^{\lambda_1 x + \lambda_2 y} dxdy \\
&\le \int_{\mathbb{R}_-}\int_{\mathbb{R}_-} e^{\lambda_1 x + \lambda_2 y} dxdy + \int_{\mathbb{R}_-}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{1,u,k_u,x}(t) > x | A_u\} e^{\lambda_1 x + \lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,k_u,y}(t) > y | A_u\} e^{\lambda_1 x + \lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{1,u,k_u,x}(t) + \chi^*_{2,u,k_u,y}(t) > x + y | A_u\} e^{\lambda_1 x + \lambda_2 y} dxdy \\
&\le \frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_2}\int_{\mathbb{R}_+} C_1 e^{-C_2 x^2} e^{\lambda_1 x} dx \qquad (2.10) \\
&\quad + \frac{1}{\lambda_1}\int_{\mathbb{R}_+} C_1 e^{-C_2 y^2} e^{\lambda_2 y} dy + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} C_1 e^{-C_2 (x+y)^2} e^{\lambda_1 x + \lambda_2 y} dxdy < \infty,
\end{aligned}$$

where (2.10) follows from [66][Thm 8.1] with some constants $C_1, C_2 > 0$. Thus the dominated convergence theorem can be applied to (2.9). Combining the weak convergence of $\chi^*_{1,u,k_u,x}(t)$ and $\chi^*_{2,u,k_u,y}(t)$ with dominated convergence theorem we obtain

$$I_u(\Delta) \sim \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\Delta]} \begin{array}{c} W_1(t) - \frac{t}{T} > x \\ W_2(t) - \frac{a_2 t}{T} > y \end{array}\right\} e^{\lambda_1 x + \lambda_2 y} dxdy.$$

□

Proof of Theorem 2.2.1.

Case (i): $a_2 < \rho$. Notice that

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \le \mathbb{P}\{\exists_{t\in[0,T]} W_1(t) - c_1u^\alpha t > u^\beta\}.$$

Let B_1, B_2 be mutually independent Brownian motions. Formula (2.4) and the fact that $\rho > 0$ imply

$$\begin{aligned}
&p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u)\\
&= \mathbb{P}\left\{\exists_{t\in[0,T]} B_1(t) - c_1u^\alpha t > u^\beta, \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) - c_2u^\alpha t > a_2u^\beta\right\}\\
&\ge \mathbb{P}\left\{\exists_{t\in[0,T]} B_1(t) - c_1u^\alpha t > u^\beta, \rho(u^\beta + c_1u^\alpha t) + \sqrt{1-\rho^2}B_2(t) - c_2u^\alpha t > a_2u^\beta\right\}\\
&\ge \mathbb{P}\{\exists_{t\in[0,T]} B_1(t) - c_1u^\alpha t > u^\beta\}\mathbb{P}\left\{\forall_{t\in[0,T]}\sqrt{1-\rho^2}B_2(t) + (\rho c_1 - c_2)u^\alpha t > (a_2-\rho)u^\beta\right\}\\
&= \mathbb{P}\{\exists_{t\in[0,T]} W_1(t) - c_1u^\alpha t > u^\beta\}\mathbb{P}\left\{\forall_{t\in[0,T]}\sqrt{1-\rho^2}B_2(t) + (\rho c_1 - c_2)u^\alpha t > (a_2-\rho)u^\beta\right\}.
\end{aligned}$$

Since $a_2 < \rho$ and $\alpha < \beta$, hence

$$\mathbb{P}\left\{\forall_{t\in[0,T]}\sqrt{1-\rho^2}B_2(t) + (\rho c_1 - c_2)u^\alpha t > (a_2-\rho)u^\beta\right\} \sim 1.$$

Thus

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \sim \mathbb{P}\{\exists_{t\in[0,T]} W_1(t) - c_1u^\alpha t > u^\beta\}.$$

Combination of the exact distribution of the running supremum given in e.g. [11][2.1.1.4, p. 250] with asymptotics of the tail distribution function of the standard normal variable completes the proof of case (i).

Case (ii): $a_2 = \rho$. Using (2.4) we have that

$$\begin{aligned}
&p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u)\\
&\ge \mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]} B_1(t) - c_1u^\alpha t > u^\beta, \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) - c_2u^\alpha t > a_2u^\beta\right\}\\
&\ge \mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]} B_1(t) - c_1u^\alpha t > u^\beta, \sqrt{1-\rho^2}B_2(t) + (c_1\rho - c_2)u^\alpha t > 0\right\}\\
&\ge \mathbb{P}\{\exists_{t\in[T(1-u^{-2\beta}),T]} B_1(t) - c_1u^\alpha t > u^\beta\}\mathbb{P}\left\{\forall_{t\in[T(1-u^{-2\beta}),T]}\sqrt{1-\rho^2}B_2(t) + (c_1\rho - c_2)u^\alpha t > 0\right\}.
\end{aligned}$$

Let $c^* = \frac{c_1\rho - c_2}{\sqrt{1-\rho^2}}$. Then for $\epsilon \in (\alpha,\beta)$ we have

$$\mathbb{P}\left\{\forall_{t\in[T(1-u^{-2\beta}),T]}\sqrt{1-\rho^2}B_2(t) + (c_1\rho - c_2)u^\alpha t > 0\right\}$$

$$
\begin{aligned}
&\geq \mathbb{P}\big\{\forall_{t\in[T(1-u^{-2\beta}),T]} B_2(t)+c^*u^\alpha t>0 | B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\}\\
&\quad \times \mathbb{P}\big\{B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\}\\
&\geq \mathbb{P}\big\{\forall_{t\in[0,Tu^{-2\beta}]} B_2(t)+c^*u^\alpha t>-u^{-\epsilon}\big\}\mathbb{P}\big\{B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\}\\
&= \mathbb{P}\big\{\forall_{t\in[0,T]} B_2(t)+c^*u^{\alpha-\beta} t>-u^{\beta-\epsilon}\big\}\mathbb{P}\big\{B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\}.
\end{aligned}
$$

Since $\beta>\epsilon$ we have $\mathbb{P}\big\{\forall_{t\in[0,T]} B_2(t)+c^*u^{\alpha-\beta}t>-u^{\beta-\epsilon}\big\}\to 1$, as $u\to\infty$. We want to show that

$$
\mathbb{P}\big\{B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\} \;\sim\; \mathbb{P}\{B_2(T)+c^*u^\alpha T>0\}.
$$

We equivalently shall prove that as $u\to\infty$

$$
\frac{e^{-\frac{(u^{-\epsilon}-c^*u^\alpha(1-u^{-2\beta})T)^2}{2T(1-u^{-2\beta})}}}{e^{-\frac{(c^*u^\alpha T)^2}{2T}}}\to 1.
$$

Note that since $0<\alpha<\epsilon<\beta$, then as $u\to\infty$

$$
\begin{aligned}
\frac{(u^{-\epsilon}-c^*u^\alpha(1-u^{-2\beta})T)^2}{2T(1-u^{-2\beta})} &= \frac{1}{2T(1-u^{-2\beta})}\big((c^*u^\alpha T)^2+(u^{-\epsilon}+c^*u^{\alpha-2\beta}T)^2\\
&+ 2(u^{-\epsilon}+c^*u^{\alpha-2\beta}T)c^*u^\alpha T\big)\\
&= \frac{1}{2T(1-u^{-2\beta})}((c^*u^\alpha T)^2+O(u^{-2\epsilon}+u^{-\epsilon+\alpha-2\beta}+u^{2\alpha-4\beta})+\\
&\quad +O(u^{-\epsilon+\alpha}+u^{2\alpha-2\beta}))\\
&\sim \frac{1}{2T}(c^*u^\alpha T)^2, \qquad (2.11)
\end{aligned}
$$

where the (2.11) is due to all other exponents being negative. Therefore

$$
\mathbb{P}\big\{B_2(T(1-u^{-2\beta}))+c^*u^\alpha T(1-u^{-2\beta})>u^{-\epsilon}\big\} \;\sim\; \mathbb{P}\{B_2(T)+c^*u^\alpha T>0\},
$$

where additionally we use $\alpha<\epsilon$. Straightforward calculations give, as $u\to\infty$

$$
\mathbb{P}\{B_2(T)+c^*u^\alpha T>0\}\sim\begin{cases}1 & \rho c_1>c_2\\ \frac{1}{2} & \rho c_1=c_2\\ \Psi\left(-\frac{\rho c_1-c_2}{\sqrt{(1-\rho^2)T}}Tu^\alpha\right) & \rho c_1<c_2\end{cases}. \qquad (2.12)
$$

On the other hand

$$
\begin{aligned}
p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) &\leq \mathbb{P}\big\{\exists_{t\in[T(1-u^{-2\beta}),T]} W_1(t)-c_1u^\alpha t>u^\beta, W_2(t)-c_2u^\alpha t>a_2u^\beta\big\}\\
&\quad +\mathbb{P}\big\{\exists_{t\in[0,T(1-u^{-2\beta})]} W_1(t)-c_1u^\alpha t>u^\beta, W_2(t)-c_2u^\alpha t>a_2u^\beta\big\}.
\end{aligned}
$$

With the above we get that

$$\log\left(\frac{\mathbb{P}\{\exists_{t\in[0,T(1-u^{-2\beta})]} W_1(t)-c_1u^\alpha t>u^\beta, W_2(t)-c_2u^\alpha t>a_2u^\beta\}}{\mathbb{P}\{B_2(T)+c^*u^\alpha T>0\}\mathbb{P}\{\exists_{t\in[T(1-u^{-2\beta}),T]}B_1(t)-c_1u^\alpha t>u^\beta\}}\right)$$
$$\sim\ -\frac{1}{2}\left(q_a^*(T(1-u^{-2\beta}))u^{2\beta}-\frac{u^{2\beta}}{T}-T(c^*)^2u^{2\alpha}\right).$$

Further, direct calculations lead to

$$-\frac{1}{2}\left(q_a^*(T(1-u^{-2\beta}))u^{2\beta}-\frac{u^{2\beta}}{T}-T(c^*)^2u^{2\alpha}\right)\ =\ -\frac{c_1(1-a_2\rho)}{1-\rho^2}u^{\alpha+\beta}+o(u^{\alpha+\beta}).$$

Set $\epsilon\in(\alpha,\beta)$. We further notice that

$$\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\begin{array}{c}W_1(t)-c_1u^\alpha t>u^\beta\\ W_2(t)-c_2u^\alpha t>a_2u^\beta\end{array}\right\}\ \le\ \mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\begin{array}{c}u^\beta+u^{-\epsilon}>W_1(t)-c_1u^\alpha t>u^\beta\\ W_2(t)-c_2u^\alpha t>a_2u^\beta\end{array}\right\}$$
$$+\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\begin{array}{c}W_1(t)-c_1u^\alpha t>u^\beta+u^{-\epsilon}\\ W_2(t)-c_2u^\alpha t>a_2u^\beta\end{array}\right\}.$$

We have that

$$\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\begin{array}{c}u^\beta+u^{-\epsilon}>W_1(t)-c_1u^\alpha t>u^\beta\\ W_2(t)-c_2u^\alpha t>a_2u^\beta\end{array}\right\}$$
$$\le\ \mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\begin{array}{c}u^\beta+u^{-\epsilon}>B_1(t)-c_1u^\alpha t>u^\beta\\ \sqrt{1-\rho^2}B_2(t)+(c_1\rho-c_2)u^\alpha t>-\rho u^{-\epsilon}\end{array}\right\}$$
$$\le\ \mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}u^\beta+u^{-\epsilon}>B_1(t)-c_1u^\alpha t>u^\beta\right\}$$
$$\times\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\sqrt{1-\rho^2}B_2(t)+(c_1\rho-c_2)u^\alpha t>-\rho u^{-\epsilon}\right\}.$$

Notice that

$$\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}\sqrt{1-\rho^2}B_2(t)+(c_1\rho-c_2)u^\alpha t>-\rho u^{-\epsilon}\right\}\sim\mathbb{P}\left\{\sqrt{1-\rho^2}B_2(T)+(c_1\rho-c_2)u^\alpha T>0\right\}$$

and further from [11][2.1.1.4, p. 250] we have

$$\log\left(\frac{\mathbb{P}\{\exists_{t\in[T(1-u^{-2\beta}),T]}B_1(t)-c_1u^\alpha t>u^\beta+u^{-\epsilon}\}}{\mathbb{P}\{\exists_{t\in[T(1-u^{-2\beta}),T]}B_1(t)-c_1u^\alpha t>u^\beta\}}\right)=-\left(\frac{1}{2T}(u^\beta+u^{-\epsilon})^2-\frac{1}{2T}u^{2\beta}\right)\sim-\frac{u^{\beta-\epsilon}}{T}$$

and hence

$$\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}u^\beta+u^{-\epsilon}>B_1(t)-c_1u^\alpha t>u^\beta\right\}\sim\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}B_1(t)-c_1u^\alpha t>u^\beta\right\}.$$

Moreover, similarly as above, for some $C>0$

$$\log\left(\frac{\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}W_1(t)-c_1u^\alpha t>u^\beta+u^{-\epsilon},W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}}{\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}W_1(t)-c_1u^\alpha t>u^\beta,W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}}\right)=-Cu^{\beta-\epsilon}+o(u^{\beta-\epsilon}).$$

Therefore the upper and lower bound agree and hence

$$\mathbb{P}\left\{\exists_{t\in[T(1-u^{-2\beta}),T]}B_1(t)-c_1u^\alpha t>u^\beta\right\}\sim\mathbb{P}\left\{\exists_{t\in[0,T]}B_1(t)-c_1u^\alpha t>u^\beta\right\}\sim 2\Psi\left(\frac{u^\beta+c_1u^\alpha T}{\sqrt{T}}\right).$$

Combination of the above asymptotics with (2.12) completes the proof.

Case (iii): $a_2>\rho$. Observe that

$$\begin{aligned}p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) &\leq \mathbb{P}\left\{\exists_{t\in[0,T-\frac{\log(u^\beta)}{u^\beta}]}W_1(t)-c_1u^\alpha t>u^\beta,W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}\\ &\quad+\mathbb{P}\left\{\exists_{t\in[T-\frac{\log(u^\beta)}{u^\beta},T]}W_1(t)-c_1u^\alpha t>u^\beta,W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}\\ &:= \mathcal{R}_1+\mathcal{R}_2.\end{aligned}$$

From Lemma 2.2.3 (i) with $f(u)=\frac{\log(u^\beta)}{u^\beta}$ we have that for some $C>0$

$$\mathcal{R}_1 \leq Ce^{-\frac{\tau}{2}u^\beta\log(u^\beta)}e^{-\frac{q^*_{\boldsymbol{a}}(T)}{2}u^{2\beta}}=o\left(u^{-2\beta}\varphi_T(u^\beta+c_1u^\alpha T,a_2u^\beta+c_2u^\alpha T)\right),$$

Further notice that $\tilde{\boldsymbol{\lambda}}:=\lim_{u\to\infty}\boldsymbol{\lambda}=\frac{1}{T}(\frac{1-a_2\rho}{1-\rho^2},\frac{a_2-\rho}{1-\rho^2})>(0,0)$, hence for $\Delta>0$, by Lemma 2.2.4 (i), we have

$$\begin{aligned}p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) &\geq \mathbb{P}\left\{\exists_{t\in[T-\frac{\Delta}{u^{2\beta}},T]}W_1(t)-c_1u^\alpha t>u^\beta,W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}\\ &\sim I(\Delta)u^{-2\beta}\varphi_T(u^\beta+c_1u^\alpha T,a_2u^\beta+c_2u^\alpha T),\end{aligned}$$

where

$$I(\Delta)=\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}\begin{array}{c}W_1(t)-\frac{t}{T}>x\\ W_2(t)-\frac{a_2t}{T}>y\end{array}\right\}e^{\tilde{\lambda}_1x+\tilde{\lambda}_2y}dxdy.$$

Using Taylor expansion we get as $u\to\infty$

$$u^{2\beta}(q_{\boldsymbol{a}}(k_u)-q_{\boldsymbol{a}}(T))=-\tau_1(k-1)\Delta+\tau_2(k-1)\Delta+o\left(\frac{1}{u^\beta}\right)=-\tau_1(k-1)\Delta+O(\Delta u^{2(\alpha-\beta)})+o\left(\frac{1}{u^\beta}\right),$$

where $\tau_1=\frac{a_2^2-2a_2\rho+1}{T^2(1-\rho^2)}>0,\tau_2=\frac{c_1^2-2c_1c_2\rho+c_2^2}{1-\rho^2}u^{2(\alpha-\beta)}=O(u^{2(\alpha-\beta)})=o(1)$.

Let $N_u:=\lfloor\frac{u^\beta\log(u^\beta)}{\Delta}\rfloor$. By combination of the above with Lemma 2.2.4 (i), we get as $u\to\infty$

$$\mathcal{R}_2 \leq \sum_{k=1}^{N_u}\mathbb{P}\left\{\exists_{t\in E_{k,u}}W_1(t)-c_1u^\alpha t>u^\beta,W_2(t)-c_2u^\alpha t>a_2u^\beta\right\}$$

$$\leq \sum_{k=1}^{N_u} I(\Delta) u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T) e^{-\frac{u^{2\beta}}{2}(q_{\boldsymbol{a}}(k_u) - q_{\boldsymbol{a}}(T))}$$

$$\leq \sum_{k=1}^{N_u} e^{-\frac{-\tau_1(k-1)\Delta}{2}} I(\Delta) u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T)(1 + o(1))$$

$$\leq \frac{1}{1 - e^{-\frac{\tau_1 \Delta}{2}}} I(\Delta) u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T)(1 + o(1)).$$

Hence, passing $\Delta \to \infty$, and using that by [20][proof of Thm 2.1]

$$\mathcal{P}_{\tilde{\boldsymbol{\lambda}}} := \lim_{\Delta \to \infty} I(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{t \in [0,\infty)} \begin{array}{c} W_1(t) - \frac{t}{T} > x \\ W_2(t) - \frac{a_2 t}{T} > y \end{array} \right\} e^{\tilde{\lambda}_1 x + \tilde{\lambda}_2 y} dx dy \in (0, \infty)$$

we obtain

$$\lim_{u \to \infty} \frac{p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u)}{\mathcal{P}_{\tilde{\boldsymbol{\lambda}}} u^{-2\beta} \varphi_T(u^\beta + c_1 u^\alpha T, a_2 u^\beta + c_2 u^\alpha T)} = 1.$$

This completes the proof. □

2.3 Case $\alpha = \beta$.

The results of this section are closely connected with [16], thus we use the notation for sets I, J, K used therein. For a two-dimensional matrix M and two index sets A, B let

$$M_{AB} = (m_{ab})_{a \in A, b \in B}$$

and for a vector v and index set A let

$$v_A = (v_a)_{a \in A}.$$

Let $I(t)$ be the essential index set, defined in [16] [Lemma 2.1] as follows.

Lemma 2.3.1 *([16] [Lemma 2.1]) For the quadratic optimization problem*

$$q(t) = \min_{\boldsymbol{x} > (\boldsymbol{a} + \boldsymbol{c}t)} \boldsymbol{x} \Sigma_t^{-1} \boldsymbol{x}^\top \tag{2.13}$$

there exists unique $\tilde{\boldsymbol{a}}$ *and non-empty index set* $I(t) \subset \{1, 2\}$ *such that*

$$\tilde{\boldsymbol{a}}_{I(t)} = \boldsymbol{a}_{I(t)} \neq 0_{I(t)}, \tilde{\boldsymbol{a}}_{I^{\complement}(t)} = (\Sigma_t)_{I^{\complement}(t) I(t)} (\Sigma_t^{-1})_{I(t) I(t)} \boldsymbol{a}_{I(t)}$$

so

$$(\Sigma_t^{-1})_{I(t) I(t)} \boldsymbol{a}_{I(t)} > 0_{I(t)}$$

and

$$\min_{\boldsymbol{x}>(\boldsymbol{a}+\boldsymbol{c}t)} \boldsymbol{x}\Sigma_t^{-1}\boldsymbol{x}^\top = \boldsymbol{a}_{I(t)}(\Sigma_t^{-1})_{I(t)I(t)}\boldsymbol{a}_{I(t)}^\top$$

with $I^\complement(t)$ *being the complement of set* $I(t)$ *of the set* $\{1,2\}$.

We further define

$$K(t) := \{k \in I^\complement(t) : (\Sigma_t)_{kI(t)}(\Sigma_t^{-1})_{I(t)I(t)}\boldsymbol{a}_{I(t)} = a_k(t)\},$$

$$J(t) := \{j \in I^\complement(t) : (\Sigma_t)_{jI(t)}(t)(\Sigma_t^{-1})_{I(t)I(t)}\boldsymbol{a}_{I(t)} > a_j(t)\},$$

where $(\Sigma_t)_{jI(t)} = ((\sigma_t)_{ji})_{j=j,i\in I(t)}, \boldsymbol{a}_{I(t)}(t) = (\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}$. Let

$$t_0^* := \min(t_0, T) \text{ with } t_0 := \sqrt{\frac{\boldsymbol{a}_I\Sigma_{II}^{-1}\boldsymbol{a}_I^T}{\boldsymbol{c}_I\Sigma_{II}^{-1}\boldsymbol{c}_I^T}} > 0 \tag{2.14}$$

and$I := I(t_0^*), K := K(t_0^*)$ and $J := J(t_0^*)$. We call I the *essential index set*, K the *weakly essential index set*, since it only contributes to the constant part of the asymptotics and J is called the *unessential index set* since it does not contribute to the asymptotics. One can note that

$$a_1 - a_2\rho > (\rho c_2 - c_1)\min(t_0, T) \text{ or } a_2 - a_1\rho > (\rho c_1 - c_2)\min(t_0, T). \tag{2.15}$$

On the basis of this observation, we divide the obtained results into two scenarios: a) full-dimensional case where both inequalities in (2.15) hold (then both processes W_1, W_2 in (2.1) affect the asymptotics) and b) dimension-reduction case where only one inequality in (2.15) occurs (which implies that one coordinate asymptotically dominates the other).
Due to symmetry of the results, in order to simplify notation, in the following theorem we write $\neg 1 := 2$, $\neg 2 := 1$ and (a_1, a_2) instead of $(1, a_2)$.

Theorem 2.3.2 (Dimension-reduction case) *Let* $\alpha = \beta$.
(i) If $I = \{i\}$, $J = \emptyset$ *and* $K = \{\neg i\}$, *then with* $t_0 = \frac{a_i}{c_i}$

$$p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \sim \begin{cases} (\frac{T}{a_i - c_iT} + \frac{T}{a_i + c_iT})u^{-\alpha}\varphi_T((a_i + c_iT)u^\alpha) & t_0 > T \\ \sqrt{t_0}\sqrt{\frac{\pi}{2}}\varphi_{t_0}(2a_iu^\alpha) & t_0 = T \cdot \\ \sqrt{t_0}\sqrt{2\pi}\varphi_{t_0}(2a_iu^\alpha) & t_0 < T \end{cases}$$

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(ii) If $I = \{i\}$, $J = \{\neg i\}$ and $K = \emptyset$ then with $t_0 = \frac{a_i}{c_i}$

$$p_{\alpha,\alpha,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \sim \begin{cases} \frac{1}{2}(\frac{T}{a_i - c_i T} + \frac{T}{a_i + c_i T}) u^{-\alpha} \varphi_T((a_i + c_i T) u^\alpha) & t_0 > T \\ \sqrt{t_0} \int_0^\infty \Psi(-\frac{c_{\neg i} - \rho c_i}{\sqrt{(1-\rho^2) c_i}} x) e^{-\frac{x^2}{2}} dx \varphi_{t_0}(2 a_i u^\alpha) & t_0 = T \\ \sqrt{t_0} \int_{-\infty}^\infty \Psi(-\frac{c_{\neg i} - \rho c_i}{\sqrt{1-\rho^2} c_i} x) e^{-\frac{x^2}{2}} dx \varphi_{t_0}(2 a_i u^\alpha) & t_0 < T \end{cases}.$$

The heuristics behind the asymptotics derived in Theorem 2.3.2 is that one of the coordinates dominates the other and the results are up to constant the same as in the one-dimensional case; see, e.g., [27].

Theorem 2.3.3 (Full-dimensional case) *Let $\alpha = \beta$.*
If $I = \{1, 2\}$, $J = \emptyset$ and $K = \emptyset$, then

$$p_{\alpha,\alpha,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \sim \begin{cases} \mathcal{P}_{\boldsymbol{\lambda}} u^{-2\alpha} \varphi_T((1 + c_1 T) u^\alpha, (a_2 + c_2 T) u^\alpha) & t_0 > T \\ \frac{1}{2} \sqrt{\frac{2\pi (t_0)^3 (1-\rho^2)}{a_2^2 - 2 a_2 \rho + 1}} \mathcal{H}_{\boldsymbol{\lambda}} u^{-\alpha} \varphi_{t_0}((1 + c_1 t_0) u^\alpha, (a_2 + c_2 t_0) u^\alpha) & t_0 = T \\ \sqrt{\frac{2\pi (t_0)^3 (1-\rho^2)}{a_2^2 - 2 a_2 \rho + 1}} \mathcal{H}_{\boldsymbol{\lambda}} u^{-\alpha} \varphi_{t_0}((1 + c_1 t_0) u^\alpha, (a_2 + c_2 t_0) u^\alpha) & t_0 < T \end{cases},$$

where

$$\mathcal{P}_{\boldsymbol{\lambda}} = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{t \in [0,\infty)} \begin{array}{c} W_1(t) - \frac{t}{T} > x \\ W_2(t) - \frac{a_2 t}{T} > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dx dy \in (0, \infty)$$

and

$$\mathcal{H}_{\boldsymbol{\lambda}} = \lim_{\Delta \to \infty} \frac{1}{\Delta} \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{t \in [0,\Delta]} \begin{array}{c} W_1(t) - c_1 t > x, \\ W_2(t) - c_2 t > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dx dy \in (0, \infty),$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) := (\frac{1 - a_2 \rho + (c_1 - \rho c_2) \min(t_0, T)}{(1-\rho^2) \min(t_0, T)}, \frac{a_2 - \rho + (c_2 - \rho c_1) \min(t_0, T)}{(1-\rho^2) \min(t_0, T)})$.

The intuitive explanation of the asymptotics of (2.1) presented in Theorem 2.3.3 is that both components have the same order of magnitude and therefore have a significant effect on the results.

Remark 2.3.4 *Theorems 2.3.2 and 2.3.3 are also valid for the infinite time interval, i.e. $T = \infty$. In that case we necessarily have $t_0 < T$.*

2.3.1 Proof of Theorem 2.3.2 and Theorem 2.3.3

We begin the proof with the search of the optimal point of the function $q(t)$ defined in (2.13).

Lemma 2.3.5 *Let* $t_0^* := \min(t_0, T)$ *with* $t_0 := \sqrt{\frac{\boldsymbol{a}_I \Sigma_{II}^{-1} \boldsymbol{a}_I^T}{\boldsymbol{c}_I \Sigma_{II}^{-1} \boldsymbol{c}_I^T}} > 0$. *Then* t_0^* *is the unique point minimizing the function* $q(t)$ *in the interval* $[0, T]$. *Furthermore,*

(i) if $t_0 > T$, *then*

$$q(T-t) = q(T) + q'(T)t(1+o(1)), \ \text{as } t \to 0+, \tag{2.16}$$

with $q'(t) = \frac{-\boldsymbol{a}_I \Sigma_{II}^{-1} \boldsymbol{a}_I + \boldsymbol{c}_I \Sigma_{II}^{-1} \boldsymbol{c}_I t^2}{t^2}$.

(ii) if $t_0 \le T$, *then*

$$q(t_0 \pm t) = q(t_0) + \frac{q''(t_0\pm)}{2} t^2 (1+o(1)), \ \text{as } t \to 0, \tag{2.17}$$

with $q''(t_0\pm) = 2\frac{\boldsymbol{a}_{I\pm} \Sigma_{I\pm I\pm}^{-1} \boldsymbol{a}_{I\pm}}{(t_0)^3}$.

The proof of part (i) of Lemma 2.3.5 follows straightforwardly from Taylor expansion of function $q(T)$, while part (ii) of Lemma 2.3.5 is a direct application of [16][Lemma 2.2].

Lemma 2.3.6 *Let* t_0 *be the unique point that minimizes the value of* $q(t)$ *on interval* $[0, \infty)$.

(i) If $t_0 > T$, *then*

$$\mathbb{P}\left\{\exists_{t \in [0,T] \setminus [T - \frac{\log(u^\alpha)}{u^\alpha}, T]} \boldsymbol{W}(t) - \boldsymbol{c} u^\alpha t > \boldsymbol{a} u^\alpha \right\} \le C e^{-\frac{\tau}{2} \log(u^\alpha) u^\alpha} e^{-\frac{q(T)}{2} u^{2\alpha}},$$

for some $\tau, C > 0$.

(ii) If $t_0 = T$, *then*

$$\mathbb{P}\left\{\exists_{t \in [0,T] \setminus [T - \frac{\log(u^\alpha)}{u^\alpha}, T]} \boldsymbol{W}(t) - \boldsymbol{c} u^\alpha t > \boldsymbol{a} u^\alpha \right\} \le C e^{-\frac{\tau}{2} \log^2(u^{2\alpha})} e^{-\frac{q(T)}{2} u^{2\alpha}},$$

for some $\tau, C > 0$.

(iii) If $t_0 < T$, *then*

$$\mathbb{P}\left\{\exists_{t \in [0,\infty) \setminus [t_0 - \frac{\log(u^\alpha)}{u^\alpha}, t_0 + \frac{\log(u^\alpha)}{u^\alpha}]} \boldsymbol{W}(t) - \boldsymbol{c} u^\alpha t > \boldsymbol{a} u^\alpha \right\} \le C e^{-\frac{\tau}{2} \log^2(u^{2\alpha})} e^{-\frac{q(t_0)}{2} u^{2\alpha}},$$

for some $\tau, C > 0$.

Proof of Lemma 2.3.6 We present detailed proof only for case (i); the other cases follow in a similar way.

Let $\delta_{u^\alpha} := \frac{\log(u^\alpha)}{u^\alpha}$. We have that for $\boldsymbol{b}(t) = (\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}(\Sigma_t)^{-1}_{I(t)I(t)}$ with $Z_{I(t)}(t) := \frac{\boldsymbol{b}(t)(W_1(t),W_2(t))^\top_{I(t)}}{\boldsymbol{b}(t)(\boldsymbol{a}+\boldsymbol{c}t)^\top_{I(t)}}$,

$$\mathbb{P}\left\{\exists_{t\in[0,T-\delta_{u^\alpha}]}\boldsymbol{W}(t)-\boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \le \mathbb{P}\left\{\exists_{t\in[0,T-\delta_{u^\alpha}]}Z_{I(t)}(t) > u^\alpha\right\}$$

where we use the fact from [Prop 2.1](2.2) [42] that $\boldsymbol{b}(t) > \boldsymbol{0}_{I(t)}$. Straightforward calculations give $Var(Z_{I(t)}(t)) = \frac{1}{q(t)}$. Further since the process $(Z_{I(t)}(t), t\ge 0)$ has bounded sample path, by the Borell-TIS inequality (see, e.g.[Thm 2.6.1] [2]) it holds for sufficiently large u and some $\tau > 0$

$$\mathbb{P}\left\{\exists_{t\in[0,T-\delta_{u^\alpha}]}Z_{I(t)}(t) > u^\alpha\right\} \le e^{-\frac{(u^\alpha-\mu)^2}{2}\inf_{t\in[0,T-\delta_{u^\alpha}]} q(t)},$$

where $\mu := \mathbb{E}[\sup_{t\in[0,T-\delta_{u^\alpha}]} Z_{I(t)}(t)] < \infty$. Using Lemma 2.3.5 we obtain that $\inf_{t\in[0,T-\delta_{u^\alpha}]} q(t) = q(T) + \tau\delta_{u^\alpha}$. Thus, we obtain

$$\mathbb{P}\left\{\exists_{t\in[0,T-\delta_{u^\alpha}]}Z_{I(t)}(t) > u^\alpha\right\} \le e^{-\frac{q(T)}{2}u^{2\alpha}-\frac{\tau}{2}\log(u^\alpha)u^\alpha+O(u^\alpha)} \le Ce^{-\frac{\tau}{2}\log(u^\alpha)u^\alpha}e^{-\frac{q(T)}{2}u^{2\alpha}}.$$

This completes the proof. □

For $\Delta > 0$ let $k_u = T - \frac{(k-1)\Delta}{u^{2\alpha}}$ and $E_{u,k} = [(k+1)_u, k_u]$. Moreover let

$$\boldsymbol{M}_{\{1,2\}} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{M}_{\{1\}} := \begin{pmatrix} 1 & 0 \\ \rho & 0 \end{pmatrix}, \boldsymbol{M}_{\{2\}} := \begin{pmatrix} 0 & \rho \\ 0 & 1 \end{pmatrix}.$$

Observe that for sufficiently large $u > 0$ if $I(T) = \{1,2\}$, then $I(k_u) = \{1,2\}$.

Lemma 2.3.7 *Let $t_0 > T$, and $\Delta > 0$.*

(i) If $I = \{i\}$, $J = \{\neg i\}$ and $K = \emptyset$, then as $u \to \infty$

$$\mathbb{P}\left\{\exists_{t\in E_{u,1}}\boldsymbol{W}(t)-\boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \sim \mathcal{H}_I(\Delta)u^{-\alpha}\varphi_T((a_i+c_iT)u^\alpha),$$

where $\mathcal{H}_I(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in[0,\Delta]}W_i(t) - \frac{a_it}{T} > x\right\}e^{\frac{a_i+c_iT}{T}x}dx$.

(ii) If $I = \{i\}$, $J = \emptyset$ and $K = \{\neg i\}$, then, as $u \to \infty$

$$\mathbb{P}\left\{\exists_{t\in E_{u,1}}\boldsymbol{W}(t)-\boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \sim \frac{1}{2}\mathcal{H}_I(\Delta)u^{-\alpha}\varphi_T((a_i+c_iT)u^\alpha),$$

where $\mathcal{H}_I(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in[0,\Delta]}W_i(t) - \frac{a_it}{T} > x\right\}e^{\frac{a_i+c_iT}{T}x}dx$.

(iii) If $I = \{1,2\}$, $J = \emptyset$ and $K = \emptyset$, then, as $u \to \infty$

$$\mathbb{P}\left\{\exists_{t\in E_{u,k}}\boldsymbol{W}(t)-\boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \sim \mathcal{H}_I(\Delta)u^{-2\alpha}\varphi_T((a_1+c_1T)u^\alpha,(a_2+c_2T)u^\alpha)e^{\frac{q'(T)}{2}(k-1)\Delta},$$

where $\mathcal{H}_I(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\Delta]}\boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\}e^{(\boldsymbol{a}+\boldsymbol{c}T)\Sigma_T^{-1}\boldsymbol{x}^T}d\boldsymbol{x}$.

PROOF OF LEMMA 2.3.7 Let $\widetilde{I} = I(k_u), \widetilde{J} = J(k_u), \widetilde{K} = K(k_u), n = |\widetilde{I}|$, $A_{\widetilde{I},u} := \Big\{ \boldsymbol{W}_{\widetilde{I}}(T) = ((\boldsymbol{a}+\boldsymbol{c}k_u)u^{\alpha} - \boldsymbol{x}u^{-\alpha})_{\widetilde{I}} \Big\}$. Then for $\mathcal{P}_k := \mathbb{P}\big\{ \exists_{t\in E_{u,k}} \boldsymbol{W}(t) - \boldsymbol{c}u^{\alpha}t > \boldsymbol{a}u^{\alpha} \big\}$, and $E = [-\Delta, 0]$, by using the total probability formula, we have

$$\begin{aligned}
\mathcal{P}_k &= \int_{\mathbb{R}^n} \mathbb{P}\left\{ \exists_{t\in E} \boldsymbol{W}(\frac{t}{u^{2\alpha}} + k_u) - \boldsymbol{c}(\frac{t}{u^{2\alpha}} + k_u)u^{\alpha} > \boldsymbol{a}u^{\alpha} \middle| A_{\widetilde{I},u} \right\} u^{-\alpha n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_{\widetilde{I}}u^{\alpha} - \frac{\boldsymbol{x}_{\widetilde{I}}}{u^{\alpha}}) d\boldsymbol{x}_{\widetilde{I}} \\
&= \int_{\mathbb{R}^n} \mathbb{P}\left\{ \exists_{t\in E} \boldsymbol{W}(\frac{t}{u^{2\alpha}} + k_u) - \boldsymbol{M}_{\widetilde{I}}\boldsymbol{W}(T) + \boldsymbol{M}_{\widetilde{I}}((\boldsymbol{a}+\boldsymbol{c}k_u)u^{\alpha} - \frac{\boldsymbol{x}}{u^{\alpha}}) - \boldsymbol{c}(\frac{t}{u^{2\alpha}} + k_u)u^{\alpha} > \boldsymbol{a}u^{\alpha} \middle| A_{\widetilde{I},u} \right\} \\
&\quad \times u^{-\alpha n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_I u^{\alpha} - \frac{\boldsymbol{x}_I}{u^{\alpha}}) d\boldsymbol{x}_I \\
&= \int_{\mathbb{R}^n} \mathbb{P}\left\{ \exists_{t\in E} \boldsymbol{W}(\frac{t}{u^{2\alpha}} + k_u) - \boldsymbol{M}_{\widetilde{I}}\boldsymbol{W}(k_u) - \boldsymbol{c}\frac{t}{u^{\alpha}} > (\boldsymbol{Id} - \boldsymbol{M}_{\widetilde{I}})(\boldsymbol{a}+\boldsymbol{c}k_u)u^{\alpha} + \frac{\boldsymbol{M}_{\widetilde{I}}\boldsymbol{x}}{u^{\alpha}} \middle| A_{\widetilde{I},u} \right\} \\
&\quad \times u^{-\alpha n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_{\widetilde{I}}u^{\alpha} - \frac{\boldsymbol{x}_{\widetilde{I}}}{u^{\alpha}}) d\boldsymbol{x}_{\widetilde{I}}.
\end{aligned}$$

Using that, as $u \to \infty$,

$$\varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_{\widetilde{I}}u^{\alpha} - \frac{\boldsymbol{x}_{\widetilde{I}}}{u^{\alpha}}) = \varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_{\widetilde{I}}u^{\alpha}) e^{\boldsymbol{\lambda}_{\widetilde{I}}\boldsymbol{x}_{\widetilde{I}}^T} e^{\frac{q'(T)}{2}\Delta} e^{O(\boldsymbol{x}_{\widetilde{I}}\boldsymbol{x}_{\widetilde{I}}^T u^{-2\alpha})}$$

with $\boldsymbol{\lambda}_{\widetilde{I}} = (\boldsymbol{a}+\boldsymbol{c}T)_{\widetilde{I}}\Sigma_{\widetilde{I}\widetilde{I}}^{-1}$. In order to prove the thesis it remains to show finiteness of

$$\begin{aligned}
\mathcal{H}_{\widetilde{I},u}(\Delta) &:= \int_{\mathbb{R}^n} \mathbb{P}\left\{ \exists_{t\in E} \boldsymbol{W}(\frac{t}{u^{2\alpha}} + k_u) - \boldsymbol{M}_{\widetilde{I}}\boldsymbol{W}(k_u) - \boldsymbol{c}\frac{t}{u^{\alpha}} > (\widetilde{\boldsymbol{I}} - \boldsymbol{M}_{\widetilde{I}})(\boldsymbol{a}+\boldsymbol{c}k_u)u^{\alpha} + \frac{\boldsymbol{M}_{\widetilde{I}}\boldsymbol{x}}{u^{\alpha}} \middle| A_{\widetilde{I},u} \right\} \\
&\quad \times e^{\boldsymbol{\lambda}_{\widetilde{I}}\boldsymbol{x}_{\widetilde{I}}^T} d\boldsymbol{x}_{\widetilde{I}},
\end{aligned}$$

where

$$((\widetilde{\boldsymbol{I}} - \boldsymbol{M}_{\widetilde{I}})(\boldsymbol{a}+\boldsymbol{c}k_u))_i = \begin{cases} 0, & \text{if } i \in \widetilde{I} \\ (a_i + c_iT) - \rho(a_{\neg i} + c_{\neg i}k_u) = 0, & \text{if } i \in \widetilde{K} \\ (a_i + c_iT) - \rho(a_{\neg i} + c_{\neg i}k_u) < 0, & \text{if } i \in \widetilde{J} \end{cases} . \tag{2.18}$$

For this purpose, define $\boldsymbol{\chi}_u(t) = (\chi_{u;1}(t), \chi_{u;2}(t)) := \boldsymbol{W}(\frac{t}{u^{2\alpha}} + k_u) - \boldsymbol{M}_{\widetilde{I}}\boldsymbol{W}(k_u) - \boldsymbol{c}\frac{t}{u^{\alpha}}$ and $\bar{\boldsymbol{\mu}}_u := \mathbb{E}[\boldsymbol{\chi}_u(t)|A_{\widetilde{I},u}]$, $\bar{\boldsymbol{\Sigma}}_u := Var[\boldsymbol{\chi}_u(t)|A_{\widetilde{I},u}]$. Then we have that

(a) if $\widetilde{I} = \{i\}$ it holds

$$\bar{\boldsymbol{\mu}}_{u;i} = \frac{a_i}{T}tu^{-\alpha} + x_iO\left(u^{-3\alpha}\right), \ \bar{\boldsymbol{\mu}}_{u;\neg i} = \frac{-c_{\neg i}T + \rho(a_i + c_iT)}{T}tu^{-\alpha} + \rho x_iO\left(u^{-3\alpha}\right),$$

$$\bar{\boldsymbol{\Sigma}}_{u;i,i} = -tu^{-2\alpha} + O\left(u^{-4\alpha}\right), \ \bar{\boldsymbol{\Sigma}}_{u;\neg i,\neg i} = (1-\rho^2)T - (1-2\rho^2)tu^{-2\alpha} + O\left(u^{-4\alpha}\right), \ \bar{\boldsymbol{\Sigma}}_{u;\neg i,i} = -\rho tu^{-2\alpha} + O\left(u^{-4\alpha}\right).$$

(b) if $\widetilde{I}=\{1,2\}$ it holds

$$\bar{\boldsymbol{\mu}}_{u;1}=\frac{a_1}{T}tu^{-\alpha}+x_1O\left(u^{-3\alpha}\right),\ \bar{\boldsymbol{\mu}}_{u;2}=\frac{a_2}{T}tu^{-\alpha}+x_2O\left(u^{-3\alpha}\right),$$

$$\bar{\boldsymbol{\Sigma}}_{u;1,1}=-tu^{-2\alpha}+O\left(u^{-4\alpha}\right),\ \bar{\boldsymbol{\Sigma}}_{u;2,2}=-tu^{-2\alpha}+O\left(u^{-4\alpha}\right),\ \bar{\boldsymbol{\Sigma}}_{u;1,2}=-\rho tu^{-2\alpha}+O\left(u^{-4\alpha}\right).$$

Let $\boldsymbol{\chi}^*_u(t)=(\chi^*_{u;1}(t),\chi^*_{u;2}(t)):=\boldsymbol{v}_{\widetilde{I},u}\boldsymbol{\chi}_u(t)$ where $\boldsymbol{v}_{\widetilde{I},u}:=(v_{\widetilde{I},u;1},v_{\widetilde{I},u;2})^T$ with $v_{\widetilde{I},u;i}=\begin{cases}u^\alpha, & \text{if } i\in\widetilde{I}\\ 1, & \text{if } i\notin\widetilde{I}\end{cases}$.

(a) Suppose that $\widetilde{I}=\{i\}$. Then for sufficiently large u

$$\begin{aligned}\mathcal{H}_{\widetilde{I},u}(\Delta) &= \int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in E}:\boldsymbol{\chi}^*_u(t)>\boldsymbol{v}_{\widetilde{I},u}((\widetilde{\boldsymbol{I}}-\boldsymbol{M}_{\widetilde{I}})(\boldsymbol{a}+\boldsymbol{c}T)u^\alpha+\frac{\boldsymbol{M}_{\widetilde{I}}\boldsymbol{x}}{u^\alpha})\middle|A_{\widetilde{I},u}\right\}e^{\lambda_i x}dx\\ &\le \int_{\mathbb{R}_-}e^{\lambda_i x_i}dx_i+\int_{\mathbb{R}_+}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}:\chi^*_{u;i}(t)>x|A_{\widetilde{I},u}\right\}e^{\lambda_i x_i}dx_i<\infty, \qquad (2.19)\end{aligned}$$

where (2.19) follows from (2.18) and [66][Thm 8.1] with some constants $C,\widetilde{C}>0$. Combining weak convergence of $\boldsymbol{\chi}^*_u(t)$ with the dominated convergence theorem, we have that as $u\to\infty$

$$\begin{aligned}\mathcal{H}_{\widetilde{I},u}(\Delta) &\sim \int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}\ \begin{matrix}W_i(t)-\frac{a_i}{T}t>x_i\\ \sqrt{T}Z>((a_{\neg i}+c_{\neg i}T)-\rho(a_i+c_iT))u^\alpha+\rho x_i\end{matrix}\right\}e^{\lambda_i x_i}dx_i\\ &\sim C_{\widetilde{J},\widetilde{K}}\int_{\mathbb{R}}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}W_i(t)-\frac{a_i}{T}t>x_i\right\}e^{\lambda_i x_i}dx_i=\mathcal{H}_I(\Delta),\end{aligned}$$

where $Z\sim N(0,1)$ is independent of $(W_i(t):t\ge 0)$ and $C_{\widetilde{J},\widetilde{K}}=\begin{cases}\frac{1}{2}, & \text{if } \neg i\in\widetilde{K}\\ 1, & \text{if } \neg i\in\widetilde{J}\end{cases}$.

(b) Suppose that $\widetilde{I}=\{1,2\}$. Then for sufficiently large u

$$\begin{aligned}\mathcal{H}_{\widetilde{I},u}(\Delta) &= \int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{t\in E}:\boldsymbol{\chi}^*_u(t)>\boldsymbol{x}\middle|A_{\widetilde{I},u}\right\}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}\\ &\le \int_{\mathbb{R}_-}\int_{\mathbb{R}_-}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}+\int_{\mathbb{R}_-}\int_{\mathbb{R}_+}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}:\chi^*_{u;1}(t)>x_1|A_{\widetilde{I},u}\right\}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}\\ &\quad+\int_{\mathbb{R}_+}\int_{\mathbb{R}_-}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}:\chi^*_{u;2}(t)>x_2|A_{\widetilde{I},u}\right\}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}\\ &\quad+\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\mathbb{P}\left\{\exists_{t\in[0,\Delta]}:\chi^*_{u;1}(t)+\chi^*_{u;2}(t)>x_1+x_2|A_{\widetilde{I},u}\right\}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}\\ &\le \frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_2}\int_{\mathbb{R}_+}Ce^{-\widetilde{C}x_1^2}e^{\lambda_1 x_1}dx_1+\frac{1}{\lambda_1}\int_{\mathbb{R}_+}Ce^{-\widetilde{C}x_2^2}e^{\lambda_2 x_2}dx_2 \qquad (2.20)\\ &\quad+\int_{\mathbb{R}^2_+}Ce^{-\widetilde{C}(x_1+x_2)^2}e^{\boldsymbol{\lambda}\boldsymbol{x}^T}d\boldsymbol{x}<\infty, \qquad (2.21)\end{aligned}$$

where (2.20) follows from (2.18) and [66][Thm 8.1] with some constants $C, \widetilde{C} > 0$. Combining weak convergence of $\boldsymbol{\chi}_u^*(t)$ with the dominated convergence theorem, we have with self-similarity of Brownian motion that as $u \to \infty$

$$\mathcal{H}_{\widetilde{I},u}(\Delta) \sim \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\Delta]} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x} = \mathcal{H}_{\widetilde{I}}(\Delta).$$

This completes the proof. □

In the following lemma we prove that the constants introduced in Lemma 2.3.7 (iii) are finite and positive.

Lemma 2.3.8 *Let $I = \{1,2\}$, $t_0 > T$. Then*

$$\lim_{\Delta\to\infty} \mathcal{H}_{\widetilde{I},u}(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x} \in (0,\infty).$$

Proof of Lemma 2.3.8 Note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x}\\
\le\ & \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[i,(i+1)]} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x}\\
\le\ & \sum_{i=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[i,(i+1)]} W_1(t) - \frac{t}{T} > x_1, \exists_{t\in[i,(i+1)]} W_2(t) - \frac{a_2 t}{T} > x_2\right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x}\\
=\ & \frac{1}{\lambda_1\lambda_2} \sum_{i=0}^{\infty} \mathbb{E}\{e^{\lambda_1 M_i + \lambda_2 M_i^*}\},
\end{aligned}$$

where

$$(M_i, M_i^*) = \left(\sup_{t\in[i,(i+1)]} W_1(t) - \frac{t}{T}, \sup_{t\in[i,(i+1)]} W_2(t) - \frac{a_2 t}{T}\right).$$

Using independence of increments of Brownian motion we obtain the following equality in distribution

$$\begin{aligned}
(M_i, M_i^*) &=^d \left(\sup_{t\in[0,1]} W_1(t) - \frac{t}{T}, \sup_{t\in[0,1]} W_2(t) - \frac{a_2 t}{T}\right) + (V_1(i) - \frac{i}{T}, V_2(i) - \frac{a_2 i}{T})\\
&=: (Q_1, Q_2) + (V_1(i) - \frac{i}{T}, V_2(i) - \frac{a_2 i}{T}),
\end{aligned}$$

with (V_1, V_2) an independent copy of (W_1, W_2). Hence

$$\sum_{i=0}^{\infty} \mathbb{E}\{e^{\lambda_1 M_i + \lambda_2 M_i^*}\} = \sum_{i=0}^{\infty} \mathbb{E}\{e^{\lambda_1 Q_1 + \lambda_2 Q_2}\} \mathbb{E}\{e^{\lambda_1 (V_1(i) - \frac{i}{T}) + \lambda_2 (V_2(i) - \frac{a_2 i}{T})}\}$$

$$
\begin{aligned}
&= \mathbb{E}\{e^{\lambda_1 Q_1+\lambda_2 Q_2}\}\sum_{i=0}^{\infty} e^{i(-\frac{\lambda_1-a_2\lambda_2}{T}+\frac{(\lambda_1+\rho\lambda_2)^2}{2}+(1-\rho^2)\frac{\lambda_2^2}{2})}\\
&= \mathbb{E}\{e^{\lambda_1 Q_1+\lambda_2 Q_2}\}\sum_{i=0}^{\infty} e^{-\frac{i}{2}\kappa},
\end{aligned}
$$

where $\kappa = \frac{2\lambda_1+2a_2\lambda_2}{T} - (\lambda_1+\rho\lambda_2)^2 - (1-\rho^2)\lambda_2^2$. Straightforward calculations give $\lambda_1+\rho\lambda_2 = \frac{1+c_1T}{T}$ and

$$
\kappa \;=\; \frac{(1-2a_2\rho+a_2^2)-(c_1^2-2\rho c_1c_2+c_2^2)T^2}{T^2(1-\rho^2)} > 0 \text{ iff } t_0 > T.
$$

Thus

$$
\int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x} \le \frac{1}{\lambda_1\lambda_2}\mathbb{E}\{e^{\lambda_1 Q_1+\lambda_2 Q_2}\}\frac{e^{\frac{\kappa}{2}}}{e^{\frac{\kappa}{2}}-1} < \infty.
$$

Applying Lebesgue's monotone convergence theorem we obtain

$$
\lim_{\Delta\to\infty} I(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x} \in (0,\infty).
$$

□

In the following lemma we show that if $t_0 < T$, then the behaviour of the asymptotics closely resembles those of the infinite interval.

Lemma 2.3.9 *If $t_0 < T$, then as $u \to \infty$*

$$
p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \;\sim\; \mathbb{P}\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\}.
$$

Proof of Lemma 2.3.9 Using self-similarity of Brownian motion we have

$$
p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \;=\; \mathbb{P}\{\exists_{t\in[0,Tu^{2\alpha}]} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\}.
$$

Clearly we have that

$$
p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \;\le\; \mathbb{P}\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\}.
$$

On the other hand, by the self-similarity of Brownian motion

$$
\begin{aligned}
p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) &\ge \mathbb{P}\{\exists_{t\in[u^{2\alpha}t_0-u^{\alpha}\log(u^{\alpha}),\,u^{2\alpha}t_0+u^{\alpha}\log(u^{\alpha})]} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\}\\
&= \mathbb{P}\Big\{\exists_{t\in[t_0-\frac{\log(u^{\alpha})}{u^{\alpha}},\,t_0+\frac{\log(u^{\alpha})}{u^{\alpha}}]} \boldsymbol{W}(t) - \boldsymbol{c}u^{\alpha}t > \boldsymbol{a}u^{\alpha}\Big\}.
\end{aligned}
$$

Let us denote $E_u := [t_0 - \frac{\log(u^\alpha)}{u^\alpha}, t_0 + \frac{\log(u^\alpha)}{u^\alpha}]$. From Lemma 2.3.6 we obtain that for some $C > 0$

$$\mathbb{P}\left\{\exists_{t\in[0,T]\setminus E_u} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \quad \leq \quad Ce^{-\frac{\tau}{2}\log^2(u^\alpha)}e^{-\frac{q(t_0)}{2}u^{2\alpha}}.$$

Hence again with self-similarity of Brownian motion we have that

$$\mathbb{P}\left\{\exists_{t\in[0,\infty)\setminus u^{2\alpha}E_u} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\right\} \quad = \quad o\left(\mathbb{P}\left\{\exists_{t\in u^{2\alpha}E_u} \boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{a}u^{2\alpha}\right\}\right).$$

This completes the proof. □

Proof of Theorem 2.3.3

Case (i): $t_0 > T$: Observe that

$$p_{\alpha,\alpha,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) \quad \geq \quad \mathbb{P}\left\{\exists_{t\in[T-\frac{\Delta}{u^{2\alpha}},T]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} =: \Pi(u).$$

On the other hand, we have

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \leq \mathbb{P}\left\{\exists_{t\in[0,T-\frac{\log(u^\alpha)}{u^\alpha}]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\}$$
$$+\mathbb{P}\left\{\exists_{t\in[T-\frac{\log(u^\alpha)}{u^\alpha},T-\frac{\Delta}{u^{2\alpha}}]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} + \Pi(u) =: \mathcal{P}_1(u) + \mathcal{P}_2(u) + \Pi(u).$$

By combination of Lemma 2.3.6 with Lemma 2.3.7 we get $\mathcal{P}_1(u) = o(\Pi(u))$, as $u \to \infty$.

Now, we calculate the upper bound for $\mathcal{P}_2(u)$. Let $N_u := \lfloor \frac{u^\alpha \log(u^\alpha)}{\Delta} \rfloor$.

We divide the rest of the proof into two cases: (a) $I = \{i\}$ and (b) $I = \{1, 2\}$ which require slightly different approaches.

(a) Suppose that $I = \{i\}$. Then

$$\mathcal{P}_2(u) \quad \leq \quad \sum_{k=2}^{N_u} \mathbb{P}\left\{\exists_{t\in[T-\frac{k\Delta}{u^{2\alpha}},T-\frac{(k-1)\Delta}{u^{2\alpha}}]} Z_{I(t)}(t) > u^\alpha\right\} =: \sum_{k=2}^{N_u} p_{2,k}(u).$$

From Lemma A.4 in [16] it follows that for some $\tilde{I}$ and all $t \in (T - \frac{\log(u^\alpha)}{u^\alpha}, T - \frac{\Delta}{u^{2\alpha}})$

$$q(t) = \frac{1}{t}\boldsymbol{c}_{\tilde{I}}^T(\Sigma_T^{-1})_{\tilde{I}\tilde{I}}c_{\tilde{I}} + 2\boldsymbol{c}_{\tilde{I}}^T(\Sigma_T^{-1})_{\tilde{I}\tilde{I}}a_{\tilde{I}} + \boldsymbol{a}_{\tilde{I}}^T(\Sigma_T^{-1})_{\tilde{I}\tilde{I}}a_{\tilde{I}}t.$$

Furthermore, for all $s, t \in (T - \frac{\log(u^\alpha)}{u^\alpha}, T - \frac{\Delta}{u^{2\alpha}})$ and some $C_1 > 0$ it holds

$$\mathbb{E}[(Z_{\tilde{I}}(t) - Z_{\tilde{I}}(s))^2] \leq C_1|t-s|.$$

Thus, by the Piterbarg inequality (see e.g. Lemma 5.1 in [19]) we conclude that

$$p_{2,k}(u) \quad \leq \quad \frac{\Delta}{u^{2\alpha}}u^\alpha e^{-\frac{u^{2\alpha}}{2}\inf_{t\in[T-\frac{k\Delta}{u^{2\alpha}},T-\frac{(k-1)\Delta}{u^{2\alpha}}]} q(t)}$$

holds for all large u. Using Lemma 2.3.5 and Taylor expansion of $q(t)$ we obtain for some $\tau > 0$

$$\inf_{t\in[T-\frac{k\Delta}{u^\alpha},T-\frac{(k-1)\Delta}{u^\alpha}]} q(t) = q(T) + \tau k \frac{\Delta}{2\alpha}.$$

Hence, we obtain for sufficiently large u

$$p_{2,k}(u) \le u^{-\alpha} e^{-\frac{q(T)}{2}} \Delta e^{-\frac{\tau\Delta k}{2}}.$$

Thus

$$\mathcal{P}_2(u) \;\le\; \sum_{k=2}^{N_u} p_{2,k}(u) \le u^{-\alpha} e^{-\frac{q(T)}{2}} \Delta \sum_{k=2}^{N_u} e^{-\frac{\tau\Delta k}{2}} = u^{-\alpha} e^{-\frac{2q(T)}{2}} \Delta e^{-\frac{\tau\Delta}{2}} \frac{1}{1-e^{-\frac{\tau\Delta}{2}}},$$

with $\tau = -q'(T) > 0$. Lemma 2.3.7 gives $\mathcal{P}_2(u) = O(\Pi(u))$, as $u \to \infty$, and letting $\Delta \to \infty$ we have

$$\lim_{\Delta\to\infty}\lim_{u\to\infty} \frac{\mathcal{P}_2(u)}{\Pi(u)} = \lim_{\Delta\to\infty} \tilde{C} e^{-\frac{\tau\Delta}{2}} = 0.$$

(b) Suppose that $I = \{1,2\}$. Then

$$\mathcal{P}_2(u) \;\le\; \sum_{k=1}^{N_u} \mathbb{P}\Big\{\exists_{t\in[T-\frac{k\Delta}{u^{2\alpha}},T-\frac{(k-1)\Delta}{u^{2\alpha}}]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\Big\}.$$

Using Lemma 2.3.7 (iii) we obtain

$$\begin{aligned}\mathcal{P}_2(u) \;&\le\; C\mathcal{H}_I(\Delta) u^{-2\alpha} \varphi_T((a_1+c_1T)u^\alpha, (a_2+c_2T)u^\alpha) \sum_{k=2}^{N_u} e^{\frac{q'(T)}{2}(k-1)\Delta} \\ &\le\; C\mathcal{H}_I(\Delta) u^{-2\alpha} \varphi_T((a_1+c_1T)u^\alpha, (a_2+c_2T)u^\alpha) e^{-\frac{\tau\Delta}{2}} \frac{1}{1-e^{-\frac{\tau\Delta}{2}}},\end{aligned}$$

with $\tau = -q'(T) > 0$. Lemma 2.3.7 gives $\mathcal{P}_2(u) = O(\Pi(u))$, as $u \to \infty$ and letting $\Delta \to \infty$ we have

$$\lim_{\Delta\to\infty}\lim_{u\to\infty} \frac{\mathcal{P}_2(u)}{\Pi(u)} = \lim_{\Delta\to\infty} \tilde{C} e^{-\frac{\tau\Delta}{2}} = 0.$$

This completes the proof of case (i).

Case (ii): $t_0 = T$. For $\Delta > 0$ let $k_u = T - \frac{(k-1)\Delta}{u^{2\alpha}}$, $E_{u,k} = [(k+1)_u, k_u]$ and $N_u = \lfloor \frac{u^\alpha \log(u^\alpha)}{\Delta} \rfloor$. Let denote

$$p_k(u) = \mathbb{P}\{\exists_{t\in E_{k,u}} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\} \text{ and } p_{k,l}(u) = \mathbb{P}\left\{\begin{array}{c} \exists_{t\in E_{k,u}} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha \\ \exists_{s\in E_{l,u}} \boldsymbol{W}(s) - \boldsymbol{c}u^\alpha s > \boldsymbol{a}u^\alpha \end{array}\right\}.$$

Observe that

$$S_1(u) - S_2(u) := \sum_{k=1}^{N_u} p_k(u) - \sum_{k=1}^{N_u} \sum_{l=k+1}^{N_u} p_{k,l}(u) \le p_{\alpha,\alpha,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \le S_1(u) + p_{N_u+1}(u).$$

The analysis of the sums $S_1(u)$ and $S_2(u)$ follows in a similar way to the proof of Theorem 3.1 in [16]. The only difference is in the number of components in the sums. In our study, $S_1(u)$ consists of $p_k(u)$ for $k \in \{1, \dots, N_u\}$, whereas in the proof of Theorem 3.1 $k \in \{-N_u, \dots, N_u\}$. $S_2(u)$ remains the same in both cases. For this reason, we omit the details of the proof.

Case (iii): $t_0 < T$. The combination of Lemma 2.3.9 with Theorem 3.1 in [16] straightforwardly gives the thesis. □

2.4 Case $\alpha > \beta$.

In this case the drift increases at a faster rate than the initial capital, which would suggest that for sufficiently large u it ultimately dominates. However, the above intuitive approach is not correct. We prove that this case simplifies to scenarios analyzed in case $\alpha = \beta$ with the speed parameter $\frac{\alpha+\beta}{2}$ and $T = \infty$ that was considered in Theorems 2.3.2 and 2.3.3; see also [16] [Theorem 3.1].

Theorem 2.4.1 *Let $\alpha > \beta$. Then as $u \to \infty$*

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \sim p_{\frac{\alpha+\beta}{2},\frac{\alpha+\beta}{2},\rho,\infty}(\boldsymbol{c}, \boldsymbol{a}, u).$$

2.4.1 Proof of Theorem 2.4.1

We prove that, as $u \to \infty$,

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \quad \sim \quad \mathbb{P}\left\{\exists_{t\in[0,\infty)} W_1(t) - c_1 t > u^{\alpha+\beta}, W_2(t) - c_2 t > a_2 u^{\alpha+\beta}\right\}.$$

Using self-similarity of Brownian motion, we obtain

$$\begin{aligned} p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) &= \mathbb{P}\left\{\exists_{t\in[0,Tu^{2\alpha}]} W_1(t) - c_1 t > u^{\alpha+\beta}, W_2(t) - c_2 t > a_2 u^{\alpha+\beta}\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[0,Tu^{\alpha-\beta}]} W_1(t) - c_1 u^{\frac{\alpha+\beta}{2}} t > u^{\frac{\alpha+\beta}{2}}, W_2(t) - c_2 u^{\frac{\alpha+\beta}{2}} t > a_2 u^{\frac{\alpha+\beta}{2}}\right\}. \end{aligned}$$

Let $u' = u^{\alpha+\beta}$. Clearly we have that

$$p_{\alpha,\beta,\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) \quad \le \quad \mathbb{P}\left\{\exists_{t\in[0,\infty)} W_1(t) - c_1 t > u', W_2(t) - c_2 t > a_2 u'\right\}$$

$$
\begin{aligned}
&\leq \mathbb{P}\Big\{\exists_{t\in[0,\infty)\setminus[u't_{u'0}-\sqrt{u'}\log(u'),u't_{u'0}+\sqrt{u'}\log(u')]} W_1(t)-c_1t>u', W_2(t)-c_2t>a_2u'\Big\}\\
&\quad+\mathbb{P}\Big\{\exists_{t\in[u't_{u'0}-\sqrt{u'}\log(u'),u't_{u'0}+\sqrt{u'}\log(u')]} W_1(t)-c_1t>u', W_2(t)-c_2t>a_2u'\Big\}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
p_{\alpha,\beta,\rho,T}(\boldsymbol{c},\boldsymbol{a},u) &\geq \mathbb{P}\Big\{\exists_{t\in[u't_{u'0}-\sqrt{u'}\log(u'),u't_{u'0}+\sqrt{u'}\log(u')]} W_1(t)-c_1t>u', W_2(t)-c_2t>a_2u'\Big\}\\
&= \mathbb{P}\left\{\exists_{t\in[t_{u'0}-\frac{\log(u')}{\sqrt{u'}},t_{u'0}+\frac{\log(u')}{\sqrt{u'}}]} W_1(t)-c_1\sqrt{u'}t>\sqrt{u'}, W_2(t)-c_2\sqrt{u'}t>a_2\sqrt{u'}\right\}.
\end{aligned}
$$

Denote $E_{u'} := [t_{u'0}-\frac{\log(u')}{\sqrt{u'}}, t_{u'0}+\frac{\log(u')}{\sqrt{u'}}]$. From Lemma 2.2.3 with $f(u')=\frac{\log(u')}{\sqrt{u'}}$ we obtain that for some $C>0$

$$
\mathbb{P}\Big\{\exists_{t\in[0,\infty)\setminus E_{u'}} W_1(t)-c_1\sqrt{u'}t>\sqrt{u'}, W_2(t)-c_2\sqrt{u'}t>a_2\sqrt{u'}\Big\} \leq Ce^{-\frac{\tau}{2}\log^2(u')}e^{-\frac{q^*_{\boldsymbol{a}}(T)}{2}u'}.
$$

With self-similarity of Brownian motion we have that

$$
\mathbb{P}\left\{\exists_{t\in[0,\infty)\setminus u'E_{u'}} \begin{array}{c} W_1(t)-c_1t>u' \\ W_2(t)-c_2t>a_2u' \end{array}\right\} = o\left(\mathbb{P}\left\{\exists_{t\in u'E_{u'}} \begin{array}{c} W_1(t)-c_1t>u' \\ W_2(t)-c_2t>a_2u' \end{array}\right\}\right).
$$

This completes the proof. □

Chapter 3

Finite time ruin probability for subordinated fractional Brownian motion

3.1 Introduction

Consider the risk process $R(t) := u + ct - X(t)$ where the risk component is modelled by a centered Gaussian process with stationary increments, with initial capital u and constant premium rate c. Since Brownian motion appears naturally as a limiting process, many studies were focusing on behaviour of Brownian motion driven risk process. The ruin probability of the one dimensional risk process in finite time horizeon is known and equal to

$$\mathbb{P}\left\{\inf_{t\in[0,T]} : R(t) < 0\right\} \;=\; \Phi\left(-\frac{u}{\sqrt{T}} - c_i\sqrt{T}\right) + e^{-2c_i u}\Phi\left(-\frac{u}{\sqrt{T}} + c_i\sqrt{T}\right).$$

Definitions of ruin of the risk process above have been extended in multiple directions, e.g. discrete approach in [48], two-dimensional setup in [18] or infinite time horizon in [23]. A related problem would be to investigate asymptotics of the ruin of the risk process for different kinds of process X. For example, in [60] $X(t)$ is modelled by a fractional Brownian motion $B_H(t), H \in (0,1)$. In order to represent the inspection times we introduce $X_i := \sum_{j=1}^{i} Z_j$, where $Z_j, j \geq 1$ are non-negative independent identically distributed random variables and then examine the behaviour of $R(X_i), i = 1, 2$. Let

$$\pi_{[0,T],H}(u) := \mathbb{P}\left\{\sup_{i\geq 0, X_i\in[0,T]} : B_H(X_i) - cX_i > u\right\}. \tag{3.1}$$

where $T > 0$. We investigate $\pi_{[0,T],H}(u)$, which models the behaviour of the risk process evaluated at random points in a finite interval. Note that if $X_1 > T$, then there are no random points in the $[0, T]$ interval and the probability above is equal to 0. The idea of subordination of Gaussian processes has been implemented in e.g. [39, 41, 70].

3.2 Main results

Let $\tau = \tau_1 := \sup\{i : X_i \leq T\}, \tau_i = \sup\{i : X_i < \tau_{i-1}\}$ denote the numeration of inspection times. We assume that Z_1 has continuous density function f_Z with $f_Z(0) \in (0, \infty)$, which allows us to formulate the following proposition. Its proof is postponed to the next section.

Proposition 3.2.1 *If Z_1 has continuous density function f_Z with $f_Z(0) \in (0, \infty)$, then X_τ has continuous density function f_{X_τ} with $f_{X_\tau}(T) \in (0, \infty)$, for any $T \in (0, \infty)$.*

The next theorem constitutes the main finding of this chapter.

Theorem 3.2.2 *Let Z_1 have continuous density f_Z with $f_Z(0) \in (0, \infty)$. Then as $u \to \infty$*

$$\pi_{[0,T],H}(u) \sim \frac{f_{X_\tau}(T)T^{2H}}{H} u^{-2} \Psi\left(\frac{u + cT}{T^H}\right).$$

Recall that from [34] for large enough u we have that in the continuous time setting the asymptotics of the ruin probability is driven by the variability at the end of $[0, T]$ interval. Similar behaviour is observed in this chapter. The asymptotics is driven by the probability of ruin at the end of the interval. Since the density f_{X_τ} is independent on u, the most important factor is the variance of the fractional Brownian motion itself and the random inspection times only contribute to the constant. We present below a sample usage of the theorem above to the classical case of Poisson process.

Corollary 3.2.3 *Suppose that Z_i are exponentially distributed with parameter $\lambda > 0$. Then*

$$\pi_{[0,T],H}(u) \sim \frac{e^{-\lambda(T-1)}\lambda T^{2H+1}}{H} u^{-2} \Psi\left(\frac{u + cT}{T^H}\right).$$

Proof of Corollary 3.2.3 In order to obtain the closed form of the asymptotics we have to calculate $f_{X_\tau}(T)$. We have that

$$f_{X_\tau}(T) = \sum_{k=1}^{\infty} f_{X_k}(T)\mathbb{P}\{N = k\},$$

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where since Z_i are exponential, N has a Poisson distribution with parameter $T\lambda$. With [68] [Thm 5.2] we have that X_k conditioned on k point in the interval $[0,T]$ has distribution as $\max(U_1,\dots,U_k)$ with U_i distributed uniformly on $[0,T]$. Therefore

$$\begin{aligned} f_{X_\tau}(T) &= \sum_{k=1}^{\infty} kT^{k-1}\frac{e^{-\lambda T}(\lambda T)^k}{k!} \\ &= T\frac{e^{-\lambda T}}{e^{-\lambda}}\sum_{k=1}^{\infty} k\frac{e^{-\lambda}(\lambda)^k}{k!} \\ &= Te^{-\lambda(T-1)}\lambda, \end{aligned}$$

which combined with Theorem 3.2.2 completes the proof. □

3.3 Proofs

Proof of Proposition 3.2.1 Let $\delta > 0$. Since f_Z is continuous with $f_Z(0) \in (0,\infty)$ we have that there exists $\epsilon_\delta > 0$ such that $\forall_{t\in[0,\epsilon_\delta]} f_Z(t) > \delta$. Let $k > \lfloor \frac{2T}{\epsilon_\delta} \rfloor$. Additionally, since for large enough k we have $\mathbb{P}\{\tau = k\} > 0$, then we pick k, which fulfills both conditions. Then we have with

$$\mathcal{B}_k = \{(x_1,\dots,x_k) : 0 < x_2 < \epsilon_\delta, \dots, 0 < x_k < \epsilon_\delta, 0 < T - \sum_{l=2}^{k} x_l < \epsilon_\delta\}$$

$$\begin{aligned} f_{X_k}(T) &= f_{Z_1+Z_2+\dots+Z_k}(T) \\ &= \int_0^T f_{Z_1+Z_2+\dots+Z_{k-1}}(T-x_k) f_Z(x_k)dx_k \\ &= \int_0^T \int_0^{T-x_k} \cdots \int_0^{T-\sum_{l=2}^k x_l} f_Z(T-\sum_{l=2}^{k} x_l) f_Z(x_2)\dots f_Z(x_k)dx_2\dots dx_k \\ &\geq \int_{\mathcal{B}_k} \delta^k dx_2\dots dx_k = \delta^k vol(\mathcal{B}_k). \end{aligned}$$

Since $k > \lfloor \frac{2T}{\epsilon_\delta} \rfloor$, then $vol(\mathcal{B}_k) > 0$ and together with $\mathbb{P}\{\tau = k\} > 0$ we have that

$$f_{X_\tau}(T) > \delta^k vol(\mathcal{B}_k)\mathbb{P}\{\tau = k\} > 0.$$

Furthermore, notice that from the continuity assumption together with finitness of $f_Z(0)$ we have $f_Z(x) < M$ for all $x \in [0,T]$ and some constant $M > 0$. Therefore $f_{X_1}(T) < \infty$ and for any $k > 1$

$$f_{X_k}(T) = f_{Z_1+Z_2+\dots+Z_k}(T)$$

$$\begin{aligned} &= \int_0^T f_{Z_1+Z_2+\cdots+Z_{k-1}}(T-x_k) f_Z(x_k) dx_k \\ &< M \int_0^T f_{Z_1+Z_2+\cdots+Z_{k-1}}(T-x_k) dx_k \le M. \end{aligned}$$

With the above we have

$$\begin{aligned} f_{X_\tau}(T) &= \sum_{k=1}^{\infty} f_{X_k}(T)\mathbb{P}\{\tau = k\} \\ &< M \sum_{k=1}^{\infty} \mathbb{P}\{\tau = k\} \\ &= M < \infty. \end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 3.2.2 Since the properties of fractional Brownian motion are vastly different for $H \ge \frac{1}{2}$ and $H < \frac{1}{2}$ the proof is divided onto two parts. Let $\mathcal{T}_u = [t_u, T]$ for $t_u = T(1 - \log^2(u)u^{-2})$.

Case: $H \ge \frac{1}{2}$. We have that

$$\pi_{[0,T],H}(u) \ge \pi_{\mathcal{T}_u,H}(u)$$

and

$$\pi_{[0,T],H}(u) \le \pi_{\mathcal{T}_u,H}(u) + \pi_{[0,T]\setminus\mathcal{T}_u,H}(u).$$

We begin with investigating the behaviour of $\pi_{\mathcal{T}_u,H}(u)$. Notice that

$$\pi_{\mathcal{T}_u,H}(u) \ge \mathbb{P}\{B_H(X_\tau) - cX_\tau > u, X_\tau \in \mathcal{T}_u\}.$$

With Proposition 3.2.1 we have

$$\begin{aligned} \mathbb{P}\{B_H(X_\tau) - cX_\tau > u, X_\tau \in \mathcal{T}_u\} &= \int_0^{T\frac{\log^2(u)}{u^2}} \mathbb{P}\{B_H(t+t_u) - c(t+t_u) > u\} f_{X_\tau}(t+t_u) dt \\ &\sim f_{X_\tau}(T) \int_0^{T\frac{\log^2(u)}{u^2}} \mathbb{P}\{B_H(t+t_u) - c(t+t_u) > u\} dt \\ &=: f_{X_\tau}(T) I_{\mathcal{T}_u}. \end{aligned}$$

Notice that uniformly for $t \in [0,1]$ as $u \to \infty$

$$\begin{aligned} \log\left(\mathbb{P}\left\{B_H(t+t_u) - c(t+t_u) > u + \frac{1}{\sqrt{u}}\right\}\right) &= \log\left(\mathbb{P}\left\{B_H(1) > \frac{u + \frac{1}{\sqrt{u}} + c(t+t_u)}{(t+t_u)^H}\right\}\right) \\ &= -\frac{1}{2}\left(\frac{u + \frac{1}{\sqrt{u}} + c(t+t_u)}{(t+t_u)^H}\right)^2 + O(1) \end{aligned}$$

$$= -\frac{1}{2}\left(\frac{u+c(t+t_u)}{(t+t_u)^H}\right)^2 - \frac{\sqrt{u}}{(t+t_u)^{2H}} + O(1).$$

Therefore uniformly for $t \in [0,1]$ as $u \to \infty$

$$\mathbb{P}\left\{B_H(t+t_u) - c(t+t_u) > u + \frac{1}{\sqrt{u}}\right\} = o(\mathbb{P}\{B_H(t+t_u) - c(t+t_u) > u\}).$$

Using Taylor expansion and the above, we have as $u \to \infty$

$$\begin{aligned}
I_{\mathcal{T}_u} &= \int_0^{\frac{\log^2(u)}{u^2}} \int_u^{\infty} \frac{1}{\sqrt{2\pi T^{2H}(1-t)^{2H}}} e^{-\frac{(x+cT(1-t))^2}{2T^{2H}(1-t)^{2H}}} dxdt \\
&\sim \int_0^{\frac{\log^2(u)}{u^2}} \int_u^{u+\frac{1}{\sqrt{u}}} \frac{1}{\sqrt{2\pi T^{2H}(1-t)^{2H}}} e^{-\frac{(x+cT(1-t))^2}{2T^{2H}(1-t)^{2H}}} dxdt \\
&= \int_0^{\frac{\log^2(u)}{u^2}} \int_u^{u+\frac{1}{\sqrt{u}}} \frac{1}{\sqrt{2\pi T^{2H}(1-t)^{2H}}} e^{-\frac{(x+cT)^2}{2T^{2H}}} e^{-\frac{Htx^2}{T^{2H}}} e^{O(t^2x^2+tx)} dxdt \\
&\le \int_0^{\frac{\log^2(u)}{u^2}} \int_u^{u+\frac{1}{\sqrt{u}}} \frac{1}{\sqrt{2\pi T^{2H}(1-t)^{2H}}} e^{-\frac{(x+cT)^2}{2T^{2H}}} e^{-\frac{Htu^2}{T^{2H}}} dxdt \\
&\sim \int_0^{\frac{\log^2(u)}{u^2}} \frac{1}{\sqrt{2\pi T^{2H}}} e^{-\frac{(x+cT)^2}{2T^{2H}}} dx \int_0^{\log^2(u)} e^{-\frac{Ht}{T^{2H}}} dt \\
&\sim \mathbb{P}\{B_H(T) - cT > u\} u^{-2} \int_0^{\log^2(u)} e^{-\frac{Ht}{T^{2H}}} dt \\
&= \frac{T^{2H}}{H}(1 - e^{-\log^2(u)HT^{-2H}}) u^{-2} \mathbb{P}\{B_H(T) - cT > u\} \\
&\sim \frac{T^{2H}}{H} u^{-2} \mathbb{P}\{B_H(T) - cT > u\}.
\end{aligned}$$

Similarly, by substituting $x = u + \frac{1}{\sqrt{u}}$, we obtain for large enough u

$$\begin{aligned}
I_{\mathcal{T}_u} &\ge \int_0^{\frac{\log^2(u)}{u^2}} \int_u^{u+\frac{1}{\sqrt{u}}} \frac{1}{\sqrt{2\pi T^{2H}(1-t)^{2H}}} e^{-\frac{(x+cT)^2}{2T^{2H}}} e^{-\frac{Ht(u+\frac{1}{\sqrt{u}})^2}{T^{2H}}} dxdt \\
&\sim \mathbb{P}\{B_H(T) - cT > u\} u^{-2} \int_0^{\log^2(u)} e^{-\frac{Ht}{T^{2H}}} e^{-\frac{Ht(\frac{2}{\sqrt{u^3}}+\frac{1}{u^3})}{T^{2H}}} dt \\
&\sim \frac{T^{2H}}{H}(1 - e^{-\log^2(u)HT^{-2H}}) u^{-2} \mathbb{P}\{B_H(T) - cT > u\} \\
&\sim \frac{T^{2H}}{H} u^{-2} \mathbb{P}\{B_H(T) - cT > u\}.
\end{aligned}$$

Hence

$$\lim_{u\to\infty} \frac{\mathbb{P}\{B_H(X_\tau) - cX_\tau > u, X_\tau \in \mathcal{T}_u\}}{f_{X_\tau}(T) T^{2H} H^{-1} u^{-2} \mathbb{P}\{B_H(T) - cT > u\}} = 1.$$

On the other hand

$$\begin{aligned}
\pi_{\mathcal{T}_u,B_H}(u) &\leq \sum_{k=1}^{\infty}\mathbb{P}\left\{\sup_{i\in\{1,\dots,k\}} B_H(X_{\tau_i})-cX_{\tau_i}>u, X_{\tau_k}\in\mathcal{T}_u\right\}\\
&\sim f_{X_\tau}(T)I_{\mathcal{T}_u}+\sum_{k=2}^{\infty}\mathbb{P}\left\{\sup_{i\in\{1,\dots,k\}} B_H(X_{\tau_i})-cX_{\tau_i}>u, X_{\tau_k}\in\mathcal{T}_u\right\}\\
&:= f_{X_\tau}(T)I_{\mathcal{T}_u}+S_u.
\end{aligned}$$

Further we have with $(Z_2',...)$ i.i.d. with the same distribution as Z_1

$$\begin{aligned}
S_u &\leq \sum_{k=2}^{\infty}\mathbb{P}\left\{\sup_{t\in\mathcal{T}_u} B_H(t)-ct>u, X_{\tau_k}\in\mathcal{T}_u\right\}\\
&\leq \mathbb{P}\left\{\sup_{t\in[0,T]} B_H(t)-ct>u\right\}\sum_{k=2}^{\infty}\mathbb{P}\{X_{\tau_k}\in\mathcal{T}_u\}\\
&\leq \mathbb{P}\left\{\sup_{t\in[0,T]} B_H(t)-ct>u\right\}\sum_{k=2}^{\infty}\mathbb{P}\left\{X_\tau\in\mathcal{T}_u, Z_2'<\frac{\log^2(u)}{u^2},\dots,Z_k'<\frac{\log^2(u)}{u^2}\right\}.
\end{aligned}$$

By the assumption that Z_i have continuous density function f_Z with $f_Z(0)\in(0,\infty)$ and Proposition 3.2.1 we have that for large enough u

$$\begin{aligned}
&\sum_{k=2}^{\infty}\mathbb{P}\left\{X_\tau\in\mathcal{T}_u, Z_2'<\frac{\log^2(u)}{u^2},\dots,Z_k'<\frac{\log^2(u)}{u^2}\right\}\\
&\sim \frac{T\log^2(u)}{u^2}f_{X_\tau}(T)\sum_{k=2}^{\infty}\mathbb{P}\left\{Z_2'<\frac{T\log^2(u)}{u^2}\right\}\dots\mathbb{P}\left\{Z_k'<\frac{T\log^2(u)}{u^2}\right\}\\
&\sim \frac{T\log^2(u)}{u^2}f_{X_\tau}(T)\sum_{k=1}^{\infty}\left(\frac{T\log^2(u)}{u^2}f_Z(0)\right)^k\\
&= \frac{T\log^2(u)}{u^2}f_{X_\tau}(T)\frac{\frac{T\log^2(u)}{u^2}f_Z(0)}{1-\frac{T\log^2(u)}{u^2}f_Z(0)}\\
&\sim f_{X_\tau}(T)f_Z(0)\frac{T^2\log^4(u)}{u^4}.
\end{aligned}$$

Since for $H\geq\frac{1}{2}$ from [35] [Prop 3.1] we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} B_H(t)-ct>u\right\}\sim C\mathbb{P}\{B_H(T)-cT>u\}$$

for some $C>0$ as $u\to\infty$, then for some $C'>0$

$$\lim_{u\to\infty}\frac{S_u}{I_{\mathcal{T}_u}} = \lim_{u\to\infty}C'\frac{\log^4(u)}{u^2}=0.$$

Therefore as $u\to\infty$

$$\pi_{\mathcal{T}_u,H}(u)\sim\frac{T^{2H}}{H}f_{X_\tau}(T)u^{-2}\mathbb{P}\{B_H(T)-cT>u\}.$$

Note that [35] [Prop 3.1] holds also for $T = t_u$, since condition A, C and D therein are satisfied by the same argument based on properties of fractional Brownian motion and condition B holds with $b = \frac{H}{T}$ since $t_u \to T$. Therefore we have that for some $C > 0$ and large enough u

$$\begin{aligned}
\pi_{[0,T]\setminus\mathcal{T}_u,H}(u) &\leq P(\sup_{t\in[0,T]\setminus\mathcal{T}_u} B_H(t) - ct > u) \\
&\sim CP(B_H(t_u) - ct_u > u) \\
&\sim C\frac{1}{\sqrt{2\pi}t_u^H}e^{-\frac{(u+ct_u)^2}{2t_u^{2H}}} \\
&\sim C\frac{1}{\sqrt{2\pi}T^H}e^{-\frac{(u+cT)^2}{2T^{2H}}}e^{-\frac{2H\log^2(u)}{u^2}\frac{(u+cT)^2}{2T^{2H}}+o\left(\frac{1}{u}\right)} \\
&\sim C\mathbb{P}\{B_H(T) - cT > u\}e^{-\frac{H}{T^{2H}}\log^2(u)} \\
&= o(u^{-2}\mathbb{P}\{B_H(T) - cT > u\}).
\end{aligned}$$

Hence we have

$$\pi_{[0,T],H}(u) \sim \frac{T^{2H}}{H} f_{X_\tau}(T) u^{-2}\mathbb{P}\{B_H(T) - cT > u\},$$

which completes the proof of case $H \geq \frac{1}{2}$.

Case: $H < \frac{1}{2}$. Due to negative correlation of increments of fractional Brownian motion for $H < \frac{1}{2}$ we need a different approach then used for the case $H \geq \frac{1}{2}$. For $\Delta > 0$ denote $\mathcal{T}_{u,l} := [t_u^{(l+1)}, t_u^{(l)}]$, $N_u = \frac{\log^2(u)u^{\frac{1-2H}{H}}}{\Delta}$, $t_u^{(l)} = T(1-(l-1)\Delta u^{-\frac{1}{H}})$ and $\tau_i^{(l)} = \sup(i : X_i \leq \tau_{i-1}^{(l)})$, $\tau^{(l)} = \sup(i : X_i \leq t_u^{(l)})$, $l = 1 \ldots N_u$. Notice that if Z_1 has continuous density function f_Z with $f_Z(0) \in (0, \infty)$, then $X_{\tau^{(l)}}$ has continuous density function $f_{X_{\tau^{(l)}}}$ with $f_{X_{\tau^{(l)}}}(T) > 0$, which can be proven in the same way as Proposition 3.2.1. Similarly as in the case $H \geq \frac{1}{2}$ we have

$$\pi_{[0,T],H}(u) \geq \pi_{\mathcal{T}_u,H}(u)$$

and

$$\pi_{[0,T],H}(u) \leq \pi_{\mathcal{T}_u,H}(u) + \pi_{[0,T]\setminus\mathcal{T}_u,H}(u).$$

Then

$$\pi_{\mathcal{T}_u,H}(u) \leq \sum_{l=1}^{N_u}\sum_{k=1}^{\infty}\mathbb{P}\left\{\sup_{i\in\{1,\ldots,k\}} B_H(X_{\tau_i^{(l)}}) - cX_{\tau_i^{(l)}} > u, X_{\tau_k^{(l)}} \in \mathcal{T}_{u,l}\right\} =: \sum_{l=1}^{N_u}\pi_{\mathcal{T}_{u,l},H}(u).$$

Further

$$\pi_{\mathcal{T}_{u,l},H}(u) = \mathbb{P}\{B_H(X_{\tau^{(l)}}) - cX_{\tau^{(l)}} > u, X_{\tau^{(l)}} \in \mathcal{T}_{u,l}\}$$

$$
\begin{aligned}
&+\sum_{k=2}^{\infty}\mathbb{P}\left\{\sup_{i\in\{1,\ldots,k\}} B_H(X_{\tau_i^{(l)}}) - cX_{\tau_i^{(l)}} > u, X_{\tau_k^{(l)}} \in \mathcal{T}_{u,l}\right\} \\
=: \;\; & \mathcal{P}_{u,l} + S_{u,l}.
\end{aligned}
$$

We have for large enough u

$$
\begin{aligned}
\mathcal{P}_{u,l} &= \int_0^{T\Delta u^{-\frac{1}{H}}} P(B_H(t+t_u^{(l+1)})) - c(t+t_u^{(l+1)})) > u) f_{X_{\tau^{(l)}}}(t+t_u^{(l+1)})dt \\
&\sim f_{X_{\tau^{(l)}}}(t_u^{(l)}) \int_0^{T\Delta u^{-\frac{1}{H}}} P(B_H(t+t_u^{(l+1)})) - c(t+t_u^{(l+1)})) > u)dt \\
&:= f_{X_{\tau^{(l)}}}(t_u^{(l)}) I_{\mathcal{T}_u^{(l)}}.
\end{aligned}
$$

Since for any l density function $f_{X_{\tau^{(l)}}}$ is continuous and $\lim_{u\to\infty} t_u^{(l)} = T$, we have

$$
\mathcal{P}_{u,l} \sim f_{X_{\tau^{(1)}}}(T) I_{\mathcal{T}_u^{(l)}}.
$$

With $\tau^{(1)} = \tau$ and $\lim_{u\to\infty} t_u^{(l)} = T$ we have that

$$
\begin{aligned}
\sum_{l=1}^{N_u} \mathcal{P}_{u,l} &\sim f_{X_{\tau^{(1)}}}(T) \sum_{l=1}^{N_u} I_{\mathcal{T}_u^{(l)}} \\
&= f_{X_{\tau^{(1)}}}(T) I_{\mathcal{T}_u} \\
&\sim f_{X_\tau}(T) \frac{T^{2H}}{H} u^{-2} \mathbb{P}\{B_H(T) - cT > u\},
\end{aligned}
$$

where $I_{\mathcal{T}_u}$ is defined as in the case $H \geq \frac{1}{2}$. Observe that from [35] [Prop 3.1] for $H < \frac{1}{2}$ we have for some $C > 0$

$$
\mathbb{P}\left\{\sup_{t\in[0,T\Delta u^{-\frac{1}{H}}]} B_H(t+t_u^{(l+1)}) - c(t+t_u^{(l+1)}) > u\right\} \leq Cu^{\frac{1-2H}{H}} \mathbb{P}\{B_H(T) - cT > u\}.
$$

Since Z_i have continuous density f_Z with $f_Z(0) \in (0,\infty)$ we have with $Z_i', t \geq 2$ i.i.d. with the same distribution as Z_i for some $C > 0$

$$
\begin{aligned}
S_{u,l} &\leq \sum_{k=2}^{\infty}\mathbb{P}\left\{\sup_{t\in\mathcal{T}_u^{(l)}} B_H(t) - ct > u, X_{\tau_k^{(l)}} \in \mathcal{T}_u^{(l)}\right\} \\
&\leq \mathbb{P}\left\{\sup_{t\in[0,T\Delta u^{-\frac{1}{H}}]} B_H(t+t_u^{(l+1)}) - c(t+t_u^{(l+1)}) > u\right\} \sum_{k=2}^{\infty}\mathbb{P}\left\{X_{\tau_k^{(l)}} \in \mathcal{T}_u^{(l)}\right\} \\
&\leq Cu^{\frac{1-2H}{H}} \mathbb{P}\{B_H(T) - cT > u\} \sum_{k=2}^{\infty}\mathbb{P}\left\{X_{\tau^{(l)}} \in \mathcal{T}_u^{(l)}, Z_2' < T\Delta u^{-\frac{1}{H}}, \ldots, Z_k' < T\Delta u^{-\frac{1}{H}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\sim \; Cu^{\frac{1-2H}{H}}\mathbb{P}\{B_H(T)-cT>u\}T\Delta u^{-\frac{1}{H}}f_{X_{\tau(l)}}(T)\sum_{k=2}^{\infty}\mathbb{P}\Big\{Z_2'<T\Delta u^{-\frac{1}{H}}\Big\}\dots\mathbb{P}\Big\{Z_k'<T\Delta u^{-\frac{1}{H}}\Big\}\\
&\sim \; Cu^{\frac{1-2H}{H}}\mathbb{P}\{B_H(T)-cT>u\}T\Delta u^{-\frac{1}{H}}f_{X_{\tau(l)}}(T)\sum_{k=1}^{\infty}(T\Delta u^{-\frac{1}{H}}f_Z(0))^k\\
&= \; Cu^{\frac{1-2H}{H}}\mathbb{P}\{B_H(T)-cT>u\}T\Delta u^{-\frac{1}{H}}f_{X_{\tau(l)}}(T)\frac{T\Delta u^{-\frac{1}{H}}f_Z(0)}{1-T\Delta u^{-\frac{1}{H}}f_Z(0)}\\
&\sim \; Cu^{\frac{1-2H}{H}}\mathbb{P}\{B_H(T)-cT>u\}f_{X_{\tau(l)}}(T)f_Z(0)T^2\Delta^2u^{-\frac{2}{H}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{l=1}^{N_u}S_{u,l} &\leq N_uCu^{\frac{1-2H}{H}}\mathbb{P}\{B_H(T)-cT>u\}f_{X_{\tau(l)}}(T)f_Z(0)T^2\Delta^2u^{-\frac{2}{H}}\\
&\leq C\mathbb{P}\{B_H(T)-cT>u\}\Delta\log^2(u)u^{-4}\\
&= o\left(\mathbb{P}\{B_H(T)-cT>u\}u^{-2}\right)=o\left(\sum_{l=1}^{N_u}\mathcal{P}_{u,l}\right).
\end{aligned}
$$

For the lower bound we can write

$$
\begin{aligned}
\pi_{\mathcal{T}_u,H}(u) &\geq \sum_{l=1}^{N_u}\mathbb{P}\{B_H(X_{\tau(l)})-cX_{\tau(l)}>u,X_{\tau(l)}\in\mathcal{T}_{u,l}\}\\
&\quad -\sum_{l=1}^{N_u}\sum_{m=l+1}^{N_u}\mathbb{P}\left\{\begin{array}{c}B_H(X_{\tau(l)})-cX_{\tau(l)}>u,X_{\tau(l)}\in\mathcal{T}_{u,l}\\ B_H(X_{\tau(m)})-cX_{\tau(m)}>u,X_{\tau(m)}\in\mathcal{T}_{u,m}\end{array}\right\}\\
&=: \sum_{l=1}^{N_u}\mathcal{P}_{u,l}-\sum_{l=1}^{N_u}\sum_{m=l+1}^{N_u}D_{u,l,m}.
\end{aligned}
$$

Further notice that for $m\geq l+2$

$$
\begin{aligned}
D_{u,l,m} &\leq \sup_{(s,t)\in\mathcal{T}_{u,l}\times\mathcal{T}_{u,m}}\mathbb{P}\{B_H(s)-cs>u,B_H(t)-ct>u,X_{\tau(l)}\in\mathcal{T}_{u,l},X_{\tau(m)}\in\mathcal{T}_{u,m}\}\\
&\leq \sup_{(s,t)\in\mathcal{T}_{u,l}\times\mathcal{T}_{u,m}}\mathbb{P}\left\{\begin{array}{c}B_H(s)-cs>u\\ B_H(t)-ct>u\end{array}\right\}\mathbb{P}\Big\{X_{\tau(l)}\in\mathcal{T}_{u,l},Z_1<(m-l+1)\left(T\Delta u^{-\frac{1}{H}}\right)\Big\}\\
&\sim f_{X_{\tau(l)}}(T)f_Z(0)(m-l+1)\left(T\Delta u^{-\frac{1}{H}}\right)^2\sup_{(s,t)\in\mathcal{T}_{u,l}\times\mathcal{T}_{u,m}}\mathbb{P}\left\{\begin{array}{c}B_H(s)-cs>u\\ B_H(t)-ct>u\end{array}\right\}.
\end{aligned}
$$

We have for large enough u and $\Sigma_{s,t}$ being the covariance matrix of $(B_H(s),B_H(t))$

$$
\mathbb{P}\left\{\begin{array}{c}B_H(s)-cs>u\\ B_H(t)-ct>u\end{array}\right\}\sim\frac{1}{\sqrt{2\pi|\Sigma_{s,t}|}}e^{-\frac{1}{2}(u+cs,u+ct)\Sigma_{s,t}^{-1}(u+cs,u+ct)^\top}
$$

$$\sim \quad \frac{1}{\sqrt{2\pi|\Sigma_{s,t}|}} e^{-\frac{u^2}{2}(1,1)\Sigma_{s,t}^{-1}(1,1)^\top}.$$

Notice that for $s > t$ we have

$$(1,1)\Sigma_{s,t}^{-1}(1,1)^\top \quad = \quad \frac{(s-t)^{2H}}{t^{2H}s^{2H} - \frac{1}{4}(t^{2H} + s^{2H} - (s-t)^{2H})^2}.$$

Let

$$\mathcal{M}(s,t) = \left(t^{4H} - 2t^{2H}s^{2H} - 2t^{2H}(s-t)^{2H} + s^{4H} - 2s^{2H}(s-t)^{2H} + (s-t)^{4H}\right)^2 > 0.$$

Since $(s,t) \in \mathcal{T}_{u,l} \times \mathcal{T}_{u,m}$, therefore $\lim_{u\to\infty} s = \lim_{u\to\infty} t = T$ and hence with direct calculation for large enough u we obtain

$$\frac{\partial}{\partial s}(1,1)\Sigma_{s,t}^{-1}(1,1)^\top \sim \frac{8H}{\mathcal{M}(s,t)}(s-t)^{2H-1}(2t^{2H}s^{2H} - s^{4H} - t^{4H}) < 0,$$

$$\frac{\partial}{\partial t}(1,1)\Sigma_{s,t}^{-1}(1,1)^\top \sim \frac{8H}{\mathcal{M}(s,t)}(s-t)^{2H-1}(s^{4H} + t^{4H} - 2t^{2H}s^{2H}) > 0,$$

which implies that

$$\sup_{(s,t)\in\mathcal{T}_{u,l}\times\mathcal{T}_{u,m}} \mathbb{P}\left\{\begin{array}{c} B_H(s) - cs > u \\ B_H(t) - ct > u \end{array}\right\} \quad = \quad \mathbb{P}\left\{\begin{array}{c} B_H(t_u^{(l+1)}) - ct_u^{(l+1)} > u \\ B_H(t_u^{(m)}) - ct_u^{(m)} > u \end{array}\right\}. \tag{3.2}$$

Further calculations give, as $u \to \infty$,

$$\begin{aligned} (1,1)\Sigma_{t_u^{(l+1)},t_u^{(m)}}^{-1}(1,1)^\top - \frac{1}{T^{2H}} \quad &\sim \quad \frac{1}{T^{2H}}(1 + \frac{1}{4}(m-l-1)^{2H}\Delta^{2H}u^{-2} - 1) \\ &= \quad \frac{(m-l-1)^{2H}\Delta^{2H}u^{-2}}{4T^{2H}}, \end{aligned}$$

which leads to

$$\begin{aligned} \sum_{l=1}^{N_u}\sum_{m=l+2}^{N_u} D_{u,l,m} \quad &\sim \quad \sum_{l=1}^{N_u}\sum_{m=l+2}^{N_u} f_{X_{\tau(l)}}(T) f_Z(0)(m-l+1)\left(T\Delta u^{-\frac{1}{H}}\right)^2 \\ &\quad \times e^{-u^2\frac{(m-l-1)^{2H}\Delta^{2H}u^{-2}}{8T^{2H}}} \mathbb{P}\{B_H(T) - cT > u\} \\ &\leq \quad N_u^2 f_{X_{\tau(l)}}(T) f_Z(0)\left(T\Delta u^{-\frac{1}{H}}\right)^2 \mathbb{P}\{B_H(T) - cT > u\} \sum_{k=1}^{N_u} e^{-\frac{k^{2H}\Delta^{2H}}{8T^{2H}}} \\ &= \quad \log^4(u) u^{-4} f_{X_{\tau(l)}}(T) f_Z(0) T^2 \mathbb{P}\{B_H(T) - cT > u\} \sum_{k=1}^{N_u} e^{-\frac{k^{2H}\Delta^{2H}}{8T^{2H}}}. \end{aligned}$$

Notice that

$$\sum_{k=1}^{N_u} e^{-\frac{k^{2H}\Delta^{2H}}{8T^{2H}}} \quad = \quad \frac{\Delta^{2H}}{8T^{2H}} \sum_{k=1}^{N_u} \frac{8T^{2H}}{\Delta^{2H}} e^{-\frac{k^{2H}\Delta^{2H}}{8T^{2H}}}$$

$$
\begin{aligned}
&\sim \frac{\Delta^{2H}}{8T^{2H}} \int_0^\infty e^{-t^{2H}} dt \\
&= \frac{\Delta^{2H}}{8T^{2H}} \int_0^\infty 2Hx^{\frac{1-2H}{2H}} e^{-x} dx \\
&= \frac{2H\Delta^{2H}}{8T^{2H}} \Gamma\left(\frac{1}{2H}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{l=1}^{N_u} \sum_{m=l+2}^{N_u} D_{u,l,m} &\leq \log^4(u) u^{-4} f_{X_{\tau(l)}}(T) f_Z(0) T^2 \frac{2H\Delta^{2H}}{8T^{2H}} \Gamma\left(\frac{1}{2H}\right) \mathbb{P}\{B_H(T) - cT > u\} \\
&= o\left(\sum_{l=1}^{N_u} \mathcal{P}_{u,l}\right), u \to \infty.
\end{aligned}
$$

It remains to deal with the case $m = l + 1$. Observe that, for

$$
\mathcal{T}_{u,l}^{\sqrt{\Delta}} = (T(1 - l\Delta u^{-\frac{1}{H}} + \sqrt{\Delta} u^{-\frac{1}{H}}), T(1 - (l-1)\Delta u^{-\frac{1}{H}}))
$$

we have

$$
\begin{aligned}
D_{u,l,l+1} &= \mathbb{P}\left\{ \begin{array}{c} B_H(X_{\tau^{(l)}}) - cX_{\tau^{(l)}} > u, X_{\tau^{(l)}} \in \mathcal{T}_{u,l}^{\sqrt{\Delta}} \\ B_H(X_{\tau^{(l+1)}}) - cX_{\tau^{(l+1)}} > u, X_{\tau^{(l+1)}} \in \mathcal{T}_{u,l+1} \end{array} \right\} \\
&\quad + \mathbb{P}\left\{ \begin{array}{c} B_H(X_{\tau^{(l)}}) - cX_{\tau^{(l)}} > u, X_{\tau^{(l)}} \in \mathcal{T}_{u,l}^{\sqrt{\Delta}} \\ B_H(X_{\tau^{(l+1)}}) - cX_{\tau^{(l+1)}} > u, X_{\tau^{(l+1)}} \in \mathcal{T}_{u,l+1} \end{array} \right\} \\
&=: D_{u,l,l+1}^{(1)} + D_{u,l,l+1}^{(2)}.
\end{aligned}
$$

Furthermore we have that

$$
\begin{aligned}
D_{u,l,l+1}^{(1)} &\leq \sup_{(s,t) \in \mathcal{T}_{u,l}^{\sqrt{\Delta}} \times \mathcal{T}_{u,l+1}} \mathbb{P}\left\{ B_H(s) - cs > u, B_H(t) - ct > u, X_{\tau^{(l)}} \in \mathcal{T}_{u,l}^{\sqrt{\Delta}}, X_{\tau^{(l+1)}} \in \mathcal{T}_{u,l+1} \right\} \\
&\leq \sup_{(s,t) \in \mathcal{T}_{u,l} \times \mathcal{T}_{u,l+1}} \mathbb{P}\left\{ \begin{array}{c} B_H(s) - cs > u \\ B_H(t) - ct > u \end{array} \right\} \mathbb{P}\left\{ X_{\tau^{(l)}} \in \mathcal{T}_{u,l}, Z_1 < 2\left(T\Delta u^{-\frac{1}{H}}\right) \right\} \\
&\sim 2 f_{X_{\tau(l)}}(T) f_Z(0) \left(T\Delta u^{-\frac{1}{H}}\right)^2 \sup_{(s,t) \in \mathcal{T}_{u,l} \times \mathcal{T}_{u,l+1}} \mathbb{P}\left\{ \begin{array}{c} B_H(s) - cs > u \\ B_H(t) - ct > u \end{array} \right\} \\
&\leq 2 f_{X_{\tau(l)}}(T) f_Z(0) \left(T\Delta u^{-\frac{1}{H}}\right)^2 \mathbb{P}\{B_H(T) - cT > u\}.
\end{aligned}
$$

Hence

$$
\sum_{l=1}^{N_u} D_{u,l,l+1}^{(1)} \leq 2N_u f_{X_{\tau(l)}}(T) f_Z(0) \left(T\Delta u^{-\frac{1}{H}}\right)^2 \mathbb{P}\{B_H(T) - cT > u\}
$$

$$= \quad 2\log^2(u)u^{-2-\frac{1}{H}}\Delta f_{X_{\tau(l)}}(T)f_Z(0)\mathbb{P}\{B_H(T)-cT>u\} = o\left(\sum_{l=1}^{N_u}\mathcal{P}_{u,l}\right).$$

Further notice that

$$\begin{aligned}
S^{(2)}_{u,l,l+1} &\leq \mathbb{P}\Big\{B_H(X_{\tau(l)})-cX_{\tau(l)}>u, X_{\tau(l)}\in(T(1-l\Delta u^{-\frac{1}{H}}), T(1-l\Delta u^{-\frac{1}{H}}+\sqrt{\Delta}u^{-\frac{1}{H}}))\Big\} \\
&= \int_0^{T\sqrt{\Delta}u^{-\frac{1}{H}}} P(B_H(t+t_u^{(l+1)})-c(t+t_u^{(l+1)})>u)f_{X_{\tau(l)}}(t+t_u^{(l+1)})dt \\
&\sim f_{X_{\tau(l)}}(t_u^{(l)})\int_0^{T\sqrt{\Delta}u^{-\frac{1}{H}}} P(B_H(t+t_u^{(l+1)})-c(t+t_u^{(l+1)})>u)dt \\
&\leq f_{X_{\tau(l)}}(t_u^{(l)})\frac{\sqrt{\Delta}}{\Delta}\int_0^{T\Delta u^{-\frac{1}{H}}} P(B_H(t+t_u^{(l+1)})-c(t+t_u^{(l+1)})>u)dt \qquad (3.3) \\
&\sim \frac{\sqrt{\Delta}}{\Delta}P_{u,l},
\end{aligned}$$

where in (3.3) we use that for $C\in\mathbb{N}$ we have

$$\mathbb{P}\{B_H(t+t_u^{(l+1)})-c(t+t_u^{(l+1)})>u\} \leq \mathbb{P}\Big\{B_H(t+C\sqrt{\Delta}u^{-\frac{1}{H}}+t_u^{(l+1)})-c(t+C\sqrt{\Delta}u^{-\frac{1}{H}}+t_u^{(l+1)})>u\Big\}.$$

Therefore

$$\sum_{l=1}^{N_u} D^{(2)}_{u,l,l+1} \leq \sum_{l=1}^{N_u}\frac{\sqrt{\Delta}}{\Delta}P_{u,l},$$

which leads to

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\sum_{l=1}^{N_u}\sum_{m=l+1}^{N_u}D_{u,l,m}}{\sum_{l=1}^{N_u}P_{u,l}} = 0.$$

Concluding, we have that

$$\pi_{\mathcal{T}_u,H}(u)\sim\sum_{l=1}^{N_u}P_{u,l}\sim f_{X_\tau}(T)\frac{T^{2H}}{H}u^{-2}\mathbb{P}\{B_H(T)-cT>u\}.$$

Finally, by the same arguments as in proof of case $H\geq\frac{1}{2}$ of [35][Prop. 3.1] for $T=t_u$ by using $t_u\to T$, we have that for some $C>0$ and large enough u

$$\begin{aligned}
\pi_{[0,T]\setminus\mathcal{T}_u,H}(u) &\leq P(\sup_{t\in[0,T]\setminus\mathcal{T}_u}B_H(t)-ct>u) \\
&= CP(B_H(t_u)-ct_u>u)u^{\frac{1-2H}{H}} \\
&\sim Cu^{\frac{1-2H}{H}}\frac{1}{\sqrt{2\pi}t_u^H}e^{-\frac{(u+ct_u)^2}{2t_u^{2H}}}
\end{aligned}$$

$$
\begin{aligned}
&\sim \quad Cu^{\frac{1-2H}{H}} \frac{1}{\sqrt{2\pi}T^H} e^{-\frac{(u+cT)^2}{2T^{2H}}} e^{-\frac{2H\log^2(u)}{u^2}\frac{(u+cT)^2}{2T^{2H}}+o\left(\frac{1}{u}\right)} \\
&\sim \quad Cu^{\frac{1-2H}{H}} P(B_H(T) - cT > u) e^{-\frac{H}{T^{2H}}\log^2(u)} \\
&= \quad o(u^{-2} P(B_H(T) - cT > u)).
\end{aligned}
$$

This completes the proof. □

Chapter 4

Logarithmic asymptotics of Parisian ruin probability for positively correlated Brownian motions

4.1 Introduction

We define

$$W(t) := (W_1(s), W_2(t)) = (B_1(s), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)), \quad s, t \geq 0,$$

where B_1, B_2 are two independent standard Brownian motions and $\rho \in [-1, 1]$ to reflect the dependence between the components. In the context of risk theory W_i are representing incoming claims to the system, while c_i represent premiums and u_i represent initial capitals. The model above has been studied in e.g. [18], where the two-dimensional non-simultaneous ruin probability

$$\pi(c_1, c_2, u, au) := \mathbb{P}\left\{\exists_{s,t\in[0,1]} W_1^*(s) > u, W_2^*(t) > au\right\} \tag{4.1}$$

with $W_i^*(t) = W_i(t) - c_i t, i = 1, 2$ was discussed. One of the extensions to the model above is the so-called Parisian ruin, which has been studied in e.g. [15] and differs from the above approach by defining ruin by crossing the barrier (u, au) on an interval of length $\frac{C}{u^2}, C > 0$. In this chapter we investigate a non-simultaneous Parisian ruin probability for two-dimensional time to be spent over the barrier $\boldsymbol{H}(u) = (H_1(u), H_2(u))$ defined by

$$\mathcal{P}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au) := \mathbb{P}\big\{\exists_{s'\in A, t'\in B}\forall_{s\in[s',s'+H_1(u)]}\forall_{t\in[t',t'+H_2(u)]}W_1^*(s) > u, W_2^*(t) > au\big\}, \quad (4.2)$$

and cumulative Parisian ruin probability

$$\mathcal{S}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au) := \mathbb{P}\left\{\int_A \mathbf{1}(W_1^*(s) > u)ds > H_1(u), \int_B \mathbf{1}(W_2^*(t) > au)dt > H_2(u)\right\}$$

for some $H_1(u), H_2(u) \geq 0$ functions dependant on u and compact sets $A, B = [0, T]$. Since from self-similarity of Brownian motion we have that for $c_1' = \frac{c_1}{\sqrt{T}}, u' = \frac{u}{\sqrt{T}}$

$$B(tT) - c_1 t > u \overset{D}{\Leftrightarrow} \sqrt{T}B(t) - c_1 t > u \Leftrightarrow B(t) - c_1' t > u',$$

then without loss of generality one can assume $T = 1$.

4.2 Notation and preliminaries

In this section we introduce the notation used in this chapter. We define $B_i^* = B_i - c_i$ with B_i standard Brownian motion and $c_i \in \mathbb{R}$. Let

$$\Sigma_{s,t} = \begin{pmatrix} s & \rho\min(s,t) \\ \rho\min(s,t) & t \end{pmatrix}$$

be the covariance matrix of $(W_1(s), W_2(t))$. We denote $\boldsymbol{a} = (1, a)^\top$,

$$q_{\boldsymbol{a}}(s,t) := \boldsymbol{a}^\top \Sigma_{s,t}^{-1}\boldsymbol{a} \quad (4.3)$$

$$\boldsymbol{b}(s,t) := \Sigma_{s,t}^{-1}\boldsymbol{a}$$

and set

$$q_{\boldsymbol{a}}^*(s,t) = \min_{\boldsymbol{x}\geq\boldsymbol{a}} q_{\boldsymbol{x}}(s,t), \quad q_{\boldsymbol{a}}^* = \min_{s,t\in[0,1]} q_{\boldsymbol{a}}^*(s,t). \quad (4.4)$$

This quadratic optimization problem has appeared in many papers, e.g. [8, 21, 26, 32]. In [18] instead of $\boldsymbol{a}$, $\bar{\boldsymbol{a}}(s,t) = (1 + \frac{c_1 s}{u}, a + \frac{c_2 t}{u})^\top$ was used. As it will appear in proofs of Theorem 4.3.1 and Theorem 4.3.2, since we are interested in logarithmic asymptotics, using $\boldsymbol{a}$ is no different from using $\bar{\boldsymbol{a}}(s,t)$, since the logarythmic order of both asymptotics is the same. From [26] we have

$$\lim_{u\to\infty}\frac{1}{u^2}\log\mathbb{P}\{\exists_{s,t\in[0,1]}W_1^*(s) > u, W_2^*(t) > au\} = -\frac{q_{\boldsymbol{a}}^*}{2}. \quad (4.5)$$

It appears that the function q plays a crucial role in calculating the logarithmic asymptotics of ruin of two-dimensional Brownian motion process.

4.3 Main results

The results are split onto two scenarios, dependent on the behaviour of function $\boldsymbol{H}(u)$, as $u \to \infty$. The first scenario considers the case $\boldsymbol{H}(u) \to (0,0)$. Note that in [57] for Parisian ruin this scenario was investigated for the special case $\boldsymbol{H}(u) = O(\frac{1}{u^2})$. In the second scenario we investigate the case $\lim_{u\to\infty} \boldsymbol{H}(u) = (H_1, H_2) \geq (0,0)$, which we consider only for $\rho > 0$.

4.3.1 Case $\lim_{u\to\infty} \boldsymbol{H}(u) = (0,0)$

Let $H(u)$ be such that $\lim_{u\to\infty} H(u) = (0,0)$. We show that the logarithmic asymptotics limit is equal to the value of $q_{\boldsymbol{a}}(s,t)$ in its minimal point on the interval $[0,1]^2$. By differentiating in [18] [Lem 3.1] it was proven that the optimal point of the function $q^*_{\boldsymbol{a}}(s,t)$ is dependent on the relation between a and ρ through the function $A_a = \frac{1}{4a}(1 - \sqrt{8a^2+1})$.

Theorem 4.3.1 *Let $\boldsymbol{H}(u)$ be such that $\lim_{u\to\infty} \boldsymbol{H}(u) = (0,0)$. Then*

$$\lim_{u\to\infty} \frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{u^2} = -\frac{1}{2} q^*_{\boldsymbol{a}}\left(1, \min\left(1, \frac{a}{\rho(2a\rho-1)}\right)\right).$$

The proof of the theorem is postponed to the next section.

4.3.2 Case $\lim_{u\to\infty} \boldsymbol{H}(u) > (0,0), \rho > 0$

Let $\boldsymbol{H}(u)$ be such that $\lim_{u\to\infty} \boldsymbol{H}(u) > (0,0)$. Since the period spent by the process over the barrier asymptotically is independent on u, the asymptotics of $\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))$ is vastly different from what we observed in the previous section. Intuitively, with positively correlated processes, greater variance of each individual process leads to greater ruin probability, hence the area over which the ruin most likely happens is the one which is the closest possible to the end of the $[0,1]^2$.

Theorem 4.3.2 *Let $\rho \geq 0$, $\boldsymbol{H}(u)$ be such that $\lim_{u\to\infty} \boldsymbol{H}(u) = (H_1,H_2), H_1, H_2 \in (0,1)$. Then*

$$\lim_{u\to\infty} \frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{u^2} = -\frac{1}{2} q^*_{\boldsymbol{a}}(1-H_1, 1-H_2).$$

The proof of Theorem 4.3.2 is postponed to the next section. We note that the logarithmic asymptotics obtained in Theorem 4.3.2 is equal to the value in the minimal point of the function $q_{\boldsymbol{a}}(s,t)$ on the interval $[0,1-H_1]\times[0,1-H_2]$. Since $\rho>0$, then function q is monotone and hence the optimal point of the function $q^*_{\boldsymbol{a}}(s,t)$, which determines the asymptotics for the ruin based on supremum functional is the point $(1,1)$. For the Parisian ruin on a large interval the behaviour is different and the supremum approach cannot be mimicked to obtain even logarythmic asymptotics.

Remark 4.3.3 *Using Theorem 2.2 from [18] we have that*

$$\lim_{u\to\infty}\frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{\log(\pi(c_1,c_2,u,au))}=\begin{cases}1, if \lim_{u\to\infty}H_1(u)=\lim_{u\to\infty}H_2(u)=0\\ 0, \textit{otherwise}\end{cases},$$

where $\mathcal{P}$ was defined in (4.2) and π was defined in (4.1).

Remark 4.3.4 *For $\lim_{u\to\infty}H_1(u)=\lim_{u\to\infty}H_2(u)=0$ we note that*

$$\lim_{u\to\infty}\frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{\log(\mathcal{S}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))} = 1 \tag{4.6}$$

since

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}\le\mathcal{S}_{[0,1]^2,\boldsymbol{H}(u)}\le\pi(c_1,c_2,u,au)$$

and hence inequalities in both directions hold. For $\lim_{u\to\infty}H_1(u)>0$ or $\lim_{u\to\infty}H_2(u)>0$ following the proof we observe that the definition of Parisian and sojourn ruin coincide and hence (4.6) holds as well.

4.4 Proofs

Proof of Theorem 4.3.1 We split the proof into two cases: $t^*=1$ and $t^*<1$.

Case $t^*=1$. We have that

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au)\le\pi(c_1,c_2,u,au).$$

From [18][Thm 2.2, case i-iv] we have that

$$\lim_{u\to\infty}\frac{\log(\pi(c_1,c_2,u,au))}{u^2}=-\frac{1}{2}q^*_{\boldsymbol{a}}(1,1).$$

On the other hand for $H_M(u)=\max(H_1(u),H_2(u))$ we have that

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au)$$

$$
\begin{aligned}
&\geq \mathbb{P}\{\forall_{s\in[1-H_1(u),1],t\in[1-H_2(u),1]}W_1^*(s)>u,W_2^*(t)>au\}\\
&\geq \mathbb{P}\{\forall_{s\in[1-H_M(u),1]}W_1^*(s)>u,W_2^*(s)>au\}\\
&= \mathbb{P}\left\{\forall_{s\in[1-H_M(u),1]}\begin{array}{c}W_1^*(s)>u\\ W_2^*(s)>au\end{array}\middle|\begin{array}{c}W_1^*(1-H_M(u))>u+\sqrt{u}\\ W_2^*(1-H_M(u))>a(u+\sqrt{u})\end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c}W_1^*(1-H_M(u))>u+\sqrt{u}\\ W_2^*(1-H_M(u))>a(u+\sqrt{u})\end{array}\right\}\\
&\geq \mathbb{P}\left\{\forall_{s\in[1-H_M(u),1]}\begin{array}{c}W_1^*(s)-W_1^*(1-H_M(u))>-\sqrt{u}\\ W_2^*(s)-W_2^*(1-H_M(u))>-a\sqrt{u}\end{array}\middle|\begin{array}{c}W_1^*(1-H_M(u))>u+\sqrt{u}\\ W_2^*(1-H_M(u))>a(u+\sqrt{u})\end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c}W_1^*(1-H_M(u))>u+\sqrt{u}\\ W_2^*(1-H_M(u))>a(u+\sqrt{u})\end{array}\right\}\\
&= \mathbb{P}\left\{\forall_{s\in[0,H_M(u)]}\begin{array}{c}W_1^*(s)>-\sqrt{u}\\ W_2^*(s)>-a\sqrt{u}\end{array}\right\}\mathbb{P}\left\{\begin{array}{c}W_1^*(1-H_M(u))>u+\sqrt{u}\\ W_2^*(1-H_M(u))>a(u+\sqrt{u})\end{array}\right\},
\end{aligned}
$$

where in the last equality we use independence of increments of Brownian motion. Further, using self-similarity of Brownian motion, we have

$$
\begin{aligned}
\mathbb{P}\left\{\forall_{s\in[0,H_M(u)]}\begin{array}{c}W_1^*(s)>-\sqrt{u}\\ W_2^*(s)>-a\sqrt{u}\end{array}\right\} &= \mathbb{P}\left\{\forall_{s\in[0,1]}\begin{array}{c}H_M^2(u)W_1(s)-c_1sH_M(u)>-\sqrt{u}\\ H_M^2(u)W_2(s)-c_2sH_M(u)>-a\sqrt{u}\end{array}\right\}\\
&= \mathbb{P}\left\{\forall_{s\in[0,1]}\begin{array}{c}W_1(s)>\frac{-\sqrt{u}+c_1sH_M(u)}{H_M^2(u)}\\ W_2(s)>\frac{-a\sqrt{u}+c_2sH_M(u)}{H_M^2(u)}\end{array}\right\}.
\end{aligned}
$$

Since $\lim_{u\to\infty}H_M(u)=0$, we have that $\lim_{u\to\infty}\frac{-\sqrt{u}+c_1sH_M(u)}{H_M^2(u)}=-\infty,\lim_{u\to\infty}\frac{-a\sqrt{u}+c_2sH_M(u)}{H_M^2(u)}=-\infty$ and therefore

$$
\lim_{u\to\infty}\mathbb{P}\left\{\forall_{s\in[0,1]}\begin{array}{c}W_1(s)>\frac{-\sqrt{u}+c_1sH_M(u)}{H_M^2(u)}\\ W_2(s)>\frac{-a\sqrt{u}+c_2sH_M(u)}{H_M^2(u)}\end{array}\right\}=1.
$$

Hence for large enough u we have

$$
\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au)\geq\mathbb{P}\{W_1^*(1-H_M(u))>u+\sqrt{u},W_2^*(1-H_M(u))>a(u+\sqrt{u})\}.
$$

Now notice that as $u\to\infty$

$$
\begin{aligned}
&\mathbb{P}\{W_1^*(1-H_M(u))>u+\sqrt{u},W_2^*(1-H_M(u))>a(u+\sqrt{u})\}\\
&\sim\ \frac{1}{2\pi(1-H_M(u))}e^{-\frac{(u+\sqrt{u})^2}{2}q_{\boldsymbol{a}}^*(1-H_M(u),1-H_M(u))}.
\end{aligned}
$$

Since $\lim_{u\to\infty} q^*_{\boldsymbol{a}}(1-H_M(u),1-H_M(u)) = q^*_{\boldsymbol{a}}(1,1)$, then we have that

$$\lim_{u\to\infty} \frac{\log(\mathbb{P}\{W_1^*(1-H_M(u)) > u+\sqrt{u}, W_2^*(1-H_M(u)) > a(u+\sqrt{u})\})}{-\frac{(u+\sqrt{u})^2}{2} q^*_{\boldsymbol{a}}(1,1)}$$
$$= \lim_{u\to\infty} \frac{\log(\mathbb{P}\{W_1^*(1-H_M(u)) > u+\sqrt{u}, W_2^*(1-H_M(u)) > a(u+\sqrt{u})\})}{-\frac{u^2}{2} q^*_{\boldsymbol{a}}(1,1)} = 1.$$

Therefore lower and upper bound coincide and the proof of case $t^* = 1.$ is complete.

Case $t^* < 1.$ We have that

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) \le \pi(c_1,c_2,u,au).$$

From [18][Thm 2.2, case v-vi] we have that

$$\lim_{u\to\infty} \frac{\log(\pi(c_1,c_2,u,au))}{-\frac{u^2}{2} q^*_{\boldsymbol{a}}(1,t^*)} = 1.$$

On the other hand for some function $h(u)$, to be specified below, and sufficiently large u so that $t^* + H_2(u) < 1 - H_1(u)$ we have

$$\begin{aligned}
&\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au)\\
&\ge \mathbb{P}\left\{\forall_{s\in[1-H_1(u),1],t\in[t^*,t^*+H_2(u)]} W_1^*(s) > u, W_2^*(t) > au\right\}\\
&\ge \mathbb{P}\left\{\forall_{s\in[1-H_1(u),1],t\in[t^*,t^*+H_2(u)]} \begin{array}{c} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \middle| \begin{array}{c} W_1^*(1-H_1(u)) > u+h(u) \\ W_2^*(t^*) > a(u+h(u)) \end{array}\right\}\\
&\quad \times \mathbb{P}\{W_1^*(1-H_1(u)) > u+h(u), W_2^*(t^*) > a(u+h(u))\}\\
&\ge \mathbb{P}\left\{\forall_{s\in[1-H_1(u),1],t\in[t^*,t^*+H_2(u)]} \begin{array}{c} W_1^*(s) - W_1^*(1-H_1(u)) > -h(u) \\ W_2^*(t) - W_2^*(t^*) > -ah(u) \end{array} \middle| \begin{array}{c} W_1^*(1-H_1(u)) = u+h(u) \\ W_2^*(t^*) = a(u+h(u)) \end{array}\right\}\\
&\quad \times \mathbb{P}\{W_1^*(1-H_1(u)) > u+h(u), W_2^*(t^*) > a(u+h(u))\}\\
&=: \mathbb{P}\left\{\forall_{s\in[0,H_1(u)]} W_1^*(s) > -h(u)\right\} \mathbb{P}\left\{\forall_{t\in[t^*,t^*+H_2(u)]} \chi_u(t) > 0\right\}\\
&\quad \times \mathbb{P}\{W_1^*(1-H_1(u)) > u+h(u), W_2^*(t^*) > a(u+h(u))\},
\end{aligned}$$

where the last equality holds by independence of increments of Brownian motion and since $1 - H_1(y) > t^* + H_2(u)$. Using properties of Normal distribution we have that

$$\chi_u(t) := \left(W_2^*(t) - W_2^*(t^*) + ah(u) \middle| \begin{array}{c} W_1^*(1-H_1(u)) = u+h(u) \\ W_2^*(t^*) = a(u+h(u)) \end{array} \right)$$

is a Gaussian process with

$$E[\chi_u(t)] = \frac{\rho(t-t^*)(2a^2\rho^2-3a\rho+1)}{1-H_1(u)-a\rho(1-2H_1(u))}u + \frac{(t-t^*)(2a^2\rho^3-a\rho^2-2a\rho^2+\rho)}{1-H_1(u)-a\rho(1-2H_1(u))}h(u) + ah(u) + o(h(u)+u)$$

and

$$Var(\chi_u(t)) = \frac{(t-t^*)(2a\rho^3(t-t^*)+2H_1(u)a\rho-\rho^2(t-t^*)-a\rho-H_1(u)+1)}{1-H_1(u)-a\rho(1-2H_1(u))}.$$

By picking $h(u)$ such that $h(u)=o(u), uH_2(u)=o(h(u))$ and $\lim_{u\to\infty} h(u)=\infty$ we have that

$$E[\chi_u(t)] = ah(u) + o(h(u)).$$

Therefore, since $Var(\chi_u(t))$ is finite and $\lim_{u\to\infty}\mathbb{P}\left\{\inf_{t\in[t^*,t^*+H_2(u)]}\chi_u(t)<0\right\}=0$, we have

$$\lim_{u\to\infty}\mathbb{P}\left\{\forall_{t\in[t^*,t^*+H_2(u)]}\chi_u(t)>0\right\}=1.$$

With the same choice of $h(u)$ we have that

$$\lim_{u\to\infty}\mathbb{P}\left\{\forall_{s\in[0,H_1(u)]}W_1^*(s)>-h(u)\right\}=1.$$

Therefore for large enough u we can write

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) \geq \mathbb{P}\{W_1^*(1-H_1(u))>u+h(u), W_2^*(t^*)>a(u+h(u))\}.$$

Further we have

$$\begin{aligned}&\lim_{u\to\infty}\frac{\log(\mathbb{P}\{W_1^*(1-H_1(u))>u+h(u), W_2^*(t^*)>a(u+h(u))\})}{-\frac{(u+h(u))^2}{2}q_{\boldsymbol{a}}^*(1,t^*)}\\ &= \lim_{u\to\infty}\frac{\log(\mathbb{P}\{W_1^*(1-H_1(u))>u+h(u), W_2^*(t^*)>a(u+h(u))\})}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1,t^*)}=1.\end{aligned}$$

Therefore upper and lower bound coincide and the proof of the case $t^*<1$ is complete. □

PROOF OF THEOREM 4.3.2 Note that for any $\epsilon>0$ we have that for sufficiently large u

$$|H_i(u)-H_i|<\epsilon$$

and hence

$$\lim_{u\to\infty}\frac{q_{\boldsymbol{a}}^*(1-H_1,1-H_2)}{q_{\boldsymbol{a}}^*(1-H_1(u),1-H_2(u))} = 1. \tag{4.7}$$

Therefore, we consider $\boldsymbol{H}(u)=(H_1,H_2)$ for constant H_1,H_2 and by (4.7) the proof also covers any other $\boldsymbol{H}(u)$, such that $\lim_{u\to\infty}\boldsymbol{H}(u)=(H_1,H_2)$. We first consider a special case of $H_1=H_2=H$, which will allow us to simplify the calculations for the more general case.

Lower bound. We have that

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) \geq \mathcal{P}_{[1-H,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au).$$

Furthermore using independence of increments of Brownian motion we have for $(\tilde{W}_1^*(s),\tilde{W}_2^*(t))$ Brownian motion independent of $\boldsymbol{W}$ with $\tilde{\boldsymbol{W}}^* \stackrel{d}{=} -\boldsymbol{W}^*$

$$\begin{aligned}
\mathcal{P}_{[1-H,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) &\geq \mathbb{P}\left\{\forall_{s,t\in[1-H,1]^2} \begin{array}{c} W_1^*(s)>u \\ W_2^*(t)>au \end{array} \middle| \begin{array}{c} W_1^*(1-H)>u+\sqrt{u} \\ W_2^*(1-H)>a(u+\sqrt{u}) \end{array}\right\} \\
&\quad \times\mathbb{P}\left\{W_1^*(1-H)>u+\sqrt{u}, W_2^*(1-H)>a(u+\sqrt{u})\right\} \\
&\geq \mathbb{P}\left\{\forall_{s,t\in[0,H]^2}\tilde{W}_1^*(s)<\sqrt{u},\tilde{W}_2^*(t)<a\sqrt{u}\right\} \\
&\quad \times\mathbb{P}\left\{W_1^*(1-H)>u+\sqrt{u}, W_2^*(1-H)>a(u+\sqrt{u})\right\}.
\end{aligned}$$

We have that $\lim_{u\to\infty}\mathbb{P}\left\{\forall_{s,t\in[0,H]^2}\tilde{W}_1^*(s)<\sqrt{u},\tilde{W}_2^*(t)<a\sqrt{u}\right\}=1$. Moreover for $u\to\infty$

$$\mathbb{P}\left\{W_1^*(1-H)>u+\sqrt{u}, W_2^*(1-H)>a(u+\sqrt{u})\right\} \sim \frac{1}{\sqrt{2\pi}|\Sigma_{(1-H,1-H)}|}e^{-\frac{(u+\sqrt{u})^2}{2}q_{\boldsymbol{a}}^*(1-H,1-H)}.$$

Therefore

$$\lim_{u\to\infty}\frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{-\frac{(u+\sqrt{u})^2}{2}q_{\boldsymbol{a}}^*(1-H,1-H)} = \lim_{u\to\infty}\frac{\log(\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1-H,1-H)} \geq 1.$$

Upper bound. Notice that

$$\begin{aligned}
\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) &\leq \mathcal{P}_{[1-H,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au)+\pi_{[0,1-H]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) \\
&\leq \mathbb{P}\{W_1^*(1-H)>u, W_2^*(1-H)>au\}+\pi_{[0,1-H]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au).
\end{aligned}$$

Following the proof of [18][Lemma 3.1] we have that since the correlation between W_1 and W_2 is positive, then both partial derivatives of the function $q_{\boldsymbol{a}}^*(s,t)$ are positive and hence $(1-H,1-H)$ is the optimal point of the function $q_{\boldsymbol{a}}^*(s,t)$ on the interval $[0,1-H]^2$. With [26] we therefore have that

$$\lim_{u\to\infty}\frac{\log(\pi_{[0,1-H]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1-H,1-H)}=1.$$

Furthermore from the properties of two-dimensional random Normal variable we have that

$$\lim_{u\to\infty}\frac{\log(\mathbb{P}\{W_1^*(1-H)>u, W_2^*(1-H)>au\})}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1-H,1-H)}=1.$$

Therefore, the upper bound agrees with the lower bound and the proof for the case $H_1 = H_2 = H$ is complete.

We continue with the proof of case $H_1 > H_2$.

Lower bound. We have that

$$
\begin{aligned}
&\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1, c_2, u, au)\\
&\geq \mathcal{P}_{[1-H_1,1]\times[1-H_2,1],\boldsymbol{H}(u)}(c_1, c_2, u, au)\\
&\geq \mathbb{P}\left\{\forall_{s\in[1-H_1,1],t\in[1-H_2,1]} \begin{array}{r} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \middle| \begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&= \mathbb{P}\left\{\forall_{s\in[1-H_2,1],t\in[1-H_2,1]} \begin{array}{r} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \middle| \begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\geq \mathbb{P}\left\{\forall_{s\in[1-H_2,1],t\in[1-H_2,1]} \begin{array}{r} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \middle| \begin{array}{c} W_1^*(1-H_2) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&:= P_1(u)P_2(u).
\end{aligned}
$$

The asymptotics of $P_1(u)$ as $u \to \infty$ can be handled as in previous case for $H = H_2$ and hence we know that $\lim_{u\to\infty} P_1(u) = 1$. Furthermore for some $A > 0$ we have that

$$
\begin{aligned}
P_2(u) &\geq \mathbb{P}\left\{\begin{array}{c} \forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array} \middle| \begin{array}{c} W_1^*(1-H_1) > u+A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\quad\times\mathbb{P}\left\{\begin{array}{c} W_1^*(1-H_1) > u+A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\\
&\geq \mathbb{P}\left\{\forall_{r\in[1-H_1,1-H_2]} W_1^*(r) > u+\sqrt{u} \middle| \begin{array}{c} W_1^*(1-H_1) = u+A\sqrt{u} \\ W_2^*(1-H_2) = a(u+\sqrt{u}) \end{array}\right\}
\end{aligned}
$$

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$$\times\mathbb{P}\left\{\begin{array}{c} W_1^*(1-H_1) > u + A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}$$

$$:= \mathbb{P}\left\{\forall_{r\in[1-H_1,1-H_2]}\chi_u(r) > 0\right\}$$

$$\times\mathbb{P}\left\{\begin{array}{c} W_1^*(1-H_1) > u + A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\},$$

where $\chi_u(r)$ is a Gaussian process with

$$E[\chi_u(r)] = C_1(r-(1-H_1))u + C_2(r)\sqrt{u} + o(\sqrt{u})$$

and

$$Var(\chi_u(r)) = \frac{-1+r+H_1+H_2-H_2r-H_1H_2+\rho^2r-\rho^2r^2-H_1\rho^2r}{1-H_2-\rho^2(1-H_1)}$$

for $C_1 = \frac{(a-\rho)\rho}{1-H_2-\rho^2(1-H_1)}$, $C_2(r) = \frac{A(1-H_2-\rho^2r)-1+H_2+\rho^2-a\rho+a\rho r-H_1\rho^2+H_1a\rho}{1-H_2-\rho^2(1-H_1)}$. Notice that by picking large enough constant A we have that $C_2(r) > 0$ for r close to $1-H_1$. Therefore we can write that

$$E[\chi_u(r)] > C\sqrt{u}$$

for large enough u, some positive constant C and any $r \in [1-H_1, 1-H_2]$. Since variance of $\chi_u(r)$ does not depend on u and we have continuity of sample paths we therefore have that

$$\lim_{u\to\infty}\mathbb{P}\left\{\forall_{r\in[1-H_1,1-H_2]}\chi_u(r) > 0\right\} = 1.$$

Further we have that for some constant B

$$\lim_{u\to\infty}\frac{\log\left(\mathbb{P}\left\{\begin{array}{c} W_1^*(1-H_1) > u + A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\right)}{-\frac{(u+B\sqrt{u})^2}{2}q_{\boldsymbol{a}}^*(1-H_1,1-H_2)} = \lim_{u\to\infty}\frac{\log\left(\mathbb{P}\left\{\begin{array}{c} W_1^*(1-H_1) > u + A\sqrt{u} \\ W_2^*(1-H_2) > a(u+\sqrt{u}) \end{array}\right\}\right)}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1-H_1,1-H_2)} = 1.$$

Upper bound. We again have that

$$\mathcal{P}_{[0,1]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au) \le \mathbb{P}\{W_1^*(1-H_1) > u, W_2^*(1-H_2) > au\} + \pi_{[0,1-H_1]\times[0,1-H_2],\boldsymbol{H}(u)}(c_1,c_2,u,au).$$

Similarly as in case $H_1 = H_2$ we see that

$$\lim_{u\to\infty}\frac{\log(\pi_{[0,1-H]^2,\boldsymbol{H}(u)}(c_1,c_2,u,au))}{-\frac{u^2}{2}q_{\boldsymbol{a}}^*(1-H_1,1-H_2)} = 1$$

and

$$\lim_{u\to\infty} \frac{\log(\mathbb{P}\{W_1^*(1-H) > u, W_2^*(1-H) > au\})}{-\frac{u^2}{2} q_{\boldsymbol{a}}^*(1-H_1, 1-H_2)} = 1.$$

The upper bound is hence the same as lower bound and the proof of the case $H_1 > H_2$ is complete. In the case $H_1 < H_2$ proof would follow the same steps as the proof of the case $H_1 > H_2$ and hence is omitted. □

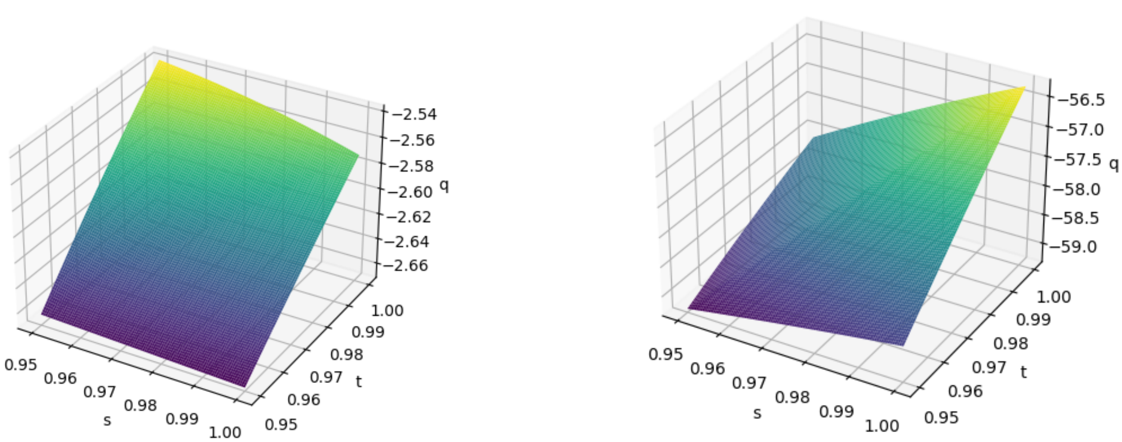
4.5 Simulations

The aim of this section is to shed some light on the behavior of the ruin probability defined in (4.1) and to study the practical aspect of calculated asymptotic probability. Recall that

$$q_{\boldsymbol{a}}(s,t) \quad := \quad \boldsymbol{a}^\top \Sigma_{s,t}^{-1} \boldsymbol{a}. \tag{4.8}$$

To cover the wide variety of the spectrum we will study in depth three cases:

1. $a = 1, \rho = 0.75, c_1 = -0.5, c_2 = 0.25.$



(a) Shape of function $q_{\boldsymbol{a}}(s,t)$ with $u = 2$

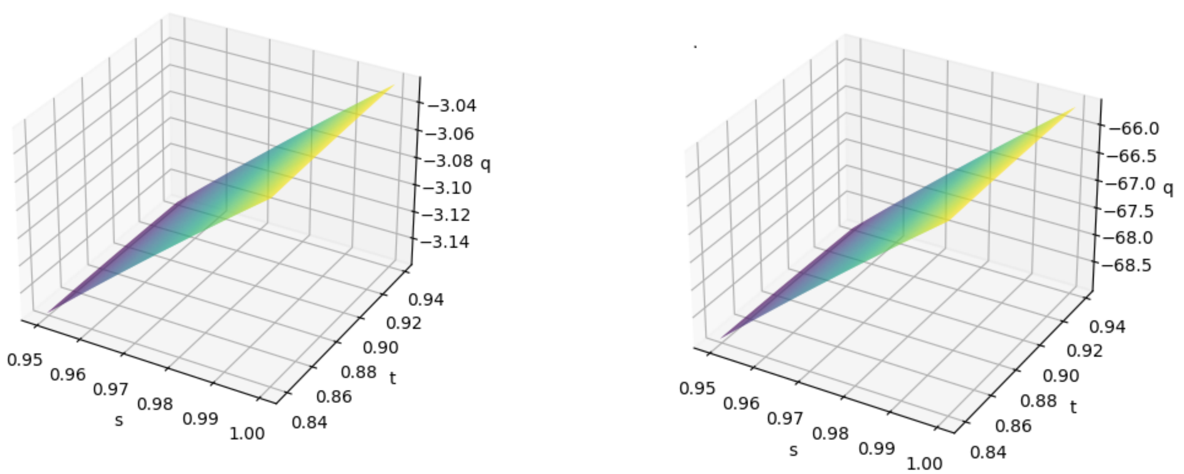
(b) Shape of function $q_{\boldsymbol{a}}(s,t)$ with $u = 10$

Figure 4.1: Plots of function $q_{\boldsymbol{a}}(s,t)$ for the case $a = 1, \rho = 0.75, c_1 = -0.5, c_2 = 0.25$.

In this case we observe a high positive correlation between the components. For small values of u we still see the impacts of drifts and hence the point maximizing (4.8) for a specific

u is different than the asymptotically optimal point (1,1). However, for sufficiently large u, impact of drifts vanishes and we see that the theoretical and simulated points maximizing function q coincide with the point $(1, 1)$ and the behavior of the function is linear-like in both coordinate directions.

2. $a = 0.25, \rho = -0.25, c_1 = -0.5, c_2 = 0.25$.

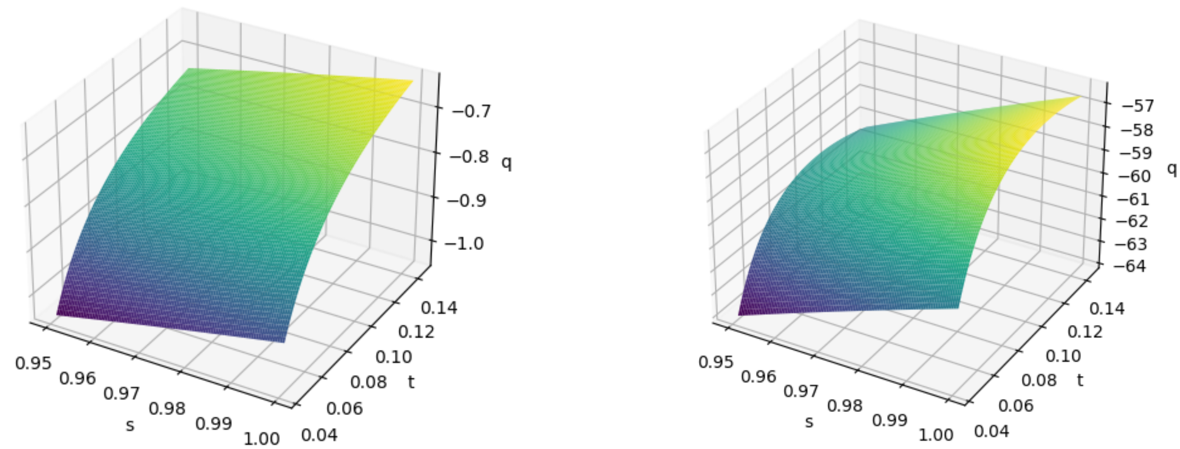


(a) Shape of function $q_{\boldsymbol{a}}(s, t)$ with $u = 2$ (b) Shape of function $q_{\boldsymbol{a}}(s, t)$ with $u = 10$

Figure 4.2: Plots of function $q_{\boldsymbol{a}}(s, t)$ for the case $a = 0.25, \rho = -0.25, c_1 = -0.5, c_2 = 0.25$.

In this case we take mildly negatively correlated processes, for which the point maximizing function q is near the point $(1, 0.9)$. In the t-axis we observe a much slower decay than on the s-axis. In the close-up we can discover that the behavior there is locally quadratic.

3. $a = 0.1, \rho = -0.9, c_1 = -1, c_2 = -1$.



(a) Shape of function $q_{\boldsymbol{a}}(s,t)$ with $u = 2$

(b) Shape of function $q_{\boldsymbol{a}}(s,t)$ with $u = 10$

Figure 4.3: Plots of function $q_{\boldsymbol{a}}(s,t)$ for the case $a = 0.1, \rho = -0.9, c_1 = -1, c_2 = -1$.

Finally let us consider the highly negative correlated processes with large drifts. For such Brownian motions we see the major difference in both the optimal point, which is around the point $(1, 0.09)$, and the impact of the drifts for small values of u.

We further present the sample realisations of the W_1^*, W_2^* processes, which we use to illustrate the most likely paths of ruin occurrence. The presented cases correspond to the cases analyzed above in the context of a shape of *generalized variance.*

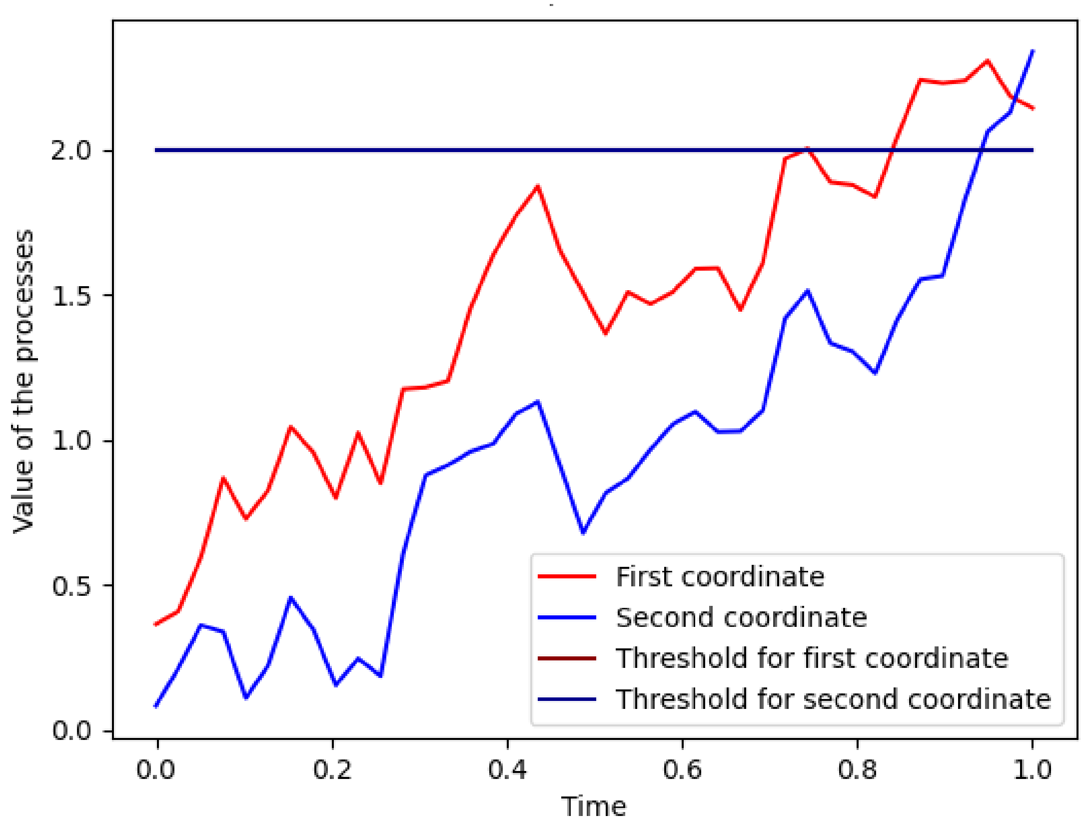


Figure 4.4: Sample realisation of the processes W_1^*, W_2^* crossing the threshold for the case $a = 1, \rho = 0.75, c_1 = -0.5, c_2 = 0.25$.

In the first case we observe that both the paths are most likely breaking the barrier at the end of the interval, since both coordinates of the process *cooperate* to cross the threshold. Note that since $a = 1$, the thresholds for both coordinates overlap. Moreover, since the variance of each components is increasing, the barrier gets easier to be crossed at the end of the interval.

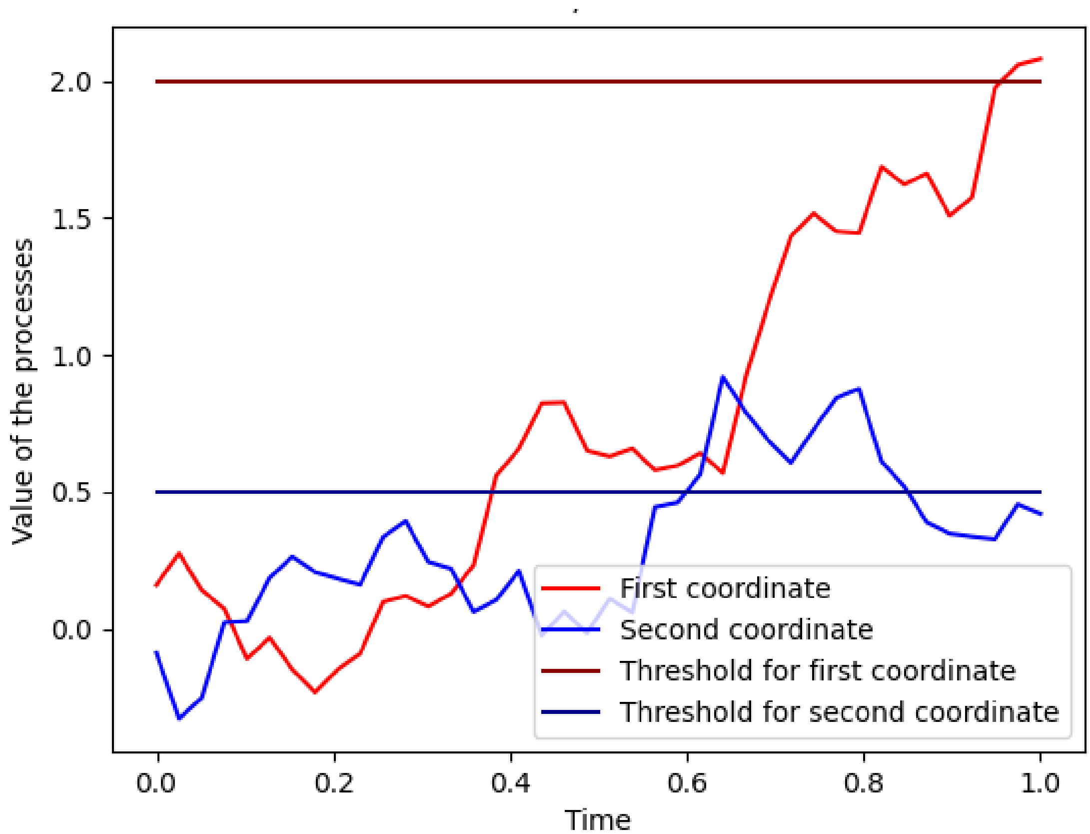


Figure 4.5: Sample realisation of the processes W_1^*, W_2^* crossing the threshold for the case $a = 0.25, \rho = -0.25, c_1 = -0.5, c_2 = 0.25$.

In the second case, we usually can observe the second component breaking the barrier a bit earlier and then falling below the threshold again. This is related to the negative correlation between processes, however since it is a very mild one and the threshold that the second process has to cross is relatively small, the impact of the parameters on the optimal point is not major.

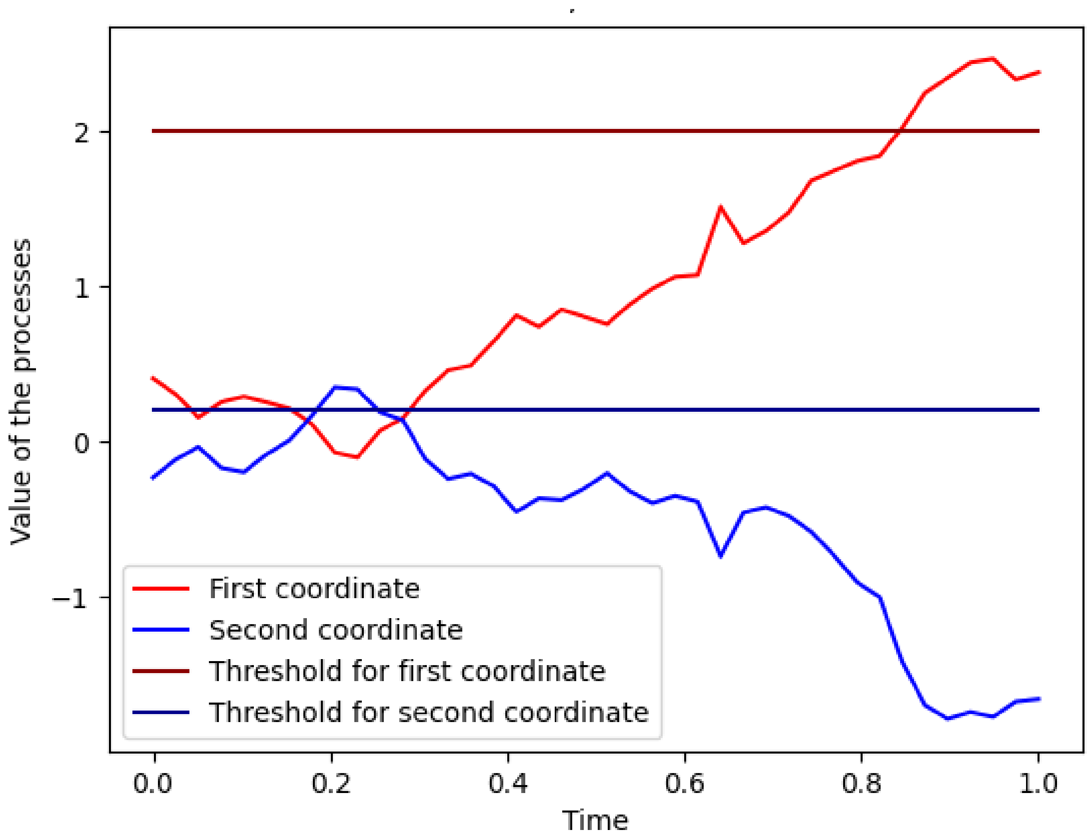


Figure 4.6: Sample realisation of the processes W_1^*, W_2^* crossing the threshold for the case $a = 0.1, \rho = -0.9, c_1 = -1, c_2 = -1$.

In the last case we see that the high negative correlation impacts the behavior of the optimal point in a very significant way. The point of crossing the barrier for the second axis has to be relatively quick, so that the negative correlation does not drag the first component in the negative direction.

4.5.1 Technical details and propositions of improvements

For the purposes of simulations we used Python package "numba" for parallel calculations, but for $u \geq 4$ due to the increasing amount of points in the grid, the calculations were taking a long time. To enhance the simulation process one can do the following

1. Exchange equidistant grid to grid concentrated nearby the optimal point of function q.

   To avoid the problem of number of points on the interval increasing with u, one can use dense enough points only in the area that is given by theoretical calculations to be crucial for the asymptotics. However, with small u this might lead to inaccurate results.

2. Use clusters of computers or cloud computing.

   In many cases the events that we are trying to simulate are very rare events. For the application to be practical, the accuracy needs to increase, hence relaxing parameter assumptions is not a viable solution. However, using stronger machines would allow to run those calculations in parallel and decently fast and could lead to very interesting results.

Chapter 5

Non-simultaneous ruin probability for positively correlated Brownian risk model

5.1 Introduction

For $d \geq 2$ consider the following Brownian risk model

$$R_i(t_i) = u_i + c_i t_i - W_i(t_i), \quad t_i > 0, i = 1, 2, \ldots, d,$$

where the vector-valued random field $\boldsymbol{W}(\boldsymbol{t}) = (W_i(t_i))_{i \in 1,2,\ldots,d}$, $\boldsymbol{t} = (t_1, t_2, \ldots, t_d)$ is multi-dimensional Brownian motion defined below and can be interpreted as the accumulated claims of appropriate business lines, $u_i > 0$ are the initial capitals and c_i are the premium rates. The correlation matrix for the d-dimensional Brownian motion is expressed as

$$\Sigma_{\boldsymbol{t}} := AA^{\top} \cdot (\min(t_i, t_j))_{i,j \in 1,2,\ldots,d} = (\rho_{ij} \min(t_i, t_j))_{i,j \in 1,2,\ldots,d}, \Sigma = \Sigma_{(1,1,\ldots,1)}$$

for $\rho_{ij} \in (-1, 1), A$ such that $\boldsymbol{W}(\boldsymbol{t}) = A\boldsymbol{B}(\boldsymbol{t})$ with $\boldsymbol{B}(\boldsymbol{t})$ a d-dimensional Brownian motion with independent components and recall that "$\cdot$" representing component-wise multiplication. Further we denote $\boldsymbol{W}^*(\boldsymbol{t}) = \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}$. Asymptotics of the multidimensional portfolios has been investigated deeply in recent years for various models of ruin. In [20] and [31] the simultaneous ruin probability for two portfolios

$$\overline{\pi}_\rho(c_1, c_2; u, au) = \mathbb{P}\{\exists s \in [0,1] : W_1^*(s) > u, W_2^*(s) > au\}$$

has been studied. The two-dimensional simultaneous time ruin model was also investigated in [56] for Parisian type ruins. Moreover, higher dimensional models for simultaneous infinite time ruin were studied in [50].

The aim of this chapter is to study the non-simultaneous ruin probability, i.e.

$$\pi_{\Sigma,\boldsymbol{\alpha}}(\boldsymbol{c}, u) = \mathbb{P}\big\{\exists \boldsymbol{t} \in [0,1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\big\}$$

as $u \to \infty$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d, \boldsymbol{c} = (c_1, c_2, \ldots, c_d)$. In the two-dimensional non-simultaneous setting, see e.g. [18], authors observe that the non-simultaneous ruin probability depends on the behaviour of the so-called generalized variance function around its maximal point. Higher dimensions create even more significantly different cases. Denote by $\boldsymbol{W}_I, \boldsymbol{t}_I, \boldsymbol{c}_I, \boldsymbol{\alpha}_I$ the restriction of $\boldsymbol{W}, \boldsymbol{t}, \boldsymbol{c}, \boldsymbol{\alpha}$ to the indices from index set $I \subseteq \{1, 2, \ldots, n\}$. Notice that (since $W_i(0) = 0$) if $\alpha_i \leq 0$, then for $I = \{1, \ldots, i-1, i+1, \ldots, d\}$ we have as $u \to \infty$

$$\mathbb{P}\big\{\exists \boldsymbol{t} \in [0,1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\big\} \sim \mathbb{P}\big\{\exists \boldsymbol{t} \in [0,1]^d : \boldsymbol{W}_I^*(\boldsymbol{t}_I) > \boldsymbol{\alpha}_I u\big\},$$

i.e. coordinate i is negligible. Hence without loss of generality in the rest of this chapter we can assume $\alpha > \mathbf{0}$. From (1.4) we have that covariance matrix Σ_t is non-singular. We define

$$q_{\boldsymbol{\alpha}}(\boldsymbol{t}) := \boldsymbol{\alpha} \Sigma_{\boldsymbol{t}}^{-1} \boldsymbol{\alpha}^\top$$

and set

$$q_{\boldsymbol{\alpha}}^*(\boldsymbol{t}) = \min_{\boldsymbol{x} \geq \boldsymbol{\alpha}} q_{\boldsymbol{x}}(\boldsymbol{t}), \quad q_{\boldsymbol{\alpha}}^* = \min_{\boldsymbol{t} \in [0,1]^d} q_{\boldsymbol{\alpha}}^*(\boldsymbol{t}). \tag{5.1}$$

As it was established in e.g. [26]

$$\lim_{u \to \infty} \frac{1}{u^2} \log \mathbb{P}\{\boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} = -\frac{q_{\boldsymbol{\alpha}}^*(\boldsymbol{t})}{2} \tag{5.2}$$

and

$$\lim_{u \to \infty} \frac{1}{u^2} \log \mathbb{P}\{\exists_{\boldsymbol{t} \in [0,1]^d} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} = -\frac{q_{\boldsymbol{\alpha}}^*}{2}. \tag{5.3}$$

Hence the shape of function $q_{\boldsymbol{\alpha}}$ plays a crucial role in understanding the behaviour of the asymptotics and will be investigated thoroughly in further parts of the paper.

5.2 Main result

Before stating the main results we present an accurate bound for $\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u)$, that can be calculated thanks to the assumption $A \gneqq 0$.

Theorem 5.2.1 *If $A \gneqq 0$ and $\boldsymbol{\alpha} > \mathbf{0}$ then*

$$\mathbb{P}\{\boldsymbol{W}^*(\mathbf{1}) > \boldsymbol{\alpha}u\} \leq \pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u) \leq C\mathbb{P}\{\boldsymbol{W}^*(\mathbf{1}) > \boldsymbol{\alpha}u\},$$

where $C = \frac{1}{\prod_{i=1}^{d} \Psi\left(\max\left(0,c_i'\right)\right)}$ with $c' = \Sigma^{-\frac{1}{2}}c$.

Let $\varphi(\cdot)$ denote the probability density function of $\boldsymbol{W}(\mathbf{1})$. The next result constitutes the main finding of this chapter.

Theorem 5.2.2 *If $A \gneqq 0, \boldsymbol{t} \in [0,1]^d$ and $\boldsymbol{\alpha}\Sigma_{\boldsymbol{t}}^{-1} > 0, \boldsymbol{\alpha} > \mathbf{0}$ then as $u \to \infty$*

$$\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u) \sim C_{\Sigma_{\boldsymbol{t}},\alpha} u^{-d} \varphi(\boldsymbol{\alpha}u + \boldsymbol{c}), \tag{5.4}$$

where

$$C_{\Sigma_{\boldsymbol{t}}} = \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0,\infty)^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \in (0,\infty).$$

The above results coincide with two-dimensional results found in [18] for $\rho > 0$. If the correlation between processes is positive, then the asymptotics focuses on the area that maximises the variance for each individual process, which in this case is $\mathbf{1}$. Similar behaviour was observed for two-dimensional process also in [20], where the asymptotics for simultaneous two-dimensional Brownian motion was studied.

5.3 Proofs

Proof of Theorem 5.2.1 The lower bound is straightforward, since we replace the suprema over the whole set with just value at point $\boldsymbol{t} = \mathbf{1}$. For the upper bound we will use similar idea as in [18][Theorem 1.1] and apply the bound

$$\mathbb{P}\left\{\sup_{t\in[0,1]} (B(t) - ct) > u\right\} \leq \frac{\mathbb{P}\{B(1) > u + c\}}{\Psi\left(\max\left(0,c\right)\right)}, \tag{5.5}$$

see [20, 54]. For $\boldsymbol{B}^*(\boldsymbol{t})$ being $d-$dimensional Brownian motion with independent components and drift $\boldsymbol{c}' = A^{-1}\boldsymbol{c}$ we have

$$\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u) = \mathbb{P}\left\{\exists_{\boldsymbol{t}\in[0,1]^d} A\boldsymbol{B}^*(\boldsymbol{t}) > \boldsymbol{\alpha}u\right\}.$$

By the assumption that $A > 0$ we have

$$\mathbb{P}\left\{\exists_{\boldsymbol{t}\in[0,1]^d} A\boldsymbol{B}^*(\boldsymbol{t}) > \boldsymbol{\alpha}u\right\} \quad \leq \quad \mathbb{P}\{A\boldsymbol{V} > \boldsymbol{\alpha}u\}$$

where $(\boldsymbol{V})_i = \sup_{t\in[0,1]} B_i^*(t)$. Further let g_i denote the joint density of $(V_1,\ldots,V_{i-1},B_{i+1}^*(1),\ldots,B_d^*(1))$, which exist due to [6][Theorem 7.1]. Then, with (5.5)

$$\begin{aligned}
\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u) &\leq \mathbb{P}\{A\boldsymbol{V} > \boldsymbol{\alpha}u\} \\
&= \int_{\mathbb{R}^{d-1}} \mathbb{P}\left\{A(x_1,\ldots,x_{d-1},V_d)^\top > \boldsymbol{\alpha}u\right\} g_d(x_1,x_2,\ldots,x_{d-1})d(x_1,x_2,\ldots,x_{d-1}) \\
&\leq \int_{\mathbb{R}^{d-1}} \frac{\mathbb{P}\left\{A(x_1,\ldots,x_{d-1},B_d^*(1))^\top > \boldsymbol{\alpha}u\right\}}{\Psi\left(\max\left(0,c_d'\right)\right)} g_d(x_1,x_2,\ldots,x_{d-1})d(x_1,x_2,\ldots,x_{d-1}) \\
&= \frac{1}{\Psi\left(\max\left(0,c_d'\right)\right)} \mathbb{P}\left\{A(V_1,\ldots,V_{d-1},B_d^*(1))^\top > \boldsymbol{\alpha}u\right\} \\
&= \frac{1}{\Psi\left(\max\left(0,c_d'\right)\right)} \int_{\mathbb{R}^{d-1}} \mathbb{P}\left\{A(x_1,\ldots,x_{d-2},V_{d-1},x_d)^\top > \boldsymbol{\alpha}u\right\} \\
&\quad g_{d-1}(x_1,x_2,\ldots,x_d)d(x_1,x_2,\ldots,x_{d-2},x_d) \\
&\leq \frac{1}{\Psi\left(\max\left(0,c_d'\right)\right)} \frac{1}{\Psi\left(\max\left(0,c_{d-1}'\right)\right)} \\
&\quad \times \int_{\mathbb{R}^{d-1}} \mathbb{P}\left\{A(x_1,\ldots,x_{d-2},B_{d-1}^*(1),x_{d-1})^\top > \boldsymbol{\alpha}u\right\} \\
&\quad g_{d-1}(x_1,x_2,\ldots,x_{d-2},x_d)d(x_1,x_2,\ldots,x_{d-2},x_d) \\
&= \frac{1}{\Psi\left(\max\left(0,c_d'\right)\right)} \frac{1}{\Psi\left(\max\left(0,c_{d-1}'\right)\right)} \mathbb{P}\left\{A(V_1,\ldots,V_{d-2},B_{d-1}^*(1),B_d^*(1))^\top > \boldsymbol{\alpha}u\right\}
\end{aligned}$$

Using iterative arguments we finally get

$$\begin{aligned}
\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c},u) &\leq \frac{1}{\prod_{i=1}^d \Psi\left(\max\left(0,c_i'\right)\right)} \mathbb{P}\{A\boldsymbol{B}^*(1) > \boldsymbol{\alpha}u\} \\
&= \frac{1}{\prod_{i=1}^d \Psi\left(\max\left(0,c_i'\right)\right)} \mathbb{P}\{\boldsymbol{W}^*(1) > \boldsymbol{\alpha}u\}.
\end{aligned}$$

This completes the proof. □

The following lemma will be useful in the proof of Theorem 5.2.2.

Lemma 5.3.1 *Assume $A \gneqq 0, \boldsymbol{\alpha}\Sigma_{\boldsymbol{t}}^{-1} > 0, \boldsymbol{t} \in [0,1]^d$. Then for any $\boldsymbol{t} \in [0,1]^d$ and any $\boldsymbol{s}$ in the neighbourhood of $\boldsymbol{t}$ there exist vectors of positive coordinates $\boldsymbol{C}_{\boldsymbol{t}} \in {\mathbb{R}_+}^d$ such that as $\boldsymbol{s} \to \boldsymbol{t}$*

$$q_{\boldsymbol{\alpha}}(\boldsymbol{s}) - q_{\boldsymbol{\alpha}}(\boldsymbol{t}) = \boldsymbol{C}_{\boldsymbol{t}} \cdot (\boldsymbol{t} - \boldsymbol{s})(1 + o(1)).$$

Proof of Lemma 5.3.1 We observe that for any $i \in (1, \dots, d)$ we have

$$\begin{aligned} \frac{\partial}{\partial t_i} q_{\boldsymbol{\alpha}}(\boldsymbol{t}) &= \boldsymbol{\alpha} \frac{\partial}{\partial t_i} \Sigma_{\boldsymbol{t}}^{-1} \boldsymbol{\alpha}^\top \\ &= -\boldsymbol{\alpha} \Sigma_{\boldsymbol{t}}^{-1} \left(\frac{\partial}{\partial t_i} \Sigma_{\boldsymbol{t}} \right) \Sigma_{\boldsymbol{t}}^{-1} \boldsymbol{\alpha}^\top < 0, \end{aligned} \tag{5.6}$$

where the last inequality follows from $\boldsymbol{\alpha}\Sigma_{\boldsymbol{t}}^{-1} = (\Sigma_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}^\top)^\top \gneqq 0$ and the fact that $\Sigma_{\boldsymbol{t}}$ by definition is non-decreasing in each coordinate. Using multidimensional Taylor expansion we get that

$$q_{\boldsymbol{\alpha}}(\boldsymbol{s}) - q_{\boldsymbol{\alpha}}(\boldsymbol{t}) = \nabla_q(t) \cdot (\boldsymbol{s} - \boldsymbol{t}) + O((\boldsymbol{t} - \boldsymbol{s}) \cdot (\boldsymbol{t} - \boldsymbol{s})),$$

where ∇_q denotes the gradient of function q. With (5.6) we have that $\nabla_q(t) < \mathbf{0}$ and hence the proof is complete. □

Remark 5.3.2 *It follows from the proof of Lemma 5.3.1 that if* $A \gneqq 0, \boldsymbol{\alpha}\Sigma_{\boldsymbol{t}}^{-1} > 0, \boldsymbol{t} \in [0,1]^d$, *then function* $q_{\boldsymbol{\alpha}}(\boldsymbol{t})$ *attains its unique minimum over set* $[0,1]^d$ *at* $\boldsymbol{t} = \mathbf{1}$.

Next for any $\Delta > 0$ let

$$\boldsymbol{k}_u = (1 - \frac{(k_1-1)\Delta}{u^2}, \dots, 1 - \frac{(k_d-1)\Delta}{u^2}) \quad (\boldsymbol{k}+\mathbf{1})_u = (1 - \frac{k_1\Delta}{u^2}, \dots, 1 - \frac{k_d\Delta}{u^2}),$$

where $\forall_{i\in 1,\dots,d} k_i \in \{1, 2, \dots, \lfloor u \log u \rfloor\}$. Denote also $E_{k_u} = [(\boldsymbol{k}+\mathbf{1})_u, \boldsymbol{k}_u]$ and $E = [-\Delta, 0]^d$. Let

$$I(\Delta) = \lim_{u\to\infty} \int_{\mathbb{R}^d} \mathbb{P}\left\{ \exists \boldsymbol{t} \in E : \boldsymbol{W}(\frac{\boldsymbol{t}}{u^2} + \boldsymbol{k}_u) - \boldsymbol{W}(\boldsymbol{k}_u) > \frac{\boldsymbol{x}}{u} \middle| \boldsymbol{W}^*(\boldsymbol{k}_u) = \boldsymbol{\alpha} u - \frac{\boldsymbol{x}}{u} \right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}.$$

Lemma 5.3.3 *Let* $\Sigma_{\boldsymbol{t}} \gneqq 0, \Delta > 0, \boldsymbol{\alpha} > 0$ *be given. Then as* $u \to \infty$

$$\mathbb{P}\{\exists \boldsymbol{t} \in E_{k_u} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} = I(\Delta) u^{-d} e^{-u^2 \boldsymbol{C}\cdot(\mathbf{1}-\boldsymbol{k}_u)} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c}),$$

with

$$I(\Delta) < \infty.$$

Moreover, for $\boldsymbol{k}_u = \mathbf{1}$, *for large enough* u *we have that*

$$I(\Delta) = \int_{\mathbb{R}^d} \mathbb{P}\{\exists \boldsymbol{t} \in [0,\Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha} \cdot \boldsymbol{t} > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}.$$

Proof of Lemma 5.3.3 By the total probability formula we have

$$
\begin{aligned}
&\mathbb{P}\{\exists \boldsymbol{t} \in E_{k_u} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} \\
= u^{-d} &\int_{\mathbb{R}^d} \mathbb{P}\Big\{\exists \boldsymbol{t} \in E_{k_u} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u \Big| \boldsymbol{W}^*(\boldsymbol{k}_u) = \boldsymbol{\alpha} u - \frac{\boldsymbol{x}}{u}\Big\} \varphi_{k_u}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u}) d\boldsymbol{x} \\
= u^{-d} &\int_{\mathbb{R}^d} \mathbb{P}\Big\{\exists \boldsymbol{t} \in E : \boldsymbol{W}(\frac{\boldsymbol{t}}{u^2} + \boldsymbol{k}_u) - \boldsymbol{W}(\boldsymbol{k}_u) > \frac{\boldsymbol{x}}{u} \Big| \boldsymbol{W}^*(\boldsymbol{k}_u) = \boldsymbol{\alpha} u - \frac{\boldsymbol{x}}{u}\Big\} \varphi_{\boldsymbol{k}_u}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u}) d\boldsymbol{x},
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_{\boldsymbol{k}_u}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u}) &:= \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma_{\boldsymbol{k}_u}|} e^{-\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma_{\boldsymbol{k}_u}|} e^{-\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top} e^{\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top - \frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma_{\boldsymbol{k}_u}|} e^{-\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top} e^{-(\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top - \frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top)} \\
&\quad \times e^{-(\boldsymbol{k}_u - \boldsymbol{1} - \frac{\boldsymbol{x}}{u})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}|\Sigma_{\boldsymbol{k}_u}|} e^{-\frac{1}{2}(\boldsymbol{\alpha} u + \boldsymbol{c})\Sigma^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c})^\top} e^{-\frac{1}{2}(q_{\boldsymbol{\alpha}}(\boldsymbol{k}_u) - q_{\boldsymbol{\alpha}}(\boldsymbol{1}))} e^{-(\frac{1}{2}\boldsymbol{c}(\Sigma_{\boldsymbol{k}_u}^{-1} - \Sigma^{-1})(\boldsymbol{\alpha} u + \boldsymbol{c})^\top} \\
&\quad \times e^{-(\boldsymbol{k}_u - \boldsymbol{1})\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top} e^{\boldsymbol{\alpha}\Sigma^{-1}\boldsymbol{x}^\top} e^{\frac{\boldsymbol{x}}{u}\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top}.
\end{aligned}
$$

Finally, using Lemma 5.3.1 we have that for some vector $\boldsymbol{C} > 0$

$$
q_{\boldsymbol{\alpha}}(\boldsymbol{k}_u) - q_{\boldsymbol{\alpha}}(\boldsymbol{1}) = \boldsymbol{C} \cdot (\boldsymbol{1} - \boldsymbol{k}_u)(1 + o(1))
$$

and hence as $u \to \infty$

$$
\lim_{u \to \infty} \frac{\varphi_{\boldsymbol{k}_u}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})}{\varphi(\boldsymbol{\alpha} u + \boldsymbol{c}) e^{-u^2 \boldsymbol{C}\cdot(\boldsymbol{1} - \boldsymbol{k}_u)} e^{\boldsymbol{\alpha}\Sigma^{-1}\boldsymbol{x}^\top}} = 1.
$$

Hence it remains to investigate the behaviour of

$$
\begin{aligned}
I(\Delta) &= \int_{\mathbb{R}^d} \mathbb{P}\Big\{\exists \boldsymbol{t} \in E : \boldsymbol{W}(\frac{\boldsymbol{t}}{u^2} + \boldsymbol{k}_u) - \boldsymbol{W}(\boldsymbol{k}_u) > \frac{\boldsymbol{x}}{u} \Big| \boldsymbol{W}^*(\boldsymbol{k}_u) = \boldsymbol{\alpha} u - \frac{\boldsymbol{x}}{u}\Big\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \\
&:= \int_{\mathbb{R}^d} \mathbb{P}\big\{\exists \boldsymbol{t} \in [0, \Delta]^d : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\big\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}
\end{aligned}
$$

with

$$
\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) := \boldsymbol{W}(\frac{\boldsymbol{t}}{u^2} + \boldsymbol{k}_u) - \boldsymbol{W}(\boldsymbol{k}_u) > \frac{\boldsymbol{x}}{u} \Big| \boldsymbol{W}^*(\boldsymbol{k}_u) = \boldsymbol{\alpha} u - \frac{\boldsymbol{x}}{u}
$$

and $\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t})$ is a Gaussian process with mean

$$
E[\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t})] = u(S\Sigma_{\boldsymbol{k}_u}^{-1}(\boldsymbol{\alpha} u + \boldsymbol{c}\boldsymbol{k}_u^\top - \frac{\boldsymbol{x}}{u})^\top)
$$

and covariance matrix

$$\Sigma_{\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t})} = u^2(\Sigma_{\boldsymbol{W}(\boldsymbol{k}_u - \frac{\boldsymbol{t}}{u^2}) - \boldsymbol{W}(\boldsymbol{k}_u)} - S\Sigma_{\boldsymbol{k}_u}^{-1} S^\top),$$

where

$$S = Cov(\boldsymbol{W}(\boldsymbol{k}_u - \frac{\boldsymbol{t}}{u^2}) - \boldsymbol{W}(\boldsymbol{k}_u), \boldsymbol{W}(\boldsymbol{k}_u)).$$

Observe that

$$S_{i,j} = \begin{cases} 0, & if\ k_i < k_j \\ -\frac{\sigma_{i,j} t_i}{u^2}, & else \end{cases}, \quad \Sigma_{\boldsymbol{W}(\boldsymbol{k}_u + \frac{\boldsymbol{t}}{u^2}) - \boldsymbol{W}(\boldsymbol{k}_u)} = \frac{1}{u^2}\Sigma_{\boldsymbol{t}}.$$

We prove finitness of the integral above by induction. Case $d = 2$ was proven in [18]. Suppose that for every $k < n$

$$\int_{\mathbb{R}^k} \mathbb{P}\big\{\exists \boldsymbol{t} \in [0, \Delta]^k : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\big\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \quad < \quad \infty. \tag{5.7}$$

We aim to show that

$$\int_{\mathbb{R}^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0, \Delta]^n : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} < \infty.$$

Denote by $J_i = 1, 2, \dots, n \setminus \{i\}$. Now observe that using that $\boldsymbol{\alpha}\Sigma^{-1} > 0$ there exists $C > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0, \Delta]^n : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \\ &\qquad \le \int_{\mathbb{R}_+^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0, \Delta]^n : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \\ &\qquad + \sum_{i=1}^n \int_{\mathbb{R}_-} e^{Cx_i} dx_i \int_{\mathbb{R}^{n-1}} \mathbb{P}\big\{\exists \boldsymbol{t} \in [0, \Delta]^{n-1} : \boldsymbol{X}_{\boldsymbol{k}_u, J_i}(\boldsymbol{t}_{J_i}) > \boldsymbol{x}_{J_i}\big\} e^{\boldsymbol{\alpha}_{J_i}(\Sigma_1^{-1})_{J_i}\boldsymbol{x}_{J_i}^\top} d\boldsymbol{x}_{J_i}, \end{aligned}$$

where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. Hence using (5.7) it remains to show that

$$\int_{\mathbb{R}_+^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0, \Delta]^n : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} < \infty.$$

Since $\boldsymbol{\alpha}\Sigma^{-1} \ge 0$ and $\Delta < \infty$, using [66][Thm 8.1] we have that there exist constants $C_1, C_2 > 0$ such that as $u \to \infty$

$$\int_{\mathbb{R}_+^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0, \Delta]^n : \boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \quad \le \quad \int_{\mathbb{R}_+^n} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0, \Delta]^n : \sum_{i=1}^n X_{\boldsymbol{k}_u, i}(t_i) > \sum_{i=1}^n x_i\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}$$

$$\leq \int_{\mathbb{R}_+^n} C_1 e^{-C_2(\sum_{i=1}^n x_i)^2} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} < \infty.$$

Finally using dominated convergence theorem, for $\boldsymbol{k}_u = \mathbf{1}$ we get as $u \to \infty$

$$S = -\frac{\boldsymbol{t}}{u^2}\Sigma,$$

hence

$$E[\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t})] = -\boldsymbol{\alpha}\cdot\boldsymbol{t} + O\left(\frac{1}{u}\right)$$

and

$$\Sigma_{\boldsymbol{X}_{\boldsymbol{k}_u}(\boldsymbol{t})} = \Sigma + O\left(\frac{1}{u^2}\right)\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

and therefore as $u \to \infty$

$$I(\Delta) \sim \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0,\Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}.$$

□

Lemma 5.3.4 *Let* $\Sigma_{\boldsymbol{t}} > 0, \boldsymbol{\alpha}\Sigma_{\boldsymbol{t}}^{-1} > 0$. *Then*

$$\int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0,\Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \in (0,\infty).$$

Proof of Lemma 5.3.4 Note that for $\boldsymbol{x} < 0$ we have

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in [0,\Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\right\} = 1.$$

Further since $\boldsymbol{\alpha}\Sigma_1^{-1} > 0$, then

$$\int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0,\Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} \geq \int_{\mathbb{R}_-{}^d} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x} > 0.$$

Recall that for $\boldsymbol{\mu} > 0$ and $\boldsymbol{X}$ a $d-$dimensional random vector with finite moment generating function we have

$$\int_{\mathbb{R}^d} \mathbb{P}\{\boldsymbol{X} > \boldsymbol{x}\} e^{\boldsymbol{\mu}\cdot\boldsymbol{x}} d\boldsymbol{x} = \frac{1}{\prod_{i=1}^n \mu_i}\mathbb{E}[e^{\boldsymbol{\mu}^\top \boldsymbol{X}}].$$

Take $\Delta = n$ and denote $\boldsymbol{\lambda} = \alpha\Sigma_1^{-1}$. Then

$$I(n) \leq \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n-1}} \int_{\mathbb{R}^d} \mathbb{P}\{\exists \boldsymbol{t} \in [\boldsymbol{i}, \boldsymbol{i}+\mathbf{1}] : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha}\cdot\boldsymbol{t} > \boldsymbol{x}\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}$$

$$= \frac{1}{\prod_{i=1}^{d}\lambda_i}\sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n-1}}\mathbb{E}[e^{\boldsymbol{\lambda}^\top M(\boldsymbol{i})}],$$

where

$$M(\boldsymbol{i}) = \sup_{\boldsymbol{s}\in[\boldsymbol{i},\boldsymbol{i}+\boldsymbol{1}]}\boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{\alpha}\cdot\boldsymbol{s}.$$

Let $\boldsymbol{i} = (i_1, i_2, \ldots, i_d)$, where without loss of generality $i_1 \le i_2 \le \cdots \le i_d$ and $(j_1, \ldots, j_m)$ be a vector of unique values of $\boldsymbol{i}$, where $j_1 < j_2 < \cdots < j_m$. Define for $b \in \{1, \ldots, m\} l_b = \min_{k\in(1,2,\ldots,d)}\{i_k = j_b\}$. Then using independence of increments of Brownian motion we have

$$\begin{aligned}
M(\boldsymbol{i}) &= \sup_{\boldsymbol{s}\in[\boldsymbol{i},\boldsymbol{i}+\boldsymbol{1}]}\boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{\alpha}\cdot\boldsymbol{s} - (0,\ldots,W_{l_m}(j_m),\ldots,W_d(j_m)) + (0,\ldots,W_{l_m}(j_m),\ldots,W_d(j_m)) \\
&= \sup_{\boldsymbol{s}\in[\boldsymbol{i},\boldsymbol{i}+\boldsymbol{1}]}(W_1(s_1),\ldots,W_{l_m-1}(s_{l_m-1}),W_l(j_m),\ldots,W_d(j_m)) - \boldsymbol{\alpha}\cdot\boldsymbol{s} \\
&\quad +(0,\ldots,0,W_{l_m}(s_{l_m}) - W_l(j_m),\ldots,W_d(s_d) - W_d(j_m)) \\
&= \sup_{\boldsymbol{s}\in[(i_1,\ldots,i_{l_m-1}),(i_1,\ldots,i_{l_m-1})+\boldsymbol{1}]}(W_1(s_1),\ldots,W_{l_m-1}(s_{l_m-1}),W_l(j_m),\ldots,W_d(j_m)) - (\alpha_1,\ldots,\alpha_{l_m-1})\cdot\boldsymbol{s} \\
&\quad -(\alpha_{l_m},\ldots,\alpha_d)\cdot(j_m,\ldots,j_m) + \sup_{\boldsymbol{t}\in[0,1]^{d-l_m+1}}(0,\ldots,0,\bar{W}_{l_m}(t_1),\ldots,\bar{W}_d(t_{d-l_m+1})) \\
&\quad -(0,\ldots,0,\alpha_{l_m},\ldots,\alpha_d)\cdot(0,\ldots,0,\boldsymbol{t}) \\
&:= \sup_{\boldsymbol{s}\in[(i_1,\ldots,i_{l_m-1}),(i_1,\ldots,i_{l_m-1})+\boldsymbol{1}]}(W_1(s_1),\ldots,W_{l_m-1}(s_{l_m-1}),W_l(j_m),\ldots,W_d(j_m)) \\
&\quad -(\alpha_1,\ldots,\alpha_d)\cdot(\boldsymbol{s},j_m,\ldots,j_m) + \boldsymbol{Q}_m,
\end{aligned}$$

where $\bar{\boldsymbol{W}}$ is an independent copy of $\boldsymbol{W}$. If $m > 1$ then again with independence of increments we get

$$\begin{aligned}
M(\boldsymbol{i}) &= \sup_{\boldsymbol{s}\in[(i_1,\ldots,i_{l_m-1}),(i_1,\ldots,i_{l_m-1})+\boldsymbol{1}]}(W_1(s_1),\ldots,W_{l_{m-1}-1}(s_{l_{m-1}-1}),W_{l_{m-1}}(j_{m-1}),\ldots,W_d(j_{m-1})) \\
&\quad -(\alpha_1,\ldots,\alpha_d)\cdot(\boldsymbol{s},j_m,\ldots,j_m) \\
&\quad +(0,\ldots,0,W_{l_m}(j_m) - W_{l_m}(j_{m-1}+1),\ldots,W_d(j_m) - W_d(j_{m-1}+1)) \\
&\quad +(0,\ldots,0,W_{l_{m-1}}(s_{l_{m-1}}) - W_h(j_{m-1}),\ldots,W_{l_m-1}(s_{l_m-1}) - W_{l_m-1}(j_{m-1}),0,\ldots,0) \\
&\quad +(0,\ldots,0,W_{l_m}(j_{m-1}+1) - W_{l_m}(j_{m-1}),\ldots,W_d(j_{m-1}+1) - W_d(j_{m-1})) + \boldsymbol{Q}_m \\
&= \sup_{\boldsymbol{s}\in[(i_1,\ldots,i_{l_{m-1}-1}),(i_1,\ldots,i_{l_{m-1}-1})+\boldsymbol{1}]}(W_1(s_1),\ldots,W_{l_{m-1}-1}(s_{l_{m-1}-1}),W_h(j_{m-1}),\ldots,W_d(j_{m-1})) \\
&\quad -(\alpha_1,\ldots,\alpha_d)\cdot(\boldsymbol{s},j_{m-1},\ldots,j_{m-1},j_m,\ldots,j_m) \\
&\quad +(0,\ldots,0,\underline{W}_{l_m}(j_m - j_{m-1} - 1),\ldots,\underline{W}_d(j_m - j_{m-1} - 1)) \\
&\quad + \sup_{\boldsymbol{t}\in[(i_{l_{m-1}},\ldots,i_{l_m-1}),(i_{l_{m-1}},\ldots,i_{l_m-1})+\boldsymbol{1}]}(0,\ldots,0,W_{l_{m-1}}(s_{l_{m-1}}) - W_{l_{m-1}}(j_{m-1}),
\end{aligned}$$

$$
\begin{aligned}
& \dots, W_{l_m-1}(s_{l_m-1}) - W_{l_m-1}(j_{m-1}), 0, \dots, 0) \\
& +(0,\dots,0, W_{l_m}(j_{m-1}+1) - W_{l_m}(j_{m-1}), \dots, W_d(j_{m-1}+1) - W_d(j_{m-1})) + \boldsymbol{Q}_m \\
= & \sup_{\boldsymbol{s}\in[(i_1,\dots,i_{l_{m-1}-1}),(i_1,\dots,i_{l_{m-1}-1})+\mathbf{1}]} (W_1(s_1),\dots,W_{l_{m-1}-1}(s_{l_{m-1}-1}), W_{l_{m-1}}(j_{m-1}),\dots,W_d(j_{m-1})) \\
& -(\alpha_1,\dots,\alpha_d)\cdot(\boldsymbol{s}, j_{m-1},\dots,j_{m-1}, j_m-1,\dots,j_m-1) \\
& +(0,\dots,0,\underline{W}_{l_m}(j_m-j_{m-1}-1),\dots,\underline{W}_d(j_m-j_{m-1}-1)) \\
& + \sup_{\boldsymbol{t}\in[0,1]^{l_m-l_{m-1}}} (0,\dots,0,\widetilde{W}_{l_{m-1}}(t_1),\dots,\widetilde{W}_{l_m-1}(t_{l_m-l_{m-1}}),0,\dots,0) \\
& +(0,\dots,0,\widetilde{W}_{l_m}(1),\dots,\widetilde{W}_d(1)) + \boldsymbol{Q}_m \\
& -(0,\dots,0,\alpha_{l_{m-1}},\dots,\alpha_{l_m-1},0,\dots,0)\cdot(0,\dots,0,\boldsymbol{t},1,\dots,1) \\
:= & \sup_{\boldsymbol{s}\in[(i_1,\dots,i_{l_{m-1}-1}),(i_1,\dots,i_{l_{m-1}-1})+\mathbf{1}]} (W_1(s_1),\dots,W_{l_{m-1}-1}(s_{l_{m-1}-1}), W_{l_{m-1}}(j_{m-1}),\dots,W_d(j_{m-1})) \\
& -(\alpha_1,\dots,\alpha_d)\cdot(\boldsymbol{s}, j_{m-1},\dots,j_{m-1}, j_m-1,\dots,j_m-1) \\
& +(0,\dots,0,\underline{W}_{l_m}(j_m-j_{m-1}-1),\dots,\underline{W}_d(j_m-j_{m-1}-1)) + \boldsymbol{Q}_{m-1} + \boldsymbol{Q}_m,
\end{aligned}
$$

where $\underline{\boldsymbol{W}}, \widetilde{\boldsymbol{W}}$ are independent copies of $\boldsymbol{W}$, also independent of $\bar{\boldsymbol{W}}$. Iterating for all m steps, in the end we get that for some $\boldsymbol{C}\in\mathbb{R}^d$

$$
\begin{aligned}
M(\boldsymbol{i}) = & \sum_{j=1}^m Q_j + (W_1(j_1),\dots,W_{l_2-1}(j_1), W_{l_2}(j_2)-1,\dots,W_{l_3-1}(j_2)-1,\dots,W_{l_m}(j_m)-(m-1)) \\
& -\boldsymbol{\alpha}\cdot(j_1,\dots,j_1,j_2-1,\dots,j_2-1,\dots,j_m-(m-1)) \\
\le & \boldsymbol{C} + \sum_{j=1}^m Q_j + (W_1^*(j_1),\dots,W_{l_2-1}^*(j_1),W_{l_2}^*(j_2),\dots,W_{l_3-1}^*(j_2),\dots,W_{l_m}^*(j_m))
\end{aligned}
$$

where

$$
\begin{aligned}
Q_j = & \sup_{\boldsymbol{t}\in[0,1]^{l_{j+1}-l_j}} (0,\dots,0,W_{l_j}(t_1),\dots,W_{l_{j+1}-1}(t_{l_{j+1}-l_j}),W_{l_{j+1}}(1),\dots,W_d(1)) \\
& -(0,\dots,0,\alpha_{l_j},\dots,\alpha_d)\cdot(0,\dots,0,\boldsymbol{t},1,\dots,1)
\end{aligned}
$$

with $l_{m+1}=d+1$. Denote $Q_{m,\boldsymbol{i}} = \sum_{j=1}^m Q_j$ and further

$$
Q^* = \max_{m\in\{1,\dots,d\}} \max_{\boldsymbol{i}\in perm(\{1,\dots,d\})} \boldsymbol{\lambda}^\top Q_{m,\boldsymbol{i}}.
$$

Since with [1][Thm. 5.1] we have that $Q_{m,\boldsymbol{i}}<\infty$, hence using that $d<\infty$ we have that

$$
I(n) \le \frac{1}{\prod_{i=1}^d \lambda_i} e^{\boldsymbol{\lambda}^\top \boldsymbol{C}} e^{Q^*} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[e^{\boldsymbol{\lambda}^\top (W_1^*(j_1),\dots,W_{l_2-1}^*(j_1),W_{l_2}^*(j_2),\dots,W_{l_3-1}^*(j_2),\dots,W_{l_m}^*(j_m))}],
$$

For $A(\boldsymbol{t})A(\boldsymbol{t})^\top = \Sigma_{\boldsymbol{t}}, \chi := (W_1(j_1), \dots, W_{l_2-1}(j_1), W_{l_2}(j_2), \dots, W_{l_3-1}(j_2), \dots, W_{l_m}(j_m))$ we have

$$\begin{aligned}
\mathbb{E}[e^{\boldsymbol{\lambda}^\top \chi}] &= \prod_{k=1}^{m} \prod_{v=0}^{l_m-1} \mathbb{E}[e^{(A(\boldsymbol{i})^\top \boldsymbol{\lambda})_{i_{j_k+v}} B_{i_{j_k+v}}(j_k)}] \\
&= \prod_{k=1}^{m} \prod_{v=0}^{l_m-1} e^{\frac{j_k (A(\boldsymbol{i})^\top \boldsymbol{\lambda})^2_{i_{j_k+v}}}{2}} = e^{\frac{\boldsymbol{\lambda}^\top \Sigma_{\boldsymbol{i}} \boldsymbol{\lambda}}{2}} \\
&\le e^{\frac{\boldsymbol{\alpha}^\top \Sigma_{\mathbf{1}}^{-1} \Sigma \Sigma_{\mathbf{1}}^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{i}}{2}} = e^{\frac{\boldsymbol{\alpha}^\top \Sigma_{\mathbf{1}}^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{i}}{2}} \\
&= e^{\frac{\boldsymbol{\lambda}^\top \boldsymbol{\alpha} \cdot \boldsymbol{i}}{2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
I(n) &\le \frac{1}{\prod_{i=1}^{d} \lambda_i} e^{\boldsymbol{\lambda}^\top \boldsymbol{C}} e^{Q^*} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} e^{\frac{\boldsymbol{\lambda}^\top \boldsymbol{\alpha} \cdot \boldsymbol{i}}{2} - \boldsymbol{\lambda}^\top \boldsymbol{\alpha} \cdot \boldsymbol{i}} \\
&= \frac{1}{\prod_{i=1}^{d} \lambda_i} e^{\boldsymbol{\lambda}^\top \boldsymbol{C}} e^{Q^*} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} e^{-\frac{\boldsymbol{\lambda}^\top \boldsymbol{\alpha} \cdot \boldsymbol{i}}{2}} < \infty.
\end{aligned}$$

By letting $n \to \infty$ we obtain the claim. □

Proof of Theorem 5.2.2 Define $\boldsymbol{b}(\boldsymbol{t}) := \boldsymbol{\alpha} \Sigma_{\boldsymbol{t}}^{-1} > 0$. Notice that for any $H \subset [0,1]^d$ we can write

$$\mathbb{P}\{\exists_{\boldsymbol{t} \in H} \boldsymbol{W}(\boldsymbol{t}) > \boldsymbol{\alpha} u\} \le \mathbb{P}\left\{\exists_{\boldsymbol{t} \in H} \frac{\boldsymbol{b}(\boldsymbol{t}) \boldsymbol{W}(\boldsymbol{t})^\top}{\boldsymbol{b}(\boldsymbol{t}) \boldsymbol{\alpha}^\top} > u\right\}.$$

Further denote by $Z(\boldsymbol{t}) := \frac{\boldsymbol{b}(\boldsymbol{t}) \boldsymbol{W}(\boldsymbol{t})}{\boldsymbol{b}(\boldsymbol{t}) \boldsymbol{\alpha}}$ and observe that

$$Var(Z(\boldsymbol{t})) = \frac{1}{q_{\boldsymbol{\alpha}}(\boldsymbol{t})}.$$

From Remark 5.3.2 we have that function $q_{\boldsymbol{\alpha}}$ is minimised at $\mathbf{1}$. For any $\epsilon > 0$ define

$$H_\epsilon = [1-\epsilon, 1]^d, \quad F_u = [1 - \frac{1}{u}, 1]^d.$$

Since $\sup_{\boldsymbol{t} \in [0,1]^d \setminus H_\epsilon} Var(Z(\boldsymbol{t})) < Var(Z(\mathbf{1}))$, hence for any small enough ϵ using Borell-TIS inequality (see e.g. [Thm 2.6.1] [2]) we have

$$\mathbb{P}\{\exists_{\boldsymbol{t} \in [0,1]^d \setminus H_\epsilon} \boldsymbol{W}(\boldsymbol{t}) > \boldsymbol{\alpha} u\} \le e^{-r \frac{u}{2Var(Z(\mathbf{1}))}},$$

for large enough u and some $r > 1$. Since $\boldsymbol{C}$ is independent of u this also gives us that

$$\mathbb{P}\{\exists_{\boldsymbol{t} \in [0,1]^d \setminus H_\epsilon} \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} \le e^{-r' \frac{u}{2Var(Z(\mathbf{1}))}}, \tag{5.8}$$

for large enough u and some $r' > 1$. Next notice that using Taylor expansion we see that for all $\boldsymbol{t} \in H_\epsilon \setminus F_u$ we have

$$Var(Z(\mathbf{1})) - Var(Z(\boldsymbol{t})) > \tau \frac{1}{u}$$

for large enough u and some $\tau > 0$. Since Z is a Hölder continous random field, then applying [66][Thm 8.1] for some $C_1, C_2, c_1 > 0$ we have

$$\begin{aligned}
\mathbb{P}\left\{\exists_{\boldsymbol{t}\in H_\epsilon\setminus F_u} \boldsymbol{W}(\boldsymbol{t}) > \boldsymbol{\alpha} u\right\} &\leq \mathbb{P}\left\{\exists_{\boldsymbol{t}\in H_\epsilon\setminus F_u} \frac{Z(\boldsymbol{t})}{\sqrt{Var(Z(\boldsymbol{t}))}} > \frac{u}{\sqrt{Var(Z(\boldsymbol{t}))}}\right\} \\
&\leq \mathbb{P}\left\{\exists_{\boldsymbol{t}\in H_\epsilon\setminus F_u} \frac{Z(\boldsymbol{t})}{\sqrt{Var(Z(\boldsymbol{t}))}} > \frac{u}{\sqrt{Var(Z(\mathbf{1})) - \frac{\tau}{u}}}\right\} \\
&\leq C_1 u^{c_1} e^{-\frac{u^2}{2(Var(Z(\mathbf{1}))-\frac{\tau}{u})}} \\
&\sim C_1 u^{c_1} e^{-\frac{u^2}{2Var(Z(\mathbf{1}))} - C_2 u}.
\end{aligned}$$

Hence we can also get for some $C_1', C_2', c_1' > 0$

$$\mathbb{P}\left\{\exists_{\boldsymbol{t}\in H_\epsilon\setminus F_u} \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\right\} \leq C_1' u^{c_1'} e^{-\frac{u^2}{2Var(Z(\mathbf{1}))} - C_2' u}. \tag{5.9}$$

Next, using Lemma 5.3.3 for any $\Delta > 0$ we get

$$\begin{aligned}
\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c}, u) &\geq \mathbb{P}\left\{\exists \boldsymbol{t} \in [1 - \frac{\Delta}{u^2}, 1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\right\} \\
&= I(\Delta) u^{-d} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c}),
\end{aligned}$$

where

$$I(\Delta) = \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists \boldsymbol{t} \in [0, \Delta]^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha} \cdot \boldsymbol{t} > \boldsymbol{x}\right\} e^{\boldsymbol{\alpha}\Sigma_1^{-1}\boldsymbol{x}^\top} d\boldsymbol{x}.$$

Inequalities (5.8) and (5.9) give us that

$$\mathbb{P}\left\{\exists_{\boldsymbol{t}\in[0,1]^d\setminus F_u} \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\right\} = o\left(\mathbb{P}\{\exists_{\boldsymbol{t}\in F_u} \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\}\right).$$

Hence with $\mathbf{N_u} = (\lfloor \frac{\mathbf{u}}{\boldsymbol{\Delta}} \rfloor, \ldots, \lfloor \frac{\mathbf{u}}{\boldsymbol{\Delta}} \rfloor)$ we have

$$\begin{aligned}
\pi_{\Sigma_{\boldsymbol{t}},\boldsymbol{\alpha}}(\boldsymbol{c}, u) &\leq \mathbb{P}\{\exists_{\boldsymbol{t}\in F_u} \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\}(1 + o(1)) \\
&\leq \sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{N_u}} \mathbb{P}\{\exists \boldsymbol{t} \in E_{\boldsymbol{i}_u} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\}(1 + o(1)).
\end{aligned}$$

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For any $\mathbf{1} < \boldsymbol{i} < \mathbf{N_u}$ denote $E'_{\boldsymbol{i}_u - \boldsymbol{k}} = [\boldsymbol{i}_u - \frac{\boldsymbol{k}}{u^2}, \boldsymbol{i}_u]$. With Lemma 2.2.4 and Lemma 5.3.1 we have that for some $\boldsymbol{C} > \mathbf{0}$

$$\begin{aligned}
\mathbb{P}\{\exists \boldsymbol{t} \in E_{\boldsymbol{i}_u} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\} &\leq \sum_{k=\mathbf{1}}^{(\Delta,\dots,\Delta)} \mathbb{P}\big\{\exists \boldsymbol{t} \in E'_{\boldsymbol{i}_u - \boldsymbol{k}} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\big\} \\
&\leq \Delta^d \max_{k \in (\mathbf{1}, (\Delta,\dots,\Delta))} \mathbb{P}\big\{\exists \boldsymbol{t} \in E'_{\boldsymbol{i}_u - \boldsymbol{k}} : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\big\} \\
&\leq \Delta^d u^{-d} e^{-u^2 \boldsymbol{C}(\mathbf{1} - \boldsymbol{i}_u)^\top} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c}) \max_{k \in (\mathbf{1}, (\Delta,\dots,\Delta))} I_{\boldsymbol{i}_u - \boldsymbol{k}},
\end{aligned}$$

where

$$I_{\boldsymbol{i}_u - \boldsymbol{k}} = \int_{\mathbb{R}^n} \mathbb{P}\{\exists \boldsymbol{t} \in [0,1]^n : \boldsymbol{X}_{\boldsymbol{i}_u - \boldsymbol{k}}(\boldsymbol{t}) > \boldsymbol{x}\} e^{\boldsymbol{\alpha} \Sigma_1^{-1} \boldsymbol{x}^\top} d\boldsymbol{x}$$

with $\boldsymbol{X}_{\boldsymbol{i}_u - \boldsymbol{k}}(\boldsymbol{t})$ defined as in Lemma 2.2.4. Using the proof of Lemma 2.2.4 for any $\boldsymbol{k} \in (\mathbf{1}, (\Delta, \dots, \Delta))$ we have that $I_{\boldsymbol{i}_u - \boldsymbol{k}} < \infty$ and hence $\max_{k \in (\mathbf{1}, (\Delta,\dots,\Delta))} I_{\boldsymbol{i}_u - \boldsymbol{k}} < \infty$. Therefore we have that

$$\begin{aligned}
\pi_{\Sigma_{\boldsymbol{t}}, \boldsymbol{\alpha}}(\boldsymbol{c}, u) &\leq \mathbb{P}\Big\{\exists \boldsymbol{t} \in [1 - \frac{\Delta}{u^2}, 1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\Big\} \\
&\quad + \sum_{\mathbf{1} < \boldsymbol{i} < N_u} \Delta^d \max_{k \in (\mathbf{1}, (\Delta,\dots,\Delta))} I_{\boldsymbol{i}_u - \boldsymbol{k}} u^{-d} e^{-u^2 \boldsymbol{C}(\mathbf{1} - \boldsymbol{i}_u)^\top} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c})(1 + o(1)) \\
&= \mathbb{P}\Big\{\exists \boldsymbol{t} \in [1 - \frac{\Delta}{u^2}, 1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\Big\} \\
&\quad + \sum_{\mathbf{1} < \boldsymbol{i} < N_u} \Delta^d \max_{k \in (\mathbf{1}, (\Delta,\dots,\Delta))} I_{\boldsymbol{i}_u - \boldsymbol{k}} u^{-d} e^{-\Delta \boldsymbol{C} \boldsymbol{i}^\top} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c})(1 + o(1)).
\end{aligned}$$

Using the above we get that with $\boldsymbol{C} = (C_1, C_2, \dots, C_d) > 0$ and some $C > 0$ we have

$$\pi_{\Sigma_{\boldsymbol{t}}, \boldsymbol{\alpha}}(\boldsymbol{c}, u) \leq u^{-d} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c}) I(\Delta)(1 + o(1)) + C u^{-d} \Delta^d \varphi(\boldsymbol{\alpha} u + \boldsymbol{c}) \prod_{j=1}^{d} \frac{e^{-C_j \Delta}}{1 - e^{-C_j \Delta}} (1 + o(1)).$$

Using Lemma 5.3.4 to observe that $\lim_{\Delta \to \infty} I(\Delta) \in (0, \infty)$, we let $\Delta \to \infty$ and get that

$$\pi_{\Sigma_{\boldsymbol{t}}, \boldsymbol{\alpha}}(\boldsymbol{c}, u) \leq \int_{\mathbb{R}^d} \mathbb{P}\big\{\exists \boldsymbol{t} \in [0, \infty)^d : \boldsymbol{W}(\boldsymbol{t}) - \boldsymbol{\alpha} \cdot \boldsymbol{t} > \boldsymbol{x}\big\} e^{\boldsymbol{\alpha} \Sigma_1^{-1} \boldsymbol{x}^\top} d\boldsymbol{x} u^{-d} \varphi(\boldsymbol{\alpha} u + \boldsymbol{c})(1 + o(1)),$$

which completes the proof. $\square$

Bibliography

[1] Adler, R. J. (1990). An introduction to continuity, extrema, and related topics for general Gaussian processes. IMS.

[2] Adler, R. J. and Taylor, J. E. (2007). *Random fields and geometry.* Springer Monographs in Mathematics. Springer, New York.

[3] Arendarczyk, M. and Dębicki, K. (2011). Asymptotics of supremum distribution of a Gaussian process over a Weibullian time. *Bernoulli*, 17(1):194–210.

[4] Avram, F., Palmowski, Z., and Pistorius, M. (2008a). A two-dimensional ruin problem on the positive quadrant. *Insurance: Mathematics and Economics*, 42(1):227–234.

[5] Avram, F., Palmowski, Z., and Pistorius, M. R. (2008b). Exit problem of a two-dimensional risk process from the quadrant: exact and asymptotic results. *The Annals of Applied Probability*, 18(6):2421–2449.

[6] Azaïs, J.-M. and Wschebor, M. (2009). *Level sets and extrema of random processes and fields.* John Wiley & Sons.

[7] Bai, L., Dębicki, K., Hashorva, E., and Ji, L. (2018). Extremes of threshold-dependent Gaussian processes. *Science China Mathematics*, 61(11):1971–2002.

[8] Bisewski, K., Dębicki, K., and Kriukov, N. (2023). Simultaneous ruin probability for multivariate Gaussian risk model. *Stochastic Processes and their Applications*, 160:386–408.

[9] Bisewski, K., Dębicki, K., and Mandjes, M. (2021). Bounds for expected supremum of fractional Brownian motion with drift. *Journal of Applied Probability*, 58(2):411–427.

[10] Bonanno, G., Lillo, F., and Mantegna, R. N. (2001). High-frequency cross-correlation in a set of stocks.

[11] Borodin, A. N. and Salminen, P. (2002). *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition.

[12] Burnecki, K. and Michna, Z. (2002). Simulation of Pickands constants. *Probability and Mathematical Statistics*, pages 193–199.

[13] Dassios, A. and Wu, S. (2008). Parisian ruin with exponential claims. *Working paper*.

[14] Dębicki, K., Hashorva, E., and Ji, L. (2015). Parisian ruin of self-similar Gaussian risk processes. *Journal of Applied Probability*, 52(3):688–702.

[15] Dębicki, K., Hashorva, E., and Ji, L. (2016). Parisian ruin over a finite-time horizon. *Science China Mathematics*, 59(3):557–572.

[16] Dębicki, K., Hashorva, E., Ji, L., and Rolski, T. (2018). Extremal behavior of hitting a cone by correlated Brownian motion with drift. *Stochastic Processes and their Applications*, 128(12):4171 – 4206.

[17] Dębicki, K., Hashorva, E., and Kriukov, N. (2021a). Pandemic-type failures in multivariate Brownian risk models. *Extremes*, pages 1–23.

[18] Dębicki, K., Hashorva, E., and Krystecki, K. (2021b). Finite-time ruin probability for correlated Brownian motions. *Scandinavian Actuarial Journal*, pages 1–26.

[19] Dębicki, K., Hashorva, E., and Liu, P. (2017). Extremes of γ-reflected Gaussian processes with stationary increments. *ESAIM: Probability and Statistics*, 21:495–535.

[20] Dębicki, K., Hashorva, E., and Michna, Z. (2020a). Simultaneous ruin probability for two-dimensional Brownian risk model. *Journal of Applied Probability*, 57(2):597–612.

[21] Dębicki, K., Hashorva, E., and Wang, L. (2020b). Extremes of vector-valued Gaussian processes. *Stochastic Processes and their Applications*, 130(9):5802–5837.

[22] Dębicki, K. and Jasnovidov, G. (2024). Extremes of reflecting gaussian processes on discrete grid. *Journal of Mathematical Analysis and Applications*, 532(1):127952.

[23] Dębicki, K., Ji, L., and Rolski, T. (2019a). Logarithmic asymptotics for probability of component-wise ruin in a two-dimensional Brownian model. *Risks*, 7(3):83.

[24] Dębicki, K., Ji, L., and Rolski, T. (2019b). Logarithmic asymptotics for probability of component-wise ruin in a two-dimensional Brownian model. *Risks*, 7(3).

[25] Dębicki, K., Ji, L., and Rolski, T. (2020c). Exact asymptotics of component-wise extrema of two-dimensional Brownian motion. *Extremes*, 23(4):569–602.

[26] Dębicki, K., Kosiński, K., Mandjes, M., and Rolski, T. (2010). Extremes of multidimensional Gaussian processes. *Stochastic Processes and their Applications*, 120(12):2289 – 2301.

[27] Dębicki, K. and Mandjes, M. (2003). Exact overflow asymptotics for queues with many Gaussian inputs. *Journal of Applied Probability*, 40(3):704–720.

[28] Dębicki, K., Michna, Z., and Rolski, T. (2003). Simulation of the asymptotic constant in some fluid models. *Stochastic Models*, 19:407–423.

[29] Dębicki, K. and Tabiś, K. (2011). Extremes of the time-average of stationary Gaussian processes. *Stochastic Process. Appl.*, 121(9):2049–2063.

[30] Dębicki, K., Hashorva, E., and Ji, L. (2014a). Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. *Extremes*, 17(3):411–429.

[31] Dębicki, K., Hashorva, E., Ji, L., and Tabiś, K. (2014b). On the probability of conjunctions of stationary Gaussian process. *Stat. Probab. Lett*, 88:141–148.

[32] Dębicki, K., Hashorva, E., Ji, L., and Tabiś, K. (2015). Extremes of vector-valued Gaussian processes: Exact asymptotics. *Stochastic Process. Appl.*, 125(11):4039–4065.

[33] Dębicki, K. and Kosiński, K. (2014). On the infimum attained by the reflected fractional Brownian motion. *Extremes*, 17(3):431–446.

[34] Dębicki, K., Michna, Z., and Rolski, T. (1998). On the supremum from Gaussian processes over infinite horizon. *Probability and Mathematical Statistics*, 18:83–100.

[35] Dębicki, K. and Rolski, T. (2002). A note on transient Gaussian fluid models. *Queueing Systems*, 41(4):321–342.

[36] Dieker, A. (2005). Extremes of Gaussian processes over an infinite horizon. *Stochastic Processes and their Applications*, 115(2):207 – 248.

[37] Elton, E. J. and Gruber, M. J. (1971). Improved forecasting through the design of homogeneous groups. *The Journal of Business*, 44(4):432–450.

[38] Feldheim, N., Feldheim, O., and Nitzan, S. (2017). Persistence of Gaussian stationary processes: a spectral perspective. *arXiv preprint arXiv:1709.00204*.

[39] Gajda, J. and Magdziarz, M. (2014). Large deviations for subordinated Brownian motion and applications. *Statistics & Probability Letters*, 88:149–156.

[40] Ginsbourger, D. and Le Riche, R. (2010). Towards Gaussian process-based optimization with finite time horizon. In *mODa 9–Advances in Model-Oriented Design and Analysis: Proceedings of the 9th International Workshop in Model-Oriented Design and Analysis held in Bertinoro, Italy, June 14-18, 2010*, pages 89–96. Springer.

[41] Grigelionis, B. (2007). On subordinated multivariate Gaussian Lévy processes. *Acta Applicandae Mathematicae*, 96(1):233–246.

[42] Hashorva, E. (2005). Asymptotics and bounds for multivariate Gaussian tails. *Journal of Theoretical Probability*, 18:79–97.

[43] Hashorva, E. and Hüsler, J. (2000). Extremes of Gaussian processes with maximal variance near the boundary points. *Methodology and Computing in Applied Probability*, 2(3):255–269.

[44] Hashorva, E. and Hüsler, J. (2003). On multivariate Gaussian tails. *Annals of the Institute of Statistical Mathematics*, 55:507–522.

[45] He, H., Keirstead, W., and Rebholz, J. (1998). Double lookbacks. *Mathematical Finance, Vol. 8, No. 3 (July 1998), 201–228.*

[46] Ievlev, P. (2023). Parisian ruin with power-asymmetric variance near the optimal point with application to many-inputs proportional reinsurance. *Stochastic Models*, page 1.

[47] Iglehart, L. D. (1969). Diffusion approximations in collective risk theory. *Journal of Applied Probability*, 6(2):285–292.

[48] Jasnovidov, G. (2020). Approximation of ruin probability and ruin time in discrete Brownian risk models. *Scandinavian Actuarial Journal*, 2020(8):718–735.

[49] Jasnovidov, G. and Shemendyuk, A. (2021). Parisian ruin for insurer and reinsurer under quata-share treaty. *arXiv preprint arXiv:2103.03213*.

[50] Ji, L. (2020). On the cumulative Parisian ruin of multi-dimensional Brownian motion risk models. *Scandinavian Actuarial Journal*, 2020(9):819–842.

[51] Ji, L. and Peng, X. (2022). Extrema of multi-dimensional Gaussian processes over random intervals. *Journal of Applied Probability*, 59(1):81–104.

[52] Ji, L. and Robert, S. (2018). Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stochastic Models*, 34(1):73–97.

[53] Kępczyński, K. (2022). Running supremum of Brownian motion in dimension 2: exact and asymptotic results. *Stochastic Models*, 38(1):116–129.

[54] Korshunov, D. and Wang, L. (2020). Tail asymptotics for Shepp-statistics of Brownian motion in $\mathbb{R}^d$. *Extremes*, 23(1):35–54.

[55] Kozik, I. and Piterbarg, V. I. (2018). High excursions of gaussian nonstationary processes in discrete time. *Fundamentalnaya i Prikladnaya Matematika*, 22(2):159–169.

[56] Kriukov, N. (2022). Parisian and cumulative Parisian ruin probability for two-dimensional Brownian risk model. *Stochastics*, 94(4):629–645.

[57] Krystecki, K. (2022). Parisian ruin probability for two-dimensional Brownian risk model. *Statistics & Probability Letters*, 182.

[58] Krystecki, K. (2023). Cumulative parisian ruin probability for two-dimensional Brownian risk model. *Probability and Mathematical Statistics*, pages 63–81.

[59] Metzler, A. (2010). On the first passage problem for correlated Brownian motion. *Statistics & Probability Letters*, 80(5-6):277–284.

[60] Michna, Z. (1998). Self-similar processes in collective risk theory. *Journal of Applied Mathematics and Stochastic Analysis*, 11(4):429–448.

[61] Michna, Z. (2020). Ruin probabilities for two collaborating insurance companies. *Probability and Mathematical Statistics*, 40(2):369–386.

[62] O'Connell, N. and Unwin, A. (1992). Collision times and exit times from cones: a duality. *Stochastic Processes and their Applications*, 43(2):291–301.

[63] Peng, X. and Luo, L. (2017). Finite time parisian ruin of an integrated Gussian risk model. *Statistics & Probability Letters*, 124:22–29.

[64] Pickands, J. (1969). Upcrossing probabilities for stationary Gaussian processes. *Transactions of the American Mathematical Society*, 145:51–73.

[65] Pickands, III, J. (1967). Maxima of stationary Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:190–223.

[66] Piterbarg, V. I. (1996). *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI.

[67] Puchala, Z. and Rolski, T. (2005). The exact asymptotic of the time to collision. *Electronic Journal of Probability*, 10:1359–1380.

[68] Ross, S. M. (2014). *Introduction to probability models*. Academic Press.

[69] Shao, J. and Wang, X. (2013). Estimates of the exit probability for two correlated Brownian motions. *Advances in Applied Probability*, 45(1):37–50.

[70] Wang, W. and Chen, Z. (2018). Large deviations for subordinated fractional Brownian motion and applications. *Journal of Mathematical Analysis and Applications*, 458(2):1678–1692.

[71] Wu, Z., Chakrabarty, A., Samorodnitsky, G., et al. (2019). High minima of non-smooth Gaussian processes. *Electronic Communications in Probability*, 24:1–12.

[72] Zubeldia, M. and Mandjes, M. (2021). Large deviations for acyclic networks of queues with correlated Gaussian inputs. *Queueing Systems*, 98(3):333–371.