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O izomorficznych i geometrycznych własnościach przestrzeni Banacha funkcji ciągłych

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On isomorphic and geometric properties of Banach spaces of continuous functions

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"LORD, you will decree peace for us,
for you have accomplished all we have done."
Isa 26:12

Streszczenie

Rozprawa jest poświęcona własnościom przestrzeni Banacha funkcji ciągłych o wartościach rzeczywistych na przestrzeniach zwartych. Rozważane przestrzenie zwarte są w naszym kontekście zazwyczaj liniowo uporządkowane, są to tzw. zwarte proste. Większość pytań formułowana jest w języku analizy funkcjonalnej, jednak ich rozwiązanie wymaga zazwyczaj stosowania topologii, teorii mnogości oraz teorii miary. W pracy analizujemy zarówno klasyczne, lecz wciąż aktywnie badane problemy, jak i takie postawione stosunkowo niedawno.

Rozpoczynamy od Rozdziału 3, w którym badamy własności operatorów rozszerzeń $E : C(K) \to C(L)$ dla pewnych par przestrzeni zwartych $K \subseteq L$. Jest to powiązane z problemem dotyczącym istnienia pewnych krótkich ciągów dokładnych, co wpisuje się w nurt stosowania metod homologicznych w teorii przestrzeni Banacha. W związku z tym wprowadzamy obiekt kombinatoryczny podobny do luk i dowodzimy niektórych jego własności, ukazując w ten sposób pewne strukturalne aspekty ciągów miar na liniach zwartych.

W Rozdziale 4 definiujemy nowy wymiar przestrzeni Banacha, który w konkretnym przypadku rozróżnia przestrzenie funkcji ciągłych na iloczynach niemetryzowalnych zwartych prostych o różnej liczbie czynników. Kontrastuje to z przypadkiem metryzowalnym, gdzie klasyczne twierdzenie Miljutina mówi, że każde dwie przestrzenie funkcji ciągłych na nieprzeliczalnych przestrzeniach zwartych metryzowalnych są izomorficzne.

Rozprawę zamyka Rozdział 5, w którym przedstawiamy oszacowania odległości Banacha-Mazura między pewnymi klasycznymi przestrzeniami funkcji ciągłych. Z jednej strony staramy się odpowiedzieć na pytania postawione przez Bessagę i Pełczyńskiego [12] o odległości w przypadku, gdy rozważane przestrzenie zwarte są przeliczalne, a z drugiej badamy odległość pomiędzy klasycznymi przestrzeniami ℓ_∞ i $L_\infty[0,1]$, co otwiera kilka interesujących kierunków badawczych.

Słowa kluczowe: przestrzeń funkcji ciągłych, zwarta prosta, operator rozszerzenia, prawie łańcuch, aksjomat Martina, przestrzenie Banacha nieizomorficzne z ich kwadratem, odległość Banacha-Mazura, przestrzeń iniektywna

Abstract

This dissertation primarily concerns Banach spaces of real-valued continuous functions on a compact space. We usually consider compact spaces that are linearly ordered, the so-called compact lines. Most of the questions are formulated in the language of functional analysis; however, their solutions usually require methods from topology, set theory, and measure theory. In this work, we analyse both classical problems that are still actively studied, as well as problems that have been posed relatively recently.

We begin with Chapter 3, in which we investigate the properties of extension operators $E : C(K) \to C(L)$ for certain pairs of compact spaces $K \subseteq L$. This topic is related to the problem of the existence of certain short exact sequences, which fits into the broader trend of applying homological methods in the Banach space theory. During the chapter, we introduce a combinatorial object similar to gaps and prove some of its properties, thus revealing certain structural aspects of sequences of measures on compact lines.

In Chapter 4, we define a new dimension for Banach spaces which, in a specific case, distinguishes between spaces of continuous functions on products of non-metrizable compact lines with different numbers of factors. This stands in contrast to the metrizable case, where Milutin's classical theorem states that any two spaces of continuous functions on uncountable compact metrizable spaces are isomorphic.

We conclude with Chapter 5, where we present estimates for the Banach–Mazur distance between certain classical spaces of continuous functions. On the one hand, we address questions posed by Bessaga and Pełczyński [**12**] concerning the distance when the compact spaces under consideration are countable. On the other hand, we study the distance between the classical spaces ℓ_∞ and $L_\infty[0,1]$, which opens several interesting directions for further research.

Keywords: space of continuous functions, compact line, extension operator, almost chain, martin's axiom, Banach spaces not isomorphic to their squares, Banach-Mazur distance, injective space

Contents

CHAPTER 1

Introduction

One of the most natural classes of Banach spaces is the class of spaces of real-valued continuous functions on a compact Hausdorff space K (we always assume that a compact space is Hausdorff), sometimes called $C(K)$-spaces. While the isometric classification of such Banach spaces reduces to the homeomorphic classification of the underlying compact spaces, the isomorphic classification is less restrictive. The first questions concerning the existence of isomorphisms between certain $C(K)$-spaces date back at least to Banach [9]. The class of $C(K)$-spaces is extremely rich; over the years, many interesting examples have been found or constructed and the area still contains many unanswered questions. The study of the structure of spaces of continuous functions requires tools from various fields of mathematics. Although most problems originate in functional analysis, it is natural to apply methods of general topology, measure theory and set theory.

In general, the structure of the space $C(K)$ is closely tied to the topological structure of the compact space K. For example, it is well known that the space $C(K)$ is separable if and only if K is metrizable. We study mainly nonseparable Banach spaces, since the full isomorphic classification of separable $C(K)$-spaces was proven by Miljutin, Bessaga and Pełczyński in the 1960s [12, 66]. Let us recall these results.

Theorem 1.0.1 (Bessaga, Pełczyński [12]). *Let K be a countable compact metric space and let α be its Cantor-Bendixson rank. Then the space $C(K)$ is isomorphic to the space $C([0,\beta])$, where $\beta = \omega^\alpha$ (here, we mean the ordinal exponentiation).*

Theorem 1.0.2 (Miljutin [66]). *Let K be an uncountable metric space. Then $C(K)$ is isomorphic to $C(2^\omega)$, where 2^ω denotes the Cantor space.*

Measure theory becomes important in the study of dual spaces. Bounded linear functionals on spaces of continuous functions have a classical description via the Riesz representation Theorem [81, Theorem 18.4.1] — every element of $C(K)^*$ is given by the integration with respect to a signed Radon measure of finite total variation. Moreover, if the space K is zero-dimensional, these measures correspond to finitely additive measures on the algebra of clopen sets.

1.1. Chapter 3: Countable discrete extensions of compact lines

The application of homological methods (such as exact sequences, commutative diagrams, etc.) is one of the modern trends in the isomorphic theory of Banach spaces; recently, there a new monograph devoted to this subject has been published [16]. Using such techniques, one can formulate new problems and constructvarious interesting examples of Banach spaces. One such problem, related to the classical Sobczyk Theorem and stated in 2003 by Cabello-Sánchez, Castillo, Kalton and Yost [17], asked whether for any nonmetrizable compact space K there is always a nontrivial twisted sum of $C(K)$ and c_0. This problem was mostly solved in 2018 by Avilés, Marciszewski and Plebanek [7]. We study the techniques developed in that paper. One of the methods for constructing a twisted sum of $C(K)$ and c_0 is to consider a countable discrete extension L of K and study its properties. Our main focus is the so-called property $(\mathcal{E})$ of L, which is equivalent to the statement that $C(L)$ is a trivial twisted sum of $C(K)$ and c_0.

First, we establish some general results concerning extension operators between compact lines and their countable discrete extensions. This notion is central to the subject, since the existence of a bounded extension operator from $C(K)$ to $C(L)$ implies that $C(L)$ is a trivial twisted sum of $C(K)$ and c_0. Our main results in this chapter are as follows (where $\mathrm{non}(\boldsymbol{\mathcal{I}})$ denotes a certain uncountable cardinal that is consistently strictly less than the continuum).

THEOREM 1.1.1. *Every separable compact line K of weight greater than or equal to $\mathrm{non}(\boldsymbol{\mathcal{I}})$ has a countable discrete extension L without property $(\mathcal{E})$.*

THEOREM 1.1.2. *Assume $\mathsf{MA}(\kappa)$. If K is a separable compact line of weight κ and $L \in \mathrm{CDE}(K)$, then L has property $(\mathcal{E})$.*

These two results show that if K is a separable compact line of weight κ strictly less than the continuum, then the existence of a countable discrete extension L of K without property $(\mathcal{E})$ is independent of ZFC. It also turns out that the relevant properties of countable discrete extensions of separable compact lines can be described in terms of combinatorial objects similar to gaps.

This chapter covers the content of the papers [6], [53] (joint work with Grzegorz Plebanek and Antonio Avilés, respectively), whose results have been merged, corrected, and in some cases rewritten to better fit the context of this dissertation.

1.2. Chapter 4: Functions on nonmetrizable cubes

In this chapter, we address the question of when the space $C(\prod_{i=1}^{n} K_i)$ is isomorphic to, or can be isomorphically embedded into, the space $C(\prod_{j=1}^{k} L_j)$ for certain compact spaces K_i, L_j. This problem can be seen as a generalisation of a classical question posed by Banach (included in his book from 1932 [9]) asking whether the spaces $C([0,1])$ and $C([0,1]^2)$ are isomorphic. Since for metrizable compact spaces these questions are answered by the theorems of Bessaga, Pełczyński and Miljutin, we focus on nonmetrizable case.

More precisely, we consider mostly nonmetrizable compact lines, motivated by results from two papers. In the first, Martínez-Cervantes and Plebanek [62] showed that products of nonmetrizable compact lines of different amount of factors are not homeomorphic (and therefore the corresponding spaces of continuous functions are not isometric). In the second, Michalak [65] proved, in particular, that if all the compact lines K_i and L_j are additionally separable, then the Banach spaces of continuous functions on products of nonmetrizable compact lines of different amount of factors are not isomorphic.

We obtained results similar to those of Michalak, but for products of compact lines of uncountable character. Our approach was inspired by the work of Semadeni [80]. Building on these ideas, Candido [20] introduced the notion of the Semadeni derivative and proved several structural theorems concerning it. First, we define Semadeni-Pełczyński derivative, which is a slight variation of Candido's definition. We then use it to introduce the Semadeni-Pełczyński dimension, which serves as our main tool in proving the main results. Inspired by Galego [41], we also provide an alternative proof of parts of his results on the isomorphic classification of spaces $C(2^\theta \times [0,\lambda^+])$ for cardinal numbers κ, λ.

This chapter covers the content of the paper [52], with minor corrections in the language and corrections of minor mathematical inaccuracies.

1.3. Chapter 5: Banach–Mazur distance

Given two isomorphic Banach spaces X and Y, the Banach–Mazur distance $d_{\mathrm{BM}}(X,Y)$ between them is defined as the infimum of the distortions $\|T\| \cdot \|T^{-1}\|$ taken over all isomorphisms $T\colon X \to Y$. This distance measures, in some sense, how far two isomorphic Banach spaces are from being isometric. Although this notion of distance is quite natural and was introduced in Banach's 1932 book [9], calculating its exact value is very complicated, even for relatively simple spaces. For instance, it is very interesting to determine the exact bounds for the Banach–Mazur distance between $C(K)$-spaces when K is metrizable. In the aforementioned article [12], Bessaga and Pełczyński asked about the Banach–Mazur distance between $C(K)$-spaces for K countable. It also appears to be very difficult to estimate the distance between $C([0,1])$ and $C(2^\omega)$.

During the 1960s and 1970s, several important results were published in this area, notably by Cambern [18, 19], Gordon [43] and Amir [3]. More recently, new articles by Candido, Cuth, Galego, Gergont, Havelka, Malec, Piasecki, Rondoš, Sari, Somaglia and Villada [21, 32, 42, 59, 74, 77] have provided estimates for the Banach–Mazur distance between $C(K)$-spaces for scattered compact spaces K, including countable ones. We present several partial results in this topic, improving some lower bounds and establishing exact values in specific cases. Due to the complexity of the inequalities involved, some of our arguments rely on computer-assisted calculations. The main results of this part are as follows.

Theorem 1.3.1.

$$3.4704 < d_{\mathrm{BM}}\big(C([0,\omega] \times 3), C[0,\omega]\big) \le \frac{4 + \sqrt[3]{73 - 6\sqrt{87}} + \sqrt[3]{73 + 6\sqrt{87}}}{3} \approx 3.87512\ldots$$

Theorem 1.3.2. *Let K be a compact space such that $K^{(m)} \neq \emptyset$ for $m \ge 2$. Then*

$$d_{\mathrm{BM}}\big(C(K), C[0,\omega]\big) \ge m + \sqrt{(m-1)(m+3)}.$$

We also shed some light on the Banach–Mazur distance between $C(K)$-spaces for uncountable compact spaces K, mainly with respect to spaces $L_\infty[0,1]$ and ℓ_∞. In the process, we obtain some connections between the Banach–Mazur distance of $C(K)$ and the topological properties of the underlying compact spaces. To achieve improved upper bounds, we revisit the classical decomposition method in order to construct more efficient isomorphisms (with respect to distortion). This approach allows us to state nontrivial bounds not only for the Banach–Mazur distance between $L_\infty[0,1]$ and ℓ_∞, but also for certain separable Banach spaces. In particular, we prove

THEOREM 1.3.3. $7.41 < d_{\mathrm{BM}}\big(\ell_\infty, L_\infty[0,1]\big) \leq (3+\sqrt{2})^2 < 19.49$.

This chapter covers the content of the papers [54] and [55] (joint work with Grzegorz Plebanek). The original material has been merged, revised, and corrected. In particular, the lower bound for the Banach–Mazur distance between $C[0,\omega]$ and $C([0,\omega]\times 3)$ contained an error due to an incorrect transcription of formulas into the computer code. After correction, the computations produced a slightly weaker lower bound than the one reported in the original preprint.

CHAPTER 2

Notation and terminology

We mostly use very standard notation, but it might be useful to recall some of it here.

2.1. Set Theory

In the sequel, we write ω for the set of natural numbers (including 0), equipped with the discrete topology. By $\mathbb{Q}$ and$\mathbb{R}$ we denote the rationals and the reals respectively. We write $|A|$ for the cardinality of a set A. ω_1 stands for the first uncountable cardinal. By κ^+ we mean the smallest cardinal strictly greater than κ.

For subsets A, B of ω, we write $A \subseteq^* B$ for almost inclusion, meaning that the set $A \setminus B$ is finite. Likewise, $A =^* B$ means that the sets A, B are almost equal, i.e., $A \subseteq^* B$ and $B \subseteq^* A$. By $f|A$ we denote the restriction of a function f to the set A.

The Zermelo-Fraenkel set theory with the axiom of choice is denoted by ZFC. A sentence φ is consistent with ZFC if its negation $\neg\varphi$ cannot be proven from ZFC. Similarly, φ is independent of ZFC if both φ and $\neg\varphi$ are consistent. We say that a partial order $\mathbb{P}$ is *ccc* if any antichain in $\mathbb{P}$ is at most countable. By $\mathsf{MA}(\kappa)$ we denote Martin's Axiom, which states that for every ccc partial order $\mathbb{P}$ every any $\mathcal{D} \subseteq \mathbb{P}$ of at most κ dense sets, there exists a filter on $\mathbb{P}$ meeting every member of $\mathcal{D}$; for a detailed description see [38].

A collection $\mathcal{I} \subseteq P(A)$ of subsets of some set A is called an ideal if $\mathcal{I}$ is closed under finite unions and taking subsets. Ideals are usually required to be proper, i.e., $A \notin \mathcal{I}$. The ideal $\mathcal{I}$ is a σ-ideal if it is also closed under countable unions. We say that a family $\mathcal{F} \subseteq P(A)$ is a filter, if the family of its complements $\{F^c : F \in \mathcal{F}\}$ is an ideal. A filter $\mathcal{U} \subseteq P(A)$ is an ultrafilter if it is maximal with respect to inclusion, so for every set $B \in P(A)$ we have either $B \in \mathcal{U}$ or $B^c \in \mathcal{U}$.

2.2. Topology

All topological spaces considered here are Hausdorff. In the sequel, we usually denote compact spaces by K and L.

Let F be any topological space. We denote by $w(F)$ the topological weight of F, that is, the minimal cardinality of a base of the topology on F. The density $d(F)$ of F, is the smallest cardinality of a dense subset of F. We always have $d(F) \leq w(F)$. The character of a point $x \in F$ is the minimal cardinality of a neighbourhood base at the point x. The character $\chi(F)$ of a topological space F is the supremum of the characters of its points.

We denote by $F^{(1)}$ the derived set of F, namely, the set of all non-isolated points of F. For any ordinal α, by iterating this operation α-many times (and taking intersection at limit ordinals), we define $F^{(\alpha)}$. A topological space F is called *scattered* if any nonempty subset $G \subseteq F$ contains a relative isolated point. For a scattered compact space K, the minimal α such that $K^{(\alpha)} = \emptyset$ is called the Cantor–Bendixson rank of K. A space F is called *zero-dimensional* if it has a base consisting of clopen sets.

2.2.1. Compactifications. Let F be a Tychonoff space. A *compactification* of F is any compact space K containing a dense subspace homeomorphic to F. The most common compactifications are the one-point compactification (sometimes called the Alexandroff compactification) and the Stone–Čech compactification, denoted βF.

Recall that a topological space F is *pseudocompact* if every real-valued continuous function on F is bounded. A space F is called *locally compact* if it is homeomorphic to an open subset of a compact space. The Stone–Čech compactification behaves well with respect to products of pseudocompact locally compact spaces, as shown by the following results.

THEOREM 2.2.1 ([37, Theorem 3.10.26]). *The Cartesian product $F \times G$ of a pseudocompact space F and a pseudocompact locally compact space G is pseudocompact.*

THEOREM 2.2.2 (Glicksberg [87, 8.12]). *If F and G are infinite, then the product space $F \times G$ is pseudocompact if and only if $\beta(F \times G) = \beta F \times \beta G$.*

2.2.2. Compact lines. For a linearly ordered set X, we use the interval notation; sets $(a, b), [a, b), (\leftarrow, b], (a, \rightarrow)$ have their usual meanings. Given a linearly ordered set $(X, <)$, the family of all half-lines

$$(\leftarrow, a) \text{ and } (a, \rightarrow), \text{ for } a \in X,$$

forms a subbase for a topology which is called the *order topology* on X. A *compact line* is a linearly ordered space whose order topology is compact. Compact lines can be viewed as linear orders in which all nonempty subsets have a supremum (equivalently, an infimum), endowed with the order topology. We recall the following fact about the topological invariants of compact lines.

FACT 2.2.3 ([37, Exercise 3.12.4]). *Let K be a compact line. Then $\chi(K) \leq d(K) \leq w(K)$.*

Since ordinal numbers are linearly ordered, we also use the interval notation for them. For any cardinal number κ, the space $[0, \kappa]$ denotes the ordinal $\kappa + 1$ considered as a compact line.

Given a compact line K, a point $x \in K$ is isolated from the left if $x \notin \overline{(\leftarrow, x)}$, and isolated from the right if $x \notin \overline{(x, \rightarrow)}$. Thus, K can be decomposed into the following sets:

(1) $K_{\leftarrow}$ the set of points that are isolated from the left but not from the right;
(2) $K_{\rightarrow}$ the set of points that are isolated from the right but not from the left;
(3) $K_{\bullet}$ the set of isolated points;
(4) K_{-} the set of points that are not isolated from either side.

The following facts are rather straightforward; for the proofs see [6, Propositions 3.1, 3.2].

FACT 2.2.4. $w(K) = \max\{d(K), |K_{\rightarrow}|\}$.

FACT 2.2.5. *The set $K \setminus K_{\leftarrow}$ is dense in K.*

More details about compact lines can be found in [37, 3.12.3–4] and [82, II.39].

2.2.3. Separable compact lines. The double arrow space

$$\mathbb{S} = \big((0,1] \times \{0\}\big) \cup \big([0,1) \times \{1\}\big),$$

ordered lexicographically, is a classical example of a compact line. The space $\mathbb{S}$ is nonmetrizable, but separable and first countable; see [**37**, Exercise 3.10.C]. There is a natural generalisation of the double arrow space: consider an arbitrary closed subset F of the unit interval and a set $X \subseteq F$, then define

$$F_X = \big(F \times \{0\}\big) \cup \big(X \times \{1\}\big).$$

As before, the space F_X, ordered lexicographically, is a separable compact line and it is nonmetrizable whenever X is uncountable. In fact, for an infinite set X, the space F_X has topological weight $|X|$; see Fact 2.2.4. It turns out that spaces of the form F_X exhaust the class of separable compact lines:

THEOREM 2.2.6 (Ostaszewski [**70**]). *A space K is a separable compact linearly ordered space if and only if it is homeomorphic to F_X for some closed set $F \subseteq [0,1]$ and a subset $X \subseteq F$.*

Similarly to the double arrow space, any separable compact line is first countable, hereditarily separable and Fréchet–Urysohn (that is, the sequential closure of a set coincides with its topological closure).

2.3. Banach space theory

We will write $\|\cdot\|$ for the norm in all Banach spaces considered. By B_X we mean the closed unit ball of a Banach space X. If Banach spaces X and Y are isomorphic, we write $X \simeq Y$. For isometrically isomorphic (or briefly isometric) Banach spaces X and Y, we write $X \cong Y$. We write $X \hookrightarrow Y$ to indicate that Y contains an isomorphic copy of X. Similarly, $Y \twoheadrightarrow X$ denotes the existence of a bounded linear surjection from Y onto X.

X^* denotes the Banach space of bounded linear functionals on X, equipped with the operator norm. We can embed X into X^{**} via the evaluation functionals, namely $\varphi_x(x^*) = x^*(x)$ sends $x \in X$ to $\varphi_x \in X^{**}$. Any operator between Banach spaces is assumed to be linear. For any operator $T\colon X \to Y$, the formula $T^*y^*(x) = y^*(Tx)$ defines the dual operator $T^*\colon Y^* \to X^*$.

If $\{X_i : i \in I\}$ is a family of Banach spaces, we denote by $c_0(I, X_i)$ the c_0-sum of this family — the Banach space consisting of all sequences $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that for every $\varepsilon > 0$, the set $\{i \in I : \|x_i\| > \varepsilon\}$ is finite, endowed with the supremum norm $\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\|$. The dual space $c_0(I, X_i)^*$ can be identified with $\ell_1(I, X_i^*)$, and the bidual $c_0(I, X_i)^{**}$ with $\ell_\infty(I, X_i^{**})$, where the ℓ_1- and ℓ_∞-sums are defined analogously to the c_0-sum. A broader description of such spaces can be found in [**1**, Section 2, page 5]. We sometimes omit the index set and write $c_0(X)$ for a countable c_0-sum of copies of X, or omit the underlying spaces and write $c_0(\Gamma)$ for a c_0-sum of $|\Gamma|$-many copies of $\mathbb{R}$.

By 0 we sometimes denote the trivial 0-dimensional Banach space. If $Z \subseteq X$ is a closed subspace of a Banach space X, we denote elements of the quotient space X/Z by $[x] = x + Z$ for $x \in X$. The following standard fact will be crucial in some of the proofs.

Lemma 2.3.1. *Assume that* X, Y *are Banach spaces and* $T\colon X \to Y$ *is a continuous linear operator. Then the map* $\widehat{T}\colon X/ker(T) \to Y$ *given by* $\widehat{T}([x]) = T(x)$ *is a continuous linear operator of the same norm. If additionally* T *is surjective, then* $\widehat{T}$ *is an isomorphism.*

2.3.1. $C(K)$**-spaces.** For a compact space K, we denote by $C(K)$ the Banach space of all real-valued continuous functions on K, equipped with the supremum norm. The topological density and weight of $C(K)$ in the norm topology are equal to the weight $w(K)$ of K. By the Riesz representation theorem, we identify the dual space $C(K)^*$ with the space of signed regular Radon measures of bounded variation, denoted $M(K)$. For a measure $\mu \in M(K)$, the Jordan decomposition theorem gives a representation $\mu = \mu^+ - \mu^-$ for some nonnegative orthogonal measures μ^+, μ^-. The norm in $M(K)$ is given by the total variation of μ, namely $\|\mu\| = \mu^+(K) + \mu^-(K)$. Given $f \in C(K)$ and $\mu \in M(K)$, we simply write $\mu(f)$ for $\int_K f \, \mathrm{d}\mu$. In the space $M(K)$, equipped with the w^* topology, the set of Dirac measures $\Delta_K = \{\delta_k : k \in K\}$ is a topological copy of K (we will sometimes identify K with Δ_K).

For any function $f\colon K \to \mathbb{R}$, the oscillation of f at a point $k \in K$ is given by the formula

$$\operatorname{osc} f(k) = \inf\left\{\sup_{y,z\in V} |f(y) - f(z)| : V \ni k \text{ open}\right\}.$$

The oscillation of a function f is the supremum of its pointwise oscillations, i.e.,

$$\operatorname{osc} f = \sup_{k\in K} \operatorname{osc} f(k).$$

It is well-known that $\operatorname{osc} f = 0$ if and only if f is continuous. This fact can be generalised.

Proposition 2.3.2 ([**11**, Proposition 1.18 (ii)]). *Let* K *be a compact space and* $f \in l_\infty(K)$. *Then the norm of* f *in the quotient space* $l_\infty(K)/C(K)$ *is equal to* $\frac{1}{2} osc\, f$.

From Proposition 2.3.2 we can easily deduce the following.

Corollary 2.3.3. *Let* A *be a dense subset of a compact space* K *and let* $f\colon A \to \mathbb{R}$ *be a bounded function. Then* f *extends to a unique continuous function* $\widetilde{f}\colon K \to \mathbb{R}$ *if and only if*

$$osc_A f(k) = \inf\left\{\sup_{y,z\in V\cap A} |f(y) - f(z)| : V \ni k \text{ open}\right\} = 0.$$

2.3.2. Extension operators and complemented subspaces. Given a pair of compact spaces $K \subseteq L$, by an *extension operator* $E\colon C(K) \to C(L)$ we mean a bounded linear operator such that $Ef|K = f$ for every $f \in C(K)$. Following [**31**], we write $\eta(K, L)$ for the infimum of the norms of all extension operators $E\colon C(K) \to C(L)$, if there are any; thus $\eta(K, L) = \infty$ means that there is no bounded extension operator. By the classical Borsuk-Dugundji extension theorem, such an operator of norm one exists whenever K is metrizable; see [**73**] or [**58**, II.4.14]. This fact can be generalised to the class of compact lines by the following classical result.

Theorem 2.3.4 (Heath and Lutzer [**44**, Theorem 2.4]). *If* $K' \subseteq K$ *are compact lines, then there exists a norm-one extension operator* $C(K') \to C(K)$.

CHAPTER 3

Countable discrete extensions of compact lines

3.1. Introduction

We say that Banach spaces X, Y, Z and continuous linear operators T, S form a *short exact sequence* of Banach spaces

$$0 \to Y \xrightarrow{T} X \xrightarrow{S} Z \to 0$$

if $\ker S = \operatorname{im} T, \ker T = 0$ and S is surjective. In such a case, we also say that X is a *twisted sum* of Y and Z. A twisted sum X is called trivial if the operator S admits a right inverse, so a continuous linear operator $R\colon Z \to X$ such that $SR = I_X$.

Twisted sums of Banach spaces have been the subject of many publications (e.g. by Cabello-Sánchez and Castillo [15], Castillo and Salguero Alarcón [24, 25], and Kochanek [50]). In particular, they have led to the construction of very interesting examples of Banach spaces, such as those by Cabello-Sánchez et al. [14] and by Enflo, Lindenstrauss, and Pisier [36]. For more information on twisted sums and exact sequences of Banach spaces, one can check the monographs [23], [5], [16].

The classical Sobczyk Theorem states that every isomorphic copy of c_0 inside a separable Banach space is always complemented. In homological language it says that for any separable Banach space X, every twisted sum of c_0 and X is trivial. Thus in 2003 Cabello-Sánchez, Castillo, Kalton and Yost [17] asked whether, for any nonmetrizable compact space K, there always exists a nontrivial twisted sum X of c_0 and $C(K)$ associated with a short exact sequence

$$0 \to c_0 \to X \to C(K) \to 0.$$

This problem has been analysed in many papers by Castillo, Correa, Drygier, Marciszewski, Plebanek and Tausk [22, 27, 29, 34, 61], for several classes of compact spaces. In [7] Avilés, Marciszewski and Plebanek showed not only that, under CH, for every nonmetrizable compact space K there is a nontrivial twisted sum of c_0 and $C(K)$, but also provided several useful characterisations of spaces admitting such twisted sum, essentially solving the problem.

One of the natural classes of nonmetrizable compact spaces is that of compact lines. In this context, we mostly consider separable compact lines, since, due to rather simple observations, nonseparable compact lines always have nontrivial twisted sums with c_0. Recall that there is a vivid trend of investigating properties of Banach spaces of continuous functions on compact lines; see, e.g. Correa and Tausk [28], Kalenda and Kubiś [46, 47], Marciszewski [60], and Michalak [64]. In this chapter, we expand the collection of known examples of separable compact lines K such that $C(K)$ admits a nontrivial twisted sum with c_0, with the additional assumption that the topological weight $w(K)$ of K is not too small, but consistently below the continuum. We also show that, under $\mathsf{MA}(\omega_1)$, for any separable compact line K of weight ω_1, the space $C(K)$ does not admit a nontrivial twisted sum with c_0 of a specific form.

As shown in [7] and [61], to construct a nontrivial twisted sum of c_0 and $C(K)$, one may consider countable discrete extensions of $(B_{C(K)^*}, w^*)$ that cannot be embedded into $(C(K)^*, w^*)$. Here, we consider only a simpler special case of this construction — namely, a countable discrete extension L of K such that there is no extension operator $E\colon C(K) \to C(L)$ (recall that K can be seen as a subset of $B_{C(K)^*}$). We then say that L has property $(\mathcal{E})$. We also consider a simpler counterpart of property $(\mathcal{E})$, called property $(\mathcal{R})$, which states that there is a retraction $r\colon L \to K$.

Our study of countable discrete extensions of separable compact lines shows that in order to understand when a certain extension admits or does not admit an extension operator of a specified norm, we need to count so-called alternations of an almost chain. This observation links our results with objects from infinitary combinatorics, such as gaps.

3.1.1. Key notions of the chapter. If K is a compact space, we call a superspace $L \supseteq K$ a *countable discrete extension* of K and write $L \in \mathrm{CDE}(K)$ if L is compact and $L \setminus K$ is a countable infinite discrete set. In such a situation, we always identify $L \setminus K$ with ω. The main subject of this chapter is the investigation of two properties of countable discrete extensions of separable compact lines. In a sense, these properties measure the complexity of the way in which isolated points are added to the initial space.

Definition 3.1.1. Given a compact space K and $L \in \mathrm{CDE}(K)$, we say that

(i) L has property $(\mathcal{R})$ if there is a continuous retraction from L onto K;
(ii) L has property $(\mathcal{E})$ if $\eta(K, L) < \infty$ (for the definition of η, see section 2.3.2).

Properties $(\mathcal{R})$ and $(\mathcal{E})$ have been studied in a series of papers by Avilés, Castillo, Drygier, Marciszewski, Plebanek and Salguero Alarcón [7, 24, 34, 61] in connection with twisted sums of Banach spaces.

3.1.2. Contents of the chapter. We present the following results concerning a separable compact line K of topological weight $w(K)$ (in fact, separability is not needed for some items). Note that if K has countable weight and $L \in \mathrm{CDE}(K)$, then an extension operator $E\colon C(K) \to C(L)$ of norm one exists by the classical result of Dugundji, cf. [73, Theorem 6.6].

Section 3.2 is dedicated to general lemmas regarding countable discrete extensions and basic characterisations of properties $(\mathcal{R})$ and $(\mathcal{E})$. In sections 3.3, 3.5 – 3.7 we prove the following.

(a) For every $L \in \mathrm{CDE}(K)$, either L has property $(\mathcal{R})$ (so $\eta(K, L) = 1$) or $\eta(K, L) \geq 3$ (Theorem 3.3.2).
(b) For every $L \in \mathrm{CDE}(K)$, if $\eta(K, L) < 5$, then $\eta(K, L) \leq 3$ (Theorem 3.3.3). We also sketch the argument for proving that $\eta(K, L)$ is either infinite or equal to an odd natural number (Theorem 3.3.4).
(c) If $w(K) \geq \omega_1$, then there is $L \in \mathrm{CDE}(K)$ such that $\eta(K, L) = 3$ (Theorem 3.5.3).
(d) Under $\mathsf{MA}(\omega_1)$, if $w(K) = \omega_1$, then for all $L \in \mathrm{CDE}(K)$ we have $\eta(K, L) \leq 3$ (Theorem 3.6.3).

(e) If $w(K) \geq \mathrm{non}(\mathcal{I})$, then there is $L \in \mathrm{CDE}(K)$ without the property $(\mathcal{E})$ (Theorem 3.7.2). We also sketch the argument that for such K, any odd value of $\eta(K, L)$ can be obtained.

Here, $\mathrm{non}(\mathcal{I})$ denotes the least cardinality of a set $X \subseteq [0,1]$ that cannot be covered by a countable family of closed null sets. For the construction of the examples mentioned above, we find it convenient to see zero-dimensional compact lines as Stone spaces of Boolean algebras generated by almost chains in a countable set. In section 3.4, we introduce some technology regarding these chains and extract a combinatorial property of almost chains, which we show to be the main reason why some countable discrete extensions of separable compact lines admit extension operators, while others do not.

In light of results on twisted sums from [**7**], [**29**], [**61**] and the contents of this chapter, the following problem seems worth considering.

Problem 3.1.2. *Is it relatively consistent that $\eta(K, L) < \infty$ for every separable compact space K of weight ω_1 and its countable discrete extension L?*

3.2. Properties of countable discrete extensions

Let us start with a simple observation.

Remark 3.2.1. If $r\colon L \to K$ is a continuous retraction, then the mapping $C(K) \ni f \to f \circ r \in C(L)$ defines an extension operator of norm one. This shows that every countable discrete extension with property $(\mathcal{R})$ has property $(\mathcal{E})$; moreover, $\eta(K, L) = 1$.

Extensions with property $(\mathcal{R})$ are, in a sense, trivial. Most easy constructions of countable discrete extensions have this property. Recall that there are spaces for which all extensions have property $(\mathcal{R})$.

Example 3.2.2. *If a space K is metrizable, then every $L \in \mathrm{CDE}(K)$ has property $(\mathcal{R})$.*

Proof. Since K is compact and metrizable, it is separable [**37**, Theorem 4.1.18] and has a countable base. It follows that the space L also has a countable base, so it is metrizable [**37**, Theorem 4.2.8]. Let us fix a compatible metric d on L. We can define a retraction r by mapping each element in L to a closest element in K (choosing any if there are multiple closest points). Such a function is well-defined since K is compact, so a closest element always exists.

To check that r is continuous, it is enough to verify that whenever a sequence $(x_n)_{n\in\omega}$ in ω converges to $x \in K$, then $r(x_n) \to r(x) = x$.

Given $\varepsilon > 0$ we have $d(x_n, x) < \varepsilon$ for almost all n, so

$$d(r(x_n), x_n) \leq d(x, x_n) < \varepsilon \text{ and } d(r(x_n), x) < 2\varepsilon,$$

for large n, by the triangle inequality. △

We recall below some useful characterisations of properties $(\mathcal{R})$ and $(\mathcal{E})$; see [**61**, Lemma 2.7] for the proof, which is fairly standard; cf. [**73**] and [**86**]. Note that for a given pair of compacta $K \subseteq L$, measures on K can also be treated as measures on L via the natural extension. In particular, we often consider sequences of measures μ_n **on K** and discuss its convergence in the w^* topology of $M(L)$.

LEMMA 3.2.3. *Let $L \in \mathrm{CDE}(K)$ for any compact space K. Then*

(a) L has property $(\mathcal{R})$ if and only if there is a sequence of points $(x_n)_{n\in\omega}$ in K such that for every function $f \in C(L)$ we have

$$\lim_{n\to\infty} \big(f(x_n) - f(n)\big) = 0.$$

(b) L has property $(\mathcal{E})$ if and only if there is a bounded sequence of signed measures $(\mu_n)_{n\in\omega}$ on K such that $\mu_n - \delta_n \to 0$ in the w^* topology of $C(L)^*$, i.e. for every $f \in C(L)$ we have

$$\lim_{n\to\infty} \Big(\int_K f \, \mathrm{d}\mu_n - f(n) \Big) = 0.$$

REMARK 3.2.4. Concerning Lemma 3.2.3(b), the norm of the extension operator E satisfies $\|E\| = \sup_{n\in\omega} \|\mu_n\|$.

Recall that there are spaces of arbitrarily large weight that do not admit countable discrete extensions without property $(\mathcal{R})$. Indeed, take any cardinal number κ and consider the Cantor cube 2^κ. Then 2^κ is an absolute retract in the class of compact zero-dimensional spaces; in particular, every $L \in \mathrm{CDE}(2^\kappa)$ has property $(\mathcal{R})$. This can be demonstrated directly as follows.

The space 2^κ has a subbase consisting of sets

$$C^i_\alpha = \{x \in 2^\kappa : x(\alpha) = i\},$$

for $\alpha < \kappa$ and $i = 0, 1$. For every α, L can be partitioned into clopen sets $\widetilde{C^i_\alpha}$ such that $\widetilde{C^i_\alpha} \cap K = C^i_\alpha$. Thus, we can define a continuous retraction r by $r|K = id_K$ and, for $n \in L \setminus 2^\kappa$, let $r(n)$ be the unique point in

$$\bigcap_{\alpha\in\kappa, i\in\{0,1\}} \{C^i_\alpha : n \in \widetilde{C^i_\alpha}\}.$$

On the other hand, it is not difficult to demonstrate that every *nonseparable* compact line K with $w(K) = \omega_1$ admits a countable discrete extension L such that $\eta(K, L) = \infty$. Namely, one can construct $L = K \cup \omega$ such that L is the closure of the set $\omega \subseteq L$. Then, if we assume that $\eta(K, L) < \infty$, it would follow from Lemma 3.2.3(b) that K must support a strictly positive measure μ. However, a compact line carrying such a measure is necessarily separable (see [**61**, section 8] for the details).

3.3. Calculating $\eta(K, L)$

Corson and Lindenstrauss [**31**] showed that if K is the one-point compactification of an uncountable discrete space, then for every compact superspace $L \supseteq K$, if $\eta(K, L) < \infty$, then $\eta(K, L)$ is an odd natural number. In this section, we show that the same phenomenon occurs in our setting.

Throughout this section we assume that K is a compact line (not necessarily separable) and $L = K \cup \omega$ is its countable discrete extension. We denote by $<$ the linear order on K.

We first give a technical but convenient criterion for the convergence of measures on $L = K \cup \omega$.

LEMMA 3.3.1. *Let $(\nu_n)_n$ be a bounded sequence in $M(K)$ such that $\lim_{n\to\infty} \nu_n(K) = 1$. Suppose that for any points $s, t \in K$, $s < t$, and any closed subsets F, H of $L = K \cup \omega$ such that $F \cap K \subseteq (\leftarrow, s]$ and $H \cap K \subseteq [t, \rightarrow)$, we have*

(i) $\nu_n[t, \rightarrow) = 0$ for almost all $n \in F \cap \omega$; and
(ii) $\nu_n(\leftarrow, s] = 0$ for almost all $n \in H \cap \omega$.

Then $\nu_n - \delta_n \to 0$ in the w^* topology of $M(L)$.

PROOF. Recall that if $(t_n)_{n\in\omega}$ is a sequence in a compact topological space T and $\mathcal{U}$ is a non-principal ultrafilter on ω, then there is a unique element $T \ni t = \lim_{n\to\mathcal{U}} t_n$ such that $\{n \in \omega : t_n \in V\} \in \mathcal{U}$ for every open set V containing t.

To prove the lemma, suppose that the assertion fails. Then the sequence of measures $\nu_n - \delta_n$ has a non-zero cluster point μ. Take an ultrafilter $\mathcal{U}$ such that

$$\mu = \lim_{n\to\mathcal{U}} (\nu_n - \delta_n) = \lim_{n\to\mathcal{U}} \nu_n - \lim_{n\to\mathcal{U}} \delta_n \neq 0.$$

Clearly, $\lim_{n\to\mathcal{U}} \delta_n = \delta_s$ for some $s \in K$. Writing $\nu = \lim_{n\to\mathcal{U}} \nu_n$ we have $\nu \neq \delta_s$. Since $\nu(K) = 1$, either $|\nu|(\leftarrow, s) > 0$ or $|\nu|(s, \rightarrow) > 0$. Suppose that the latter holds (the former case is analogous). Then there is $t_1 \in K$ with $t_1 > s$ such that $|\nu|[t_1, \rightarrow) > 0$. By the normality of L, there exists an open set $V \subseteq L$ such that

$$(\leftarrow, s] \subseteq V \subseteq \overline{V} \subseteq L \setminus [t_1, \rightarrow),$$

and note that $\sup(\overline{V} \cap K) < t_1$, so we can use $\overline{V}$ and $[t_1, \rightarrow)$ as the sets F, H from the assumptions. Now, by (i), for every $t \in [t_1, \rightarrow)$ we have $\nu_n[t, \rightarrow) \to 0$ for $n \in V \cap \omega$ (since V is an open neighbourhood of s, the set $V \cap \omega$ is in $\mathcal{U}$).

Finally, observe that any function $g \in C(K)$ vanishing on $(\leftarrow, t_1]$ can be uniformly approximated by step functions built on intervals $[t, t')$ contained in $(t_1, \rightarrow)$. We conclude that $\int_K g \, d\nu_n \to 0$ for $n \in V \cap \omega \in \mathcal{U}$ which yields $|\nu|[t_1, \rightarrow) = 0$, a contradiction. △

THEOREM 3.3.2. *If $\eta(K, L) < 3$, then L has property $(\mathcal{R})$.*

PROOF. By Lemma 3.2.3 and Remark 3.2.4, there is a sequence of measures μ_n on K such that $c = \sup_n \|\mu_n\| < 3$ and $\mu_n - \delta_n \xrightarrow{w^*} 0$ in $M(L)$ (recall that we regard the measures μ_n as measures on L). Fix $\delta > 0$ such that $c + 3\delta < 3$ and for any $x \in K$ set

$$A_x = \{n \in \omega : \mu_n^+(\leftarrow, x] \geq 1 - \delta\}.$$

Then for every $n \in \omega$ we define

$$x_n = \inf\{x \in K : n \in A_x\}.$$

Note that $\mu_n(K) \to 1$, so x_n is well-defined for almost all $n \in \omega$.

CLAIM. Consider $s, t \in K$ with $s < t$ and let F, H be closed subsets of L such that $F \cap K \subseteq (\leftarrow, s]$ and $H \cap K \subseteq [t, \rightarrow)$. Then the sets

$$I = \{n \in F : x_n \geq t\}, \quad J = \{n \in H : x_n \leq s\}$$

are finite.

PROOF OF THE CLAIM. Consider a continuous function $f : L \to [0, 1]$ such that $f(x) = 1$ for $x \leq s$ and $f(x) = 0$ for $x \geq t$, and set $g = 1 - f$.

Suppose that I is infinite. Then for $n \in I$ we have $\mu_n^+(\leftarrow, x] < 1-\delta$ whenever $x < t$, so

$$\int_K f \, \mathrm{d}\mu_n \le \int_K f \, \mathrm{d}\mu_n^+ < 1-\delta.$$

On the other hand, $\lim_{n\in I} f(n) = 1$, which contradicts $\mu_n - \delta_n \xrightarrow{w^*} 0$.

Now suppose that J is infinite. Since $\lim_{n\in J} \int_K g \, \mathrm{d}\mu_n - g(n) = 0$ and $g = 1$ on $H \cap K$, we have

$$\int_K g \, \mathrm{d}\mu_n \ge 1-\delta \text{ and } \int_K f \, \mathrm{d}\mu_n < \delta,$$

for almost all $n \in J$. At the same time, $\mu_n^+(\leftarrow, s] \ge 1-\delta$, so from the estimate for the second integral above, $\mu_n^-(\leftarrow, s] \ge 1-\delta$ must hold for all sufficiently large $n \in J$. It follows that

$$|\mu_n|(K) \ge |\mu_n|(\leftarrow, s] + |\mu_n|[t, \rightarrow) \ge 2(1-\delta) + 1 - \delta = 3 - 3\delta > c,$$

contrary to the assumption that $\|\mu_n\| \le c$. ▲

Once we have verified the Claim, Lemma 3.3.1 implies that $f(x_n) - f(n) \to 0$ for every $f \in C(L)$, and we are done. △

THEOREM 3.3.3. *If $\eta(K, L) < 5$, then there exists an extension operator $E\colon C(K) \to C(L)$ with $\|E\| \le 3$.*

PROOF. By Lemma 3.2.3 and Remark 3.2.4, there is a sequence of measures μ_n on K such that $c = \sup_n \|\mu_n\| < 5$ and $\mu_n - \delta_n \xrightarrow{w^*} 0$ in $M(L)$. Fix $\delta > 0$ such that $c + 3\delta < 5$.

We shall define a sequence of measures ν_n of norm at most 3 satisfying $\nu_n - \delta_n \xrightarrow{w^*} 0$. For the rest of the proof we assume that $\|\mu_n\| > 3 - \delta/4$ for every n; for the remaining indices we simply put $\nu_n = \mu_n$. For any $x \in K$ we define

$$A_x^0 = \{n \in \omega : \mu_n^+(\leftarrow, x] \ge 1 - \delta/4\}, \qquad x_n^0 = \inf\{x \in K : n \in A_x^0\};$$
$$A_x^1 = \{n \in A_x^0 : \mu_n^-(\leftarrow, x] \ge 1 - \delta/2\}, \qquad x_n^1 = \inf\{x \in K : n \in A_x^1\};$$
$$A_x^2 = \{n \in A_x^1 : \mu_n^+(\leftarrow, x] \ge 2 - \delta\}, \qquad x_n^2 = \inf\{x \in K : n \in A_x^2\}.$$

Note that, since $|\mu_n|(K) > 3 - \delta/4$ for every n and $\mu_n(K) \to 1$, we can assume that all three sets on the right-hand side above are nonempty for all n. Hence x_n^i are well-defined and $x_n^0 \le x_n^1 \le x_n^2$.

Consider the sequence of measures

$$\nu_n = \delta_{x_n^0} - \delta_{x_n^1} + \delta_{x_n^2}.$$

We prove that $\nu_n - \delta_n \xrightarrow{w^*} 0$ by verifying the assumptions of Lemma 3.3.1. Fix $s, t \in K$ with $s < t$, and suppose that F and H are closed sets in L such that $F \cap K \subseteq (\leftarrow, s]$ and $H \cap K \subseteq [t, \rightarrow)$. Take a continuous function $f\colon L \to [0,1]$ such that $f|(\leftarrow, s] = 1$ and $f|[t, \rightarrow) = 0$ and set $g = 1 - f$. We check the assumptions of Lemma 3.3.1 in a few steps.

STEP 1: The set $I = \{n \in F \cap \omega : s < t \le x_n^0 \le x_n^1 \le x_n^2\}$ is finite.

We know that $\mu_n - \delta_n \xrightarrow{w^*} 0$ and $\lim_{n\in F} f(n) = 1$ (whenever $F \cap \omega$ is infinite). It suffices to note that $\lim_{n\in I} \int_K f \, \mathrm{d}\mu_n = 1$ for every infinite $I \subseteq F$.

STEP 2: The set $I = \{n \in H \cap \omega : x_n^0 \le s < t \le x_n^1 \le x_n^2\}$ is finite.

If I were infinite, then $\lim_{n\in I}\int_K f\,\mathrm{d}\mu_n = 0$, while for $n \in I$ we have

$$\int_K f\,\mathrm{d}\mu_n \geq \int_{(\leftarrow,s]} f\,\mathrm{d}\mu_n^+ - \int_{(\leftarrow,t]} f\,\mathrm{d}\mu_n^- \geq 1-\delta/4+\delta/2-1=\delta/4.$$

Step 3: The set $I = \{n \in F\cap\omega : x_n^0 \leq x_n^1 \leq s < t \leq x_n^2\}$ is finite.

Indeed, for infinite I we would have $\lim_{n\in I}\int_K f\,\mathrm{d}\mu_n = 1$, while for $n \in I$

$$\int_K f\,\mathrm{d}\mu_n \leq \int_{(\leftarrow,t]} f\,\mathrm{d}\mu_n^+ - \int_{(\leftarrow,s]} f\,\mathrm{d}\mu_n^- \leq 2-\delta+\delta/2-1=1-\delta/2.$$

Step 4: The set $I = \{n \in H\cap\omega : x_n^0 \leq x_n^1 \leq x_n^2 \leq s < t\}$ is finite.

If I were infinite, then again $\lim_{n\in I}\int_K f\,\mathrm{d}\mu_n = 0$. Since $x_n^2 \leq s$ implies $\mu_n^+(\leftarrow,s] \geq 2-\delta$, it follows that $\mu_n^-(\leftarrow,s] \geq 2-\delta$ for large $n \in I$. Consequently, $|\mu_n|(\leftarrow,s] \geq 4-2\delta$ eventually holds for $n\in I$. On the other hand, $\lim_{n\in I}\int_K g\,\mathrm{d}\mu_n = 1$ implies $|\mu_n|[s,\rightarrow) \geq 1-\delta$ for almost all $n \in I$, which contradicts $\|\mu_n\| \leq c < 5-3\delta$.

Step 5: All remaining cases that could violate the assumptions of Lemma 3.3.1 are also excluded.

For example, if the set $I = \{n \in F\cap\omega : x_n^0 \leq s \leq x_n^1 \leq t \leq x_n^2\}$ were infinite, it could be partitioned into two subsets to which one of the previous cases applies.

By Lemma 3.3.1, we conclude that $\nu_n - \delta_n \xrightarrow{w^*} 0$, as required. △

Examining the proofs of Theorems 3.3.2 and 3.3.3, one concludes that the argument can be further generalised. We only sketch the general idea here.

Theorem 3.3.4. *If $\eta(K,L) < \infty$, then $\eta(K,L)$ is an odd natural number.*

Proof. Choose a natural number k such that $2k-1 \leq \eta(K,L) < 2k+1$ and fix sufficiently small $\delta > 0$. Proceeding by induction, we can assume that $\|\mu_n\| < 2k+1-\delta$, where the measures μ_n arise from an extension operator of norm less than $2k+1$. For each $x \in K$ define the sets $A_x^0, \ldots, A_x^{2k-2}$ by

$$\begin{aligned} A_x^0 &= \{n \in \omega : \mu_n^+(\leftarrow,x] \geq 1-\delta/2^k\},\\ A_x^{2j+1} &= \{n \in A_x^{2j} : \mu_n^-(\leftarrow,x] \geq j+1-\delta/2^{k-2(j+1)}\},\\ A_x^{2j} &= \{n \in A_x^{2j-1} : \mu_n^+(\leftarrow,x] \geq j+1-\delta/2^{k-2j}\}. \end{aligned}$$

After setting $x_n^i = \inf\{x \in K : n \in A_x^i\}$ for $i = 0,\ldots,2k-2$, consider the measures

$$\nu_n = \sum_{i=0}^{2k-2}(-1)^i\delta_{x_n^i}.$$

Clearly $\|\nu_n\| \leq 2k-1$, so it remains to verify that $\nu_n - \delta_n \xrightarrow{w^*} 0$. △

3.4. Countable discrete extensions and Boolean algebras

In this section, we describe a method for constructing countable discrete extensions of separable compact lines via Stone spaces of Boolean algebras of subsets of ω. We use the classical Stone duality, referring to [51] if necessary. Given a Boolean algebra $\mathfrak{A}$, denote by $\mathrm{ult}(\mathfrak{A})$ the Stone space of ultrafilters on $\mathfrak{A}$. This is a compact zero-dimensional space with a base consisting of clopen sets of the form

$$\widehat{A} = \{\mathcal{U} \in \mathrm{ult}(\mathfrak{A}) : A \in \mathcal{U}\},$$

for $A \in \mathfrak{A}$.

The basic idea is simple: If an algebra $\mathfrak{A} \subseteq P(\omega)$ contains fin, the ideal of finite subsets of ω, then $\mathrm{ult}(\mathfrak{A})$ is a compactification of ω, obtained by identifying principal ultrafilters with natural numbers. Hence, if we can represent our compact zero-dimensional space K as $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ for some Boolean algebra $\mathfrak{A}$ of subsets of ω (or of any other countable set), then $L = \mathrm{ult}(\mathfrak{A})$ is a countable discrete extension of $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$. Our first objective is to understand properties $(\mathcal{R})$ and $(\mathcal{E})$ in Boolean-algebraic terms.

Lemma 3.4.1. *Suppose that* $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ *for some Boolean algebra* $\mathfrak{A}$ *of subsets of* ω *containing* fin. *Then* $L = \mathrm{ult}(\mathfrak{A})$ *has property* $(\mathcal{R})$ *if and only if there exists a lifting* $\theta\colon \mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$.

Here $\theta\colon \mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$ is called a *lifting* if it is a Boolean algebra monomorphism such that $\pi \circ \theta = id_{\mathfrak{A}/\mathrm{fin}}$. The proof of the above lemma is standard; see, e.g. [**34**].

To state an analogous lemma for $(\mathcal{E})$, recall that if $K = \mathrm{ult}(\mathfrak{B})$ for a Boolean algebra $\mathfrak{B} \subseteq P(\omega)$, then $M(K)$ can be identified with $M(\mathfrak{B})$, the space of signed finitely additive measures on $\mathfrak{B}$ with bounded variation. In this setting, the norm of a measure $\mu \in M(\mathfrak{B})$ is given by $\|\mu\| = |\mu|(\omega)$, where the variation $|\mu|$ is defined for $A \in \mathfrak{B}$ by

$$|\mu|(A) = \sup_{B\in\mathfrak{B},\, B\subseteq A} \big(|\mu(B)| + |\mu(A \setminus B)|\big).$$

Lemma 3.4.2. *In the setting of Lemma 3.4.1,* $L \in \mathrm{CDE}(K)$ *has property* $(\mathcal{E})$ *if and only if there exists a bounded sequence* $(\mu_n)_{n\in\omega}$ *in* $M(\mathfrak{A})$ *such that*

(i) $\mu_n(I) = 0$ *for every* $I \in \mathrm{fin}$ *and every* n*;*
(ii) $\lim_{n\to\infty} \big(\mu_n(A) - \delta_n(A)\big) = 0$ *for every* $A \in \mathfrak{A}$.

Proof. This follows from Lemma 3.2.3(b) and the following observations.

There is an obvious correspondence between the finitely additive measures on $\mathfrak{A}/\mathrm{fin}$ and the finitely additive measures on $\mathfrak{A}$ vanishing on finite sets. Moreover, for any zero-dimensional compact space L and any sequence ν_n in $M(L)$, we have $\nu_n \xrightarrow{w^*} 0$ if and only if $\nu_n(C) \to 0$ for every clopen $C \subseteq L$. We apply this remark to $\nu_n = \mu_n - \delta_n \in M(\mathfrak{A})$. △

3.4.1. Real almost chains. For notation and terminology concerning infinitary combinatorics, see section 2.1.

We say that a *real almost chain* is a family of sets $\mathcal{A} = \{A_x \subseteq \omega : x \in X\}$ indexed by a set $X \subseteq \mathbb{R}$ and increasing with respect to almost inclusion, that is, $A_x \subseteq^* A_y$ when $x < y$.

It is well-known that, in terms of Stone duality, zero-dimensional compact lines correspond to chain algebras. A chain algebra is a Boolean algebra with a linearly ordered set of generators. It turns out that if we choose the set of generators to be a real almost chain together with the ideal fin, then the corresponding compact space is a countable discrete extension of a separable compact line. The following Lemma is essentially known; see [**61**, Theorem 8.7].

LEMMA 3.4.3. *Suppose that* $\mathcal{A} = \{A_x : x \in X \subseteq [0,1]\}$ *is a real almost chain of subsets of* ω *and let* $\mathfrak{A}$ *be the Boolean algebra generated by* $\mathcal{A} \cup \mathrm{fin}$.

Then $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ *is a separable compact line with* $w(K) = |X|$ *and* $L = \mathrm{ult}(\mathfrak{A})$ *is a countable discrete extension of* K.

PROOF. As explained above, L may be seen as a countable discrete extension of $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$. Then K is a compact line, as $\mathfrak{A}/\mathrm{fin}$ is generated by a chain; see, e.g. [**51**, Theorem 15.7]. This follows from the fact that every ultrafilter $\mathcal{U} \in \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ is uniquely determined by the set $X(\mathcal{U}) = \{x \in X : A_x/\mathrm{fin} \in \mathcal{U}\}$. We can order $\mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ by declaring $\mathcal{U} \leq \mathcal{V}$ when $X(\mathcal{V}) \subseteq X(\mathcal{U})$.

Finally, K is separable: take a countable set $D \subseteq X$ such that for every $x \in X$ and $\delta > 0$ there is $d \in D$ such that $x - \delta < d \leq x$. For each $x \in X$ we denote by $\mathcal{U}_x$ the unique ultrafilter in K such that x is the first element in $X(\mathcal{U}_x)$. It is straightforward to verify that the set $\{\mathcal{U}_d : d \in D\}$ is dense in K. △

We can also reverse this characterisation as follows.

LEMMA 3.4.4. *Let* K *be a zero-dimensional separable compact line and let* $L \in \mathrm{CDE}(K)$. *Then there exist a set* $X \subseteq [0,1]$ *and a real almost chain* $\mathcal{A} = \{A_x : x \in X\}$ *of subsets of a countable set* N *such that*

(i) K *is homeomorphic to* $\mathrm{ult}(\mathfrak{A}/\mathrm{fin})$ *and*
(ii) L *is homeomorphic to* $\mathrm{ult}(\mathfrak{A})$,

where $\mathfrak{A}$ *is the algebra generated by* $\mathcal{A} \cup \mathrm{fin}(N)$.

PROOF. By Theorem 2.2.6, K is homeomorphic to a space F_X for some closed set $F \subseteq [0,1]$ and a subset $X \subseteq F$, so for the proof we consider $K = F_X$. Since K is zero-dimensional, X is dense in F with respect to the usual topology. As $L \in \mathrm{CDE}(F_X)$, we have $L = F_X \cup N$ for some countable infinite set N of isolated points.

For every $x \in X$, the set $C_x = (\leftarrow, (x,1)) = (\leftarrow, (x,0)]$ is clopen in F_X, so there is a clopen set $\widetilde{C_x}$ in L such that $\widetilde{C_x} \cap F_X = C_x$. Consider $A_x = \widetilde{C_x} \cap N$.

If $x < y$ in X, then the closure of $A_x \setminus A_y$ is disjoint from F_X, so the set itself is finite. In other words, $\mathcal{A} = \{A_x : x \in X\}$ is an almost chain of subsets of N. It is not difficult to verify that (i) and (ii) hold. △

3.4.2. Alternations of almost chains. The most obvious example of a real almost chain is given by $\mathcal{A} = \{A_x : x \in \mathbb{R}\}$, where $A_x = \{q \in \mathbb{Q} : q < x\}$. This is in fact a chain: $A_x \subseteq A_y$ when $x < y$. We could easily convert $\mathcal{A}$ into an almost chain that is not a chain by adding a finite set to each A_x. However, this would not produce anything essentially new, so we call it a *finite adjustment* of the original chain.

DEFINITION 3.4.5. A real almost chain $\{B_x : x \in X\}$ is said to be a finite adjustment of $\{A_x : x \in X\}$ if $A_x =^* B_x$ for all $x \in X$.

The characterisation of $(\mathcal{R})$ via liftings in Lemma 3.4.1 naturally translates to the language of real almost chains of subsets of ω and finite adjustments, giving the following.

PROPOSITION 3.4.6. *Let* $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$, *where* $\mathfrak{A}$ *is generated by a real almost chain* $\mathcal{A} = \{A_x : x \in X\}$ *of subsets of* ω. *Then the countable discrete extension* $L = \mathrm{ult}(\mathfrak{A})$ *of*

K has property $(\mathcal{R})$ if and only if there is a finite adjustment $\{B_x : x \in X\}$ of $\mathcal{A}$ that is a chain.

Proof. We use Lemma 3.4.1: To check that the conditions are sufficient, define $\theta(A_x/\mathrm{fin}) = B_x$ and extend θ to a lifting $\mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$, since every $b \in \mathfrak{A}/\mathrm{fin}$ can be expressed as a finite union of elements of the form $(A_y/\mathrm{fin}) \setminus (A_x/\mathrm{fin})$. Necessity follows from the fact that, given a lifting $\theta\colon \mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$, the sets $B_x = \theta(A_x/\mathrm{fin})$ have the required properties. △

Real almost chains that are slightly more complicated are called *barely alternating*.

Definition 3.4.7. A real almost chain $\{A_x : x \in X\}$ is *barely alternating* if $A_{x_1} \cap A_{x_3} \subseteq A_{x_2} \cup A_{x_4}$ whenever $x_1 < x_2 < x_3 < x_4$.

Equivalently, $\{A_x : x \in X\}$ is barely alternating if there does not exist $n \in \omega$ such that $n \in A_{x_1}, A_{x_3}, n \notin A_{x_2}, A_{x_4}$ for $x_1 < x_2 < x_3 < x_4$ in X.

Barely alternating almost chains play an important role in sections 3.5 and 3.6 due to the following result, which resembles Theorem 3.3.3.

Theorem 3.4.8. *Let $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$, where $\mathfrak{A}$ is generated by a real almost chain $\mathcal{A} = \{A_x : x \in X\}$ of subsets of ω. If there is a barely alternating finite adjustment $\{B_x : x \in X\}$ of $\mathcal{A}$, then the countable discrete extension $L = \mathrm{ult}(\mathfrak{A})$ of K has property $(\mathcal{E})$ and $\eta(K, L) \leq 3$.*

Proof. Let $\mathcal{B} = \{B_x : x \in X\}$ be a barely alternating finite adjustment of $\mathcal{A}$. For any $z \in K$ we denote by $\delta_z \in M(\mathfrak{A})$ the unique finitely additive measure satisfying $\delta_z(I) = 0$ if $I \in \mathrm{fin}$ and

$$\delta_z(A_x) = \begin{cases} 1 & \text{if } A_x/\mathrm{fin} \in z \\ 0 & \text{if } A_x/\mathrm{fin} \notin z. \end{cases}$$

for $x \in X$. Recall that the order on $z \in K = \mathrm{ult}(\mathcal{A}/\mathrm{fin})$ was defined by reverse inclusion of sets $\{x \in X : A_x/\mathrm{fin} \in z\}$; see Lemma 3.4.3.

For each $n \in \omega$, define points $x_n^0, x_n^1, x_n^2 \in K$ as

$$x_n^0 = \inf\{x \in X : n \in B_x\},$$
$$x_n^1 = \inf\{x \in X : n \notin B_x \text{ and } x > x_n^0\},$$
$$x_n^2 = \inf\{x \in X : n \in B_x \text{ and } x > x_n^1\}.$$

Notice that $\inf(\emptyset) = \max(K)$. Using these points, define a sequence of measures $\mu_n = \delta_{x_n^0} - \delta_{x_n^1} + \delta_{x_n^2} \in M(\mathfrak{A})$. This sequence is well-defined for almost all $n \in \omega$. Note that $x_n^0 \leq x_n^1 \leq x_n^2$ and some of these points can be equal.

Let us check that $\lim_{n\to\infty}(\mu_n - \delta_n)(A_x) = 0$ for every $x \in X$. Since $A_x =^{w^*} B_x$, we have $(\mu_n - \delta_n)(A_x) = (\mu_n - \delta_n)(B_x)$ for every $x \in X$, so it is enough to verify the following claim.

Claim. Fix any $x \in X$ we have $\mu_n(B_x) = \delta_n(B_x)$.

Proof of the Claim. Note that for $x \leq x_n^2$ we easily have $\mu_n(B_x) = 1$ if and only if $n \in B_x$. If $x > x_n^2$, then by the definition of μ_n we have $\mu_n(B_x) = 1$. We also have $n \in B_x$, as otherwise $\mathcal{B}$ is would not be barely alternating. ▲

We can also see that $\|\mu_n\| \leq 3$ for every $n \in \omega$ and $\mu_n(I) = 0$ for any $I \in \mathrm{fin}$. Lemma 3.4.2 and Remark 3.2.4 applied for the sequence μ_n end the proof. △

As in the results at the end of section 3.3, we claim that characterisations analogous to Theorem 3.4.8 also hold for chains with more alternations.

We say that a real almost chain $\{A_x : x \in X\}$ is *k-alternating* for some $k \in \omega$, when there are no $x_1 < x_2 < \ldots < x_{2k+1} < x_{2k+2}$ and an element $n \in \omega$ satisfying $n \in A_{x_i} \iff 2 \nmid i$. In this terminology, barely alternating almost chains are called 1-alternating.

The proof of the following result is essentially the same as the previous one, so we will only sketch the general idea and omit most of the technical details.

Theorem 3.4.9. *Let $K = \mathrm{ult}(\mathfrak{A}/\mathrm{fin})$, where $\mathfrak{A}$ is generated by a real almost chain $\mathcal{A} = \{A_x : x \in X\}$ of subsets of ω. If for some $k \in \omega$ there is a k-alternating finite adjustment $\{B_x : x \in X\}$ of $\mathcal{A}$, then the countable discrete extension $L = \mathrm{ult}(\mathfrak{A})$ of K has property ($\mathcal{E}$) and $\eta(K, L) \leq 2k + 1$.*

Proof. Let $\{B_x : x \in X\}$ be a k-alternating finite adjustment of $\mathcal{A}$. For each $n \in \omega$, define points $x_n^0, x_n^1, \ldots, x_n^{2k} \in K$ by

$$x_n^0 = \inf\{x \in X : n \in B_x\},$$

$$x_n^1 = \inf\{x \in X : n \notin B_x \text{ and } x > x_n^0\},$$

$$\ldots$$

$$x_n^{2k} = \inf\{x \in X : n \in B_x \text{ and } x > x_n^{2k-1}\}.$$

Define a sequence of measures $\mu_n = \Sigma_{i=0}^{k}\delta_{x_n^{2i}} - \Sigma_{i=0}^{k-1}\delta_{x_n^{2i+1}} \in M(\mathfrak{A})$. As in the proof of Theorem 3.4.8, one can verify that the sequence of measures μ_n satisfies the assumptions of Lemma 3.4.2 and Remark 3.2.4. △

3.5. Between ($\mathcal{R}$) and ($\mathcal{E}$)

In this section, we present a construction of a countable discrete extension of a separable compact line of weight ω_1 without property ($\mathcal{R}$), but with an extension operator of norm 3. At the end of the section we will also apply this result to spaces which are not necessarily zero-dimensional.

The construction below and Theorem 3.5.2 are due to Witold Marciszewski. It will be convenient to consider a subset X of the Cantor set 2^ω rather than of $[0,1]$ and replace ω by $2^{<\omega}$. This is possible because the space 2^ω can be seen as a subset of the interval $[0,1]$.

Construction 3.5.1 (Marciszewski). Consider the full dyadic tree $T = 2^{<\omega}$. By $\preccurlyeq$ we denote the lexicographic order on $2^\omega \cup 2^{<\omega}$: $x \preccurlyeq y$ means that either x is an initial segment of y or $x(k) < y(k)$ for $k = \min\{n \in \omega : x(n) \neq y(n)\}$.

Take any set $X \subseteq 2^\omega$. For each $x \in X$ define

$$S_x = \{x|n^\frown 0 : n \in \omega, x(n) = 1\},$$

where $^\frown$ stands for concatenation of finite sequences. Consider the family $\mathcal{A}_X = \{A_x : x \in X\}$, where

$$A_x = \{t \in T : t \preccurlyeq x\} \setminus S_x.$$

Note that for $x, y \in X$ with $x \prec y$, if for $k = \min\{n \in \omega : x(n) < y(n)\}$, then

$$A_y \setminus A_x \subseteq \bigcup_{i=0}^{k} 2^i, \text{ so } A_x \subseteq^* A_y.$$

Hence $\mathcal{A}_X$, a family of subsets of a countable set T, is a real almost chain. Write $\mathfrak{A}_X$ for the Boolean algebra of subsets of T generated by $\mathcal{A}_X \cup \mathrm{fin}(T)$.

THEOREM 3.5.2 (Marciszewski). *In the setting of Construction 3.5.1, if X is uncountable, then the space $K = \mathrm{ult}(\mathfrak{A}_X/\mathrm{fin}(T))$ is a separable compact line and the space $\mathrm{ult}(\mathfrak{A}_X)$ is a countable discrete extension of K without property $(\mathcal{R})$.*

PROOF. In view of Lemma 3.4.3, it is enough to show the lack of property $(\mathcal{R})$.

Suppose otherwise, that $L = \mathrm{ult}(\mathfrak{A}_X)$ has property $(\mathcal{R})$. Then, by Lemma 3.4.6, there is a finite adjustment $\{C_x : x \in X\}$ of $\mathcal{A}_X$ which is a chain. Consider a set $X_0 \subseteq X$ such that for every $x \in X_0$ both the sets $\{n : x(n) = 0\}$ and $\{n : x(n) = 1\}$ are infinite, and define a function $\varphi\colon X_0 \to \omega$ by

$$\varphi(x) = \min\{n \in \omega : C_x \triangle A_x \subseteq \bigcup_{j<n} 2^j\}.$$

The set X_0 is uncountable (since $|X \setminus X_0| \le \omega$), so there exists $k \in \omega$ such that $Y = \varphi^{-1}(\{k\})$ is also uncountable. It follows that Y has a left-sided accumulation point $y \in Y$; that is, there is a sequence $x_n \prec y$ in Y such that $x_n \to y$.

As $y(m) = 1$ infinitely often, there is $m > k$ and $x \in Y$ satisfying

$$x|m = y|m, \ x(m) = 0 \text{ and } y(m) = 1.$$

Let $\sigma = x|(m+1)$. Then $y(m) = 1$ and $\sigma(m) = 0$, so $\sigma \in S_y$, which implies that $\sigma \notin A_y$. As $m > k$, we also have $\sigma \notin C_y$ (because $(C_y \triangle A_y) \cap 2^{m+1} = \emptyset$).

On the other hand, the very definition of σ gives $\sigma \prec x$ and $\sigma \notin S_x$; therefore $\sigma \in A_x$. Since $m > k$, it follows that $\sigma \in C_x$. Finally, $\sigma \in C_x \setminus C_y$, which contradicts the fact that the sets C_x form a chain. △

Now, we will show that the real almost chain $\mathcal{A}_X$ is barely alternating and use Theorem 3.4.8 to prove the following.

THEOREM 3.5.3. *In the setting of Construction 3.5.1, if the set X is uncountable, then the space $L = \mathrm{ult}(\mathfrak{A}_X)$ is a countable discrete extension of $K = \mathrm{ult}(\mathfrak{A}_X/\mathrm{fin}(T))$ satisfying $\eta(K, L) = 3$.*

PROOF. Fix any $\sigma \in 2^{<\omega}$ and $x_1 \prec x_2 \prec x_3 \prec x_4$ in X. If $\sigma \in (A_{x_1} \cup A_{x_3}) \setminus A_{x_2}$, then $\sigma \preccurlyeq x_1$, $\sigma \in S_{x_2}$, but $\sigma \notin S_{x_3}$. It follows that $\sigma \preccurlyeq x_4$ and $\sigma \notin S_{x_4}$, so $\sigma \in A_{x_4}$. Thus, the almost chain $\mathcal{A}_X$ is barely alternating. By Theorem 3.4.8, $\eta(K, L) \le 3$. On the other hand, by Theorem 3.5.2, $\eta(K, L) \ge 3$. △

To conclude this section, we shall turn Theorem 3.5.3 into a more general result.

Recall the following observations — the first one is a slightly modified version of [**61**, Lemma 8.6].

LEMMA 3.5.4. *Let K be a separable compact line of uncountable weight κ. Then K contains a topological copy of the space $2^\omega{}_X$, where X is a dense subset of 2^ω with $|X| = \kappa$.*

REMARK 3.5.5. If $K' \subseteq K$ are compact spaces, then any countable discrete extension L' of K' defines $L \in \mathrm{CDE}(K)$ in an obvious way: Say that $L' = K' \cup \omega$. Then the space $L = K \cup \omega$ is obtained by taking the topological disjoint union of K and $K' \cup \omega$ and identifying every point in K' with its copy in K.

THEOREM 3.5.6. *If K is a nonmetrizable separable compact line, then there is $L \in \mathrm{CDE}(K)$ satisfying $\eta(K, L) = 3$.*

PROOF. Since K is nonmetrizable, $\kappa = w(K) \geq \omega_1$. By Lemma 3.5.4, K contains a topological copy K' of the zero-dimensional space $2^{\omega}{}_X$, where $|X| = \kappa$. Then, combining Theorem 3.5.3 with Lemma 3.4.4, we obtain $L' \in \mathrm{CDE}(K')$ such that $\eta(K', L') = 3$. In turn, we get an 'obvious' countable discrete extension $L = K \cup \omega$ mentioned in Remark 3.5.5.

There is an extension operator $E' \colon C(K') \to C(L')$ of norm 3, so we can define an extension operator $E \colon C(K) \to C(L)$ by

$$Ef(x) = \begin{cases} f(x) & \text{for } x \in K, \\ E'f|K'(x) & \text{for } x \in L \setminus K = \omega. \end{cases}$$

Observe that Ef is continuous on L; moreover, $\|E\| = \|E'\| = 3$. On the other hand, by Theorem 2.3.4, $\eta(K, L) < 3$ would imply that $\eta(K', L') < 3$, which cannot hold. $\triangle$

3.6. Always ($\mathcal{E}$)

It turns out that, using Martin's Axiom, all real almost chains can be forced to be barely alternating and consequently all countable discrete extensions of separable compact lines have property ($\mathcal{E}$). This result resembles the phenomenon shown in [**8**]: although (ω_1, ω_1)-gaps exist in ZFC by Hausdorff's classical construction, there are no (κ, κ, κ)-triple gaps under Martin's Axiom $\mathsf{MA}(\kappa)$.

The forcing with its properties is described in Theorem 3.6.2. The main result of this section, Theorem 3.6.3, is about its application to countable discrete extensions of separable compact lines.

For strictly technical reasons, we recall here one classical fact from the theory of gaps. For reference, see, e.g. [**45**, Lemma 29.6.].

LEMMA 3.6.1. *There are no (ω, ω) gaps. That is, if $\{U_n \subseteq \omega : n \in \omega\}$ is an ascending almost chain and $\{V_n \subseteq \omega : n \in \omega\}$ is a descending almost chain such that $U_n \subseteq^* V_m$ for all $n, m \in \omega$, then there is a set $U \subseteq \omega$ such that $U_n \subseteq^* U \subseteq^* V_n$ for all $n \in \omega$.*

We will follow the forcing notation from [**45**]. In particular, a stronger condition is lesser in the order.

THEOREM 3.6.2. *Under $\mathsf{MA}(\kappa)$, if X is a set of cardinality κ, then every real almost chain $\{A_x : x \in X\}$ of subsets of ω has a barely alternating finite adjustment $\{B_x : x \in X\}$.*

PROOF. Using the following observations, we can assume that X is a dense subset of $[0, 1]$. As $\mathbb{R}$ is order-isomorphic to $(0, 1)$, the set X can be identified with a subset of $(0, 1)$, and then we can enlarge it by a countable dense set. The almost chain $\mathcal{A}$ can then be extended to this countable set using Lemma 3.6.1 (as $X \subseteq [0,1]$, all gaps are at most (ω,ω)).

For a dense set $X \subseteq [0,1]$, consider the following partial order:

$$\begin{aligned}\mathbb{P} = \{(F, \mathcal{B} = \{B_x \subseteq \omega : x \in F\}) : \ & F \subseteq X, F \text{ is finite}, \\ & A_x =^* B_x \text{ for } x \in F, \\ & \mathcal{B} \text{ is barely alternating}\}, \\ (F_1, \mathcal{B}_1) \leq (F_2, \mathcal{B}_2) \iff & F_2 \subseteq F_1 \wedge \mathcal{B}_2 \subseteq \mathcal{B}_1.\end{aligned}$$

Claim. The partial order $\mathbb{P}$ is ccc.

Proof of the Claim. Consider any uncountable set $P = \{(F^\alpha, \mathcal{B}^\alpha) : \alpha \in \omega_1\}$ of conditions in $\mathbb{P}$. Without loss of generality, we can assume (as P is uncountable and we can pass to uncountable subfamilies) that:

- The sets F^α form a Δ-system, so for all $\alpha \in \omega_1$ we have $F^\alpha = R \cup G_\alpha$, where the sets G_α are pairwise disjoint and have a fixed cardinality $k \in \omega$. For each $\alpha \in \omega_1$, let $G_\alpha = \{g_i^\alpha : 1 \leq i \leq k\}$ and $R = \{r_i : 1 \leq i \leq l\}$ be increasing enumerations of these sets. We also fix, for each $\alpha \in \omega_1$ and $i \in \{1, \ldots, l-1\}$, the number of points of G_α below r_1, between r_i and r_{i+1}, and above r_l.
- For each $1 \leq i \leq k-1$ there are rational points p_i, q_i satisfying $g_i^\alpha < p_i < q_i < g_{i+1}^\alpha$ for every $\alpha \in \omega_1$.
- There is a set $Z \subseteq X$ such that $R \subseteq Z$ and for each pair of consecutive points $g_i^\alpha, g_{i+1}^\alpha$ there is a point $z_i \in Z$ chosen either from R (if there is any) or from the interval (p_i, q_i).
- Since the set R is finite, we can fix sets B_r for all $r \in R$ and $\alpha < \omega_1$. We can also choose sets $B_z =^* A_z$ for $z \in Z \setminus R$ such that the almost chain $\mathcal{B}^\alpha \cup \{B_z : z \in Z \setminus R\}$ remains barely alternating.
- There is a finite set $D \subseteq \omega$ such that, for all $\alpha \in \omega_1$, the almost chain
$$\mathcal{C} = \{B_{g_i^\alpha}^\alpha : 1 \leq i \leq k\} \cup \{B_z : z \in Z\}$$
is a chain outside D; i.e., for all $A, B \in \mathcal{C}$ if $A \subseteq^* B$, then $A \setminus B \subseteq D$.
For each $d \in D$ we can also fix the indices i for which the set $B_{g_i^\alpha}^\alpha$ contains d, so that for all $\alpha, \beta \in \omega_1$ we have $d \in B_{g_i^\alpha}^\alpha$ if and only if $d \in B_{g_i^\beta}^\beta$

Now let us take two elements $(F^\alpha, \mathcal{B}^\alpha), (F^\beta, \mathcal{B}^\beta) \in P$ and show that they are compatible. It is enough to show that the almost chain $\mathcal{B}^\alpha \cup \mathcal{B}^\beta$ is barely alternating. For each $x \in F^\alpha \cup F^\beta$, write B_x for the corresponding element of $\mathcal{B}^\alpha \cup \mathcal{B}^\beta$.

Fix any $m \in \omega$ and elements $x_1 < x_2 < x_3 < x_4$ from $F^\alpha \cup F^\beta$. If $m \in D$, then $m \in B_{g_i^\alpha}^\alpha$ if and only if $m \in B_{g_i^\beta}^\beta$ for all i. As both almost chains $\mathcal{B}^\alpha, \mathcal{B}^\beta$ are barely alternating, we have $m \in B_{x_1} \cap B_{x_3} \implies m \in B_{x_2} \cup B_{x_4}$.

For $m \notin D$, if $m \in B_{x_1}$ and $m \notin B_{x_2}$, then by the definition of D we have $x_1, x_2 \notin R \subseteq Z$. Without loss of generality, we can write $x_1 = g_i^\alpha, x_2 = g_j^\beta$ for some $1 \leq i \leq j \leq k$. Now if $i < j$, then we have a contradiction with the definition of D, since $m \in B_{z_i}$ (as it is an element of $B_{g_i^\alpha}^\alpha$) and $m \notin B_{z_i}$ (as it is not an element of $B_{g_j^\beta}^\beta$). If $i = j$, then $m \in B_{z_i}$ and $m \in B_{x_3}, m \in B_{x_4}$, as $x_3, x_4 \geq z_i \in Z$. It follows that the almost chain $\mathcal{B}^\alpha \cup \mathcal{B}^\beta$ is indeed barely alternating. ▲

In fact, we have proven that $\mathbb{P}$ satisfies the Knaster condition, which is stronger than ccc; see [45, Definition 15.13].

Claim. For all $x \in X$ the set $\mathcal{D}_x = \{(F, \mathcal{B}) \in \mathbb{P} : x \in F\}$ is dense in $\mathbb{P}$.

Proof of the Claim. Fix any $x \in X$ and an condition $(F, \mathcal{B}) \in \mathbb{P}$. If $x \in F$, we are done. Otherwise, let $y, z \in F$ be the elements directly preceding and succeeding x in F and put $A = B_y, C = B_z$ (if x is below or above all elements of F, put $A = \emptyset$ or $C = \omega$ accordingly). Define $B_x = (A_x \cup A) \setminus (A_x \setminus C)$. Then $B_x =^* A_x$, since we have added or removed only a finite number of elements.

Let us check that $(F \cup \{x\}, \mathcal{B} \cup \{B_x\})$ is an element of $\mathbb{P}$, so we have to verify that it is barely alternating. For all $m \in \omega$ we have either $(m \in B_x \iff m \in A)$ or $(m \in B_x \iff m \in C)$, so the almost chain $\mathcal{B} \cup \{B_x\}$ is barely alternating. It means that for all $x \in X$, the set $\mathcal{D}_x$ is dense in $\mathbb{P}$. △

As $|X| = \kappa$, by Martin's Axiom, there is a generic filter $\mathcal{G} \subseteq \mathbb{P}$ intersecting all sets $\mathcal{D}_x$, because $\mathbb{P}$ is ccc. Then the family $\bigcup_{(F,\mathcal{B})\in\mathcal{G}} \mathcal{B}$ is the almost chain we are looking for. △

We can now translate infinitary combinatorics of the previous result to the language of compact lines.

Theorem 3.6.3. *Under MA(κ), if K is a separable compact line of weight κ and $L \in \mathrm{CDE}(K)$, then L has property ($\mathcal{E}$) and $\eta(K, L) \leq 3$.*

Proof. By Theorem 2.2.6, K is homeomorphic to the space F_X for some closed set $F \subset [0, 1]$ and a subset $X \subset F$. For the proof, we may therefore consider $K = F_X$. Fix a countable subset $X' \subseteq F_X \setminus (X \times \{0, 1\})$ dense in $F_X \setminus \overline{(X \times \{0, 1\})}$. Then the set $Y = X \times \{0\} \cup X'$ is dense in K and $|Y| = w(K)$. Notice that no point in Y is isolated only from the left and $Y \subseteq F \times \{0\}$, so it is order-isomorphic to a subset of the real line.

Fix any countable discrete extension L of K.

Claim. For every $y \in Y$ there is a continuous function $h_y \in C(L)$ such that, for every $x \in K$, we have

- $h_y(x) < 0$ if $x < y$,
- $h_y(x) > 0$ if $x > y$,
- $h_y(y) < 0$ if y is isolated from the right.

Proof of the Claim. By Tietze's extension theorem, it is enough to define h_y on K. If y is isolated from the right, define $h_y(x) = -1$ if $x \leq y$ and $h_y(x) = 1$ if $x > y$.

Otherwise, by the definition of Y, y is not isolated neither from left nor right. Since every separable line is first countable, there exist an increasing sequence $(a_n)_{n\in\omega}$ and a decreasing sequence $(b_n)_{n\in\omega}$ in K, both convergent to y. It follows that $\{x \in K : x < y\}$ and $\{x \in K : x > y\}$ are disjoint F_σ open subsets of K, so we can define a continuous function that is negative on the former set and positive on the latter. ▲

We fix functions h_y as in the Claim. We define

$$A_y = \{n < \omega : h_y(n) < 0\},$$

for $y \in Y$. Then $\{A_y : y \in Y\}$ is a real almost chain of subsets of ω. Indeed, if $y < y'$, then every $n \in A_y \setminus A_{y'}$ satisfies $h_y(n) < 0$ and $h_{y'}(n) \geq 0$. Therefore

$$K \cap \overline{A_y \setminus A_{y'}} \subseteq \{x \in K : h_y(x) \leq 0, h_{y'}(x) \geq 0\} \subseteq \{x \in K : y' \leq x \leq y\} = \emptyset,$$

so $A_y \setminus A_{y'}$ is finite.

Using Theorem 3.6.2, we can finitely adjust the almost chain $\{A_y : y \in Y\}$ to a barely alternating almost chain $\mathcal{B} = \{B_y : y \in Y\}$. Let $\mathfrak{A}$ be the Boolean algebra of subsets of ω generated by $\mathcal{B} \cup \mathrm{fin}$ and denote $K' = \mathrm{ult}(\mathfrak{A}/\mathrm{fin}), L' = \mathrm{ult}(\mathfrak{A})$. By Theorem 3.4.8, there is an extension operator $E' \colon C(K') \to C(L')$ of norm at most 3.

Now, observe that K' is essentially K with some points doubled and L' is an 'obvious' countable discrete extension as in Remark 3.5.5. Thus, there is a continuous retraction $\varphi \colon L' \to L$ such that $\varphi|K' \colon K' \to K$ is also a retraction. It follows that, we can define isometric embeddings

$$S_1 \colon C(K) \to C(K'), S_2 \colon C(L) \to C(L')$$

given by

$$S_1 f = f \circ (\varphi|K'), S_2 g = g \circ \varphi.$$

It is straightforward to verify that $E = S_2^{-1} E' S_1 \colon C(K) \to C(L)$ is an extension operator of norm at most 3. △

3.7. Outside $(\mathcal{E})$

Consider again a real almost chain $\mathcal{A} = \{A_x : x \in X\}$ of subsets of ω indexed by some $X \subseteq [0, 1]$ and compact spaces K and L as in section 3.4. It turns out that the lack of property $(\mathcal{E})$ can be characterized in terms of alternations, which can be distilled into the following combinatorial property of $\mathcal{A}$.

Lemma 3.7.1. *Suppose that for every $k \in \omega$ there are no k-alternating finite adjustments $\{C_x : x \in X\}$ of $\mathcal{A}$, i.e., for every natural number k there are $x_0 < x_1 < \ldots < x_k$ in X and $m \in \omega$ such that for all $i \leq k$ we have*

$$m \in C_{x_i} \text{ if and only if } i \text{ is even.}$$

Then there is no bounded extension operator $C(K) \to C(L)$.

Proof. By Lemma 3.4.2, it is enough to demonstrate that, whenever $(\mu_n)_{n\in\omega}$ is a sequence of finitely additive measures on $\mathfrak{A}$ satisfying

(i) $\mu_n(I) = 0$ for every $I \in \mathrm{fin}$ and every n;
(ii) for every $A \in \mathfrak{A}$ and $\varepsilon > 0$, the set $\{n \in \omega : |\mu_n(A) - \delta_n(A)| \geq \varepsilon\}$ is finite,

then $\sup_n \|\mu_n\| = \infty$.

For such a sequence μ_n, for every $x \in X$ put

$$C_x = \{n \in \omega : |\mu_n(A_x) - \delta_n(A_x)| < 1/4\},$$

so that $C_x =^* A_x$. Consider $x_0 < x_1 < \ldots < x_k$ and m given by the assumption of the lemma. As the measure μ_m vanishes on finite sets, we have

$$\mu_m(A_{x_1} \setminus A_{x_0}) = \mu_m(A_{x_1}) - \mu_m(A_{x_0}) < 1/4 - 3/4 = -1/2.$$

Similarly, $|\mu_m|(A_{x_i} \setminus A_{x_{i-1}}) \geq 1/2$ for every $i \leq k$ and thus $\|\mu_m\| \geq k/2$. Since k can be arbitrarily large, it follows that $\sup_n \|\mu_n\| = \infty$, as required. △

Theorem 3.6.3 implies that one cannot find in ZFC an almost chain of cardinality ω_1 that satisfies the assumption of Lemma 3.7.1. However, we show that such a construction becomes possible if we replace ω_1 with an appropriate cardinal invariant.

Recall that if $\mathcal{I}$ is a proper σ-ideal of subsets of, for instance, the Cantor set 2^ω, then the uniformity of $\mathcal{I}$ is given by

$$\mathrm{non}(\mathcal{I}) = \min\{|A| : A \notin \mathcal{I}\}.$$

As in [7], we consider the σ-ideal $\boldsymbol{\mathcal{I}}$ of subsets of 2^ω that can be covered by a countable number of closed sets of measure zero. Cardinal coefficients of $\boldsymbol{\mathcal{I}}$ are discussed by Bartoszyński and Shelah [10]. This ideal is usually denoted by $\mathcal{E}$ — we have changed the notation for obvious reasons. Clearly, $\boldsymbol{\mathcal{I}} \subseteq \mathcal{N} \cap \mathcal{M}$, where $\mathcal{N}$ denotes the family of λ-null subsets of 2^ω (with λ being the standard product measure) and $\mathcal{M}$ denotes the σ-ideal of meager subsets of 2^ω. Hence

$$\mathrm{non}(\boldsymbol{\mathcal{I}}) \leq \min\left(\mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M})\right)$$

and the strong inequality in the above formula is relatively consistent; see [10]. Recall that cardinal coefficients of classical σ-ideals do not change if we replace 2^ω by any uncountable Polish space (and λ by any nonatomic Borel measure on it), cf. [10] and [40].

The main goal of this section is to prove the following theorem.

THEOREM 3.7.2. *Let $\kappa \geq \mathrm{non}(\boldsymbol{\mathcal{I}})$. Then there is a zero-dimensional separable compact line K of weight κ that has a countable discrete extension without property ($\mathcal{E}$).*

Let us first present a construction leading to the space mentioned in the theorem and recall some notions used in the proof.

CONSTRUCTION 3.7.3. We consider a subtree T of $\omega^{<\omega}$ defined as

$$T = \{\sigma : \sigma(n) \leq n \text{ for every } n\},$$

and its body

$$C = \{0\} \times \{0,1\} \times \{0,1,2\} \times \dots$$

Again, as in Construction 3.5.1, we consider the lexicographic order $\preccurlyeq$ on the set $T \cup C$. Since the space ω^ω can be identified with the set of irrational numbers, we can see C as a subset of the unit interval $[0,1]$.

Take any set $X \subseteq C$. For $x \in X$, define

$$S_x = \{x|n^\frown 0 : n \in \omega, x(n) \text{ is odd}\},$$

$$A_x = \{\sigma \in T : \sigma \preccurlyeq x\} \setminus S_x$$

and consider the family $\mathcal{A}_X = \{A_x : x \in X\}$.

As in Construction 3.5.1, it follows that $\mathcal{A}_X$ is a real almost chain of subsets of T. Therefore, if we denote by $\mathfrak{B}_X$ the Boolean algebra generated by $\mathcal{A}_X \cup \mathrm{fin}(T)$, then $K = \mathrm{ult}(\mathfrak{B}_X/\mathrm{fin}(T))$ is a separable compact line with $w(K) = |X|$, and $L = \mathrm{ult}(\mathfrak{B}_X)$ is its countable discrete extension; see Lemma 3.4.3.

For the proof below, we equip C with the standard product measure λ. Thus, for every $\sigma \in T$ of length n and $i \leq n$ the measure satisfies

$$\lambda([\sigma^\frown i]) = \lambda([\sigma])/(n+1), \tag{3.7.1}$$

where $[\sigma]$ denotes all elements of C extending σ. Then we may think consider $\boldsymbol{\mathcal{I}}$ as the σ-ideal of subsets of the space C generated by closed subsets of λ-measure zero.

We shall make use of the fact that the measure λ satisfies the Lebesgue density theorem; that is, for every closed set $F \subseteq C$ we have

$$\lim_{k\longrightarrow\infty} \frac{\lambda([x|k] \cap F)}{\lambda([x|k])} = 1,$$

for λ-almost all $x \in F$. See, e.g., [**67**, Proposition 2.10] for a short proof that this property is shared by every probability measure on a Polish ultrametric space.

Proof of Theorem 3.7.2. Since $\kappa \geq \mathrm{non}(\boldsymbol{\mathcal{I}})$, let us fix a set $X \subseteq C$ of cardinality κ such that $X \notin \boldsymbol{\mathcal{I}}$. It follows that whenever $X = \bigcup_n X_n$, then we have $\lambda(\overline{X_n}) > 0$ for some n.

Let $L = \mathrm{ult}(\mathfrak{B}_X)$ be the space from Construction 3.7.3. We shall check that L is a countable discrete extension of $K = \mathrm{ult}(\mathfrak{B}_X/\mathrm{fin}(T))$ that does not have property $(\mathcal{E})$. It is enough to show that the real almost chain $\mathcal{A}_X$ satisfies the assumption in Lemma 3.7.1.

Assume that $\{C_x : x \in X\}$ is a finite adjustment of $\mathcal{A}_X$. Then we can write $X = \bigcup_n X_n$, where X_n consists of those $x \in X$ for which the statement "$\sigma \in C_x$" is equivalent to "$\sigma \in A_x$" for all σ of length at least n.

By the preparatory remarks, there is n_0 such that, writing $F = \overline{X_{n_0}}$, we have $\lambda(F) > 0$. In turn, there is a point $y \in F$ at which the set F has density one.

Let us fix a natural number k. Take $n > n_0$ such that

$$\frac{\lambda([y|n] \cap F)}{\lambda([y|n])} > \frac{2k}{2k+1}. \tag{3.7.2}$$

We can, of course, assume that $n + 1 = 2kl$ for some natural number l. Consider the set

$$I = \{i \leq n : y|n^\frown i = x|(n+1) \text{ for some } x \in F\}.$$

Claim 1. The set I satisfies

$$|I| \geq \frac{2k}{2k+1}(n+1).$$

Proof. Indeed, $i \notin I$ implies $[y|n^\frown i] \cap F = \emptyset$, so the Claim follows from (3.7.1) and (3.7.2). ▲

Divide $\{0, \ldots, n\}$ into k consecutive intervals $J_0, \ldots, J_{k-1}$, each satisfying $|J_i| = 2l$.

Claim 2. $|I \cap J_i| > l$ for every $i < k$.

Proof. Indeed, by the Claim 1 we have

$$\frac{2k}{2k+1}(n+1) \leq |I| = |I \cap J_i| + |I \setminus J_i| \leq |I \cap J_i| + n + 1 - 2l, \text{ so}$$

$$|I \cap J_i| \geq \frac{2k}{2k+1}(n+1) - (n+1) + 2l = 2l\frac{k+1}{2k+1} > l.$$

▲

It follows from the Claim 2 that for every $i \leq k$, the set $I \cap J_i$ contains at least one odd and one even number. Pick an even number $m_0 \in I \cap J_0$, an odd $m_1 \in I \cap J_1$, an even $m_2 \in I \cap J_2$ and so on.

Put $\sigma = y|n^\frown 0$. From the definition of I and the fact that $F = \overline{X_{n_0}}$ we conclude that there are $x_0, \ldots, x_{k-1} \in X_{n_0}$ such that

$$\sigma \in A_{x_0}, \sigma \notin A_{x_1}, \sigma \in A_{x_2} \ldots$$

Consequently, as A_x agree with C_x at that level, the said σ is as required. △

COROLLARY 3.7.4. *Every separable compact line K of weight greater than or equal to* non($\boldsymbol{\mathcal{I}}$) *has a countable discrete extension without property* ($\mathcal{E}$).

PROOF. We can argue as in the proof of Theorem 3.5.6: Find a zero-dimensional subspace K' of K and $L' \in \mathrm{CDE}(K')$ such that $\eta(K', L') = \infty$. Combine L' with K to obtain $L \in \mathrm{CDE}(K)$ and note that $\eta(K, L)$ is also infinite by Theorem 2.3.4. △

Let us also note that a modification of the proof of Theorem 3.7.2 that is thinning out the tree used there (so that the almost chain is exactly k-alternating) should give $L \in \mathrm{CDE}(K)$ with $\eta(K, L)$ finite but arbitrarily large.

CHAPTER 4

Functions on nonmetrizable cubes

4.1. Introduction

In this chapter, we provide a partial answer to the following questions regarding the Banach spaces $C(\prod_{i=1}^n K_i)$ of all real-valued continuous functions defined on a finite product of compact lines.

QUESTION 1. *Consider compact lines* $K_1, \ldots, K_n,\ L_1, \ldots, L_k$ *for* $n > k$.
Are Banach spaces $C(\prod_{i=1}^n K_i)$ *and* $C(\prod_{j=1}^k L_j)$ *isomorphic?*
Is there an isomorphic embedding $C(\prod_{i=1}^n K_i) \hookrightarrow C(\prod_{j=1}^k L_j)$*?*
Is there a continuous linear surjection $C(\prod_{j=1}^k L_j) \twoheadrightarrow C(\prod_{i=1}^n K_i)$*?*

Question 1 is a generalisation of a famous problem whether the Banach spaces $C([0,1])$ and $C([0,1]^2)$ are isomorphic (stated in Banach's book [9]). We consider mostly nonmetrizable compact lines, as the complete isomorphic characterisation of spaces of continuous functions on metrizable compact spaces has been known since the 1960s, thanks to the results of Miljutin [66], Bessaga and Pełczyński [12].

Let $K_1, \ldots, K_n, L_1, \ldots, L_k$ be nonmetrizable compact lines for some $n \neq k$. Due to the result of Martínez-Cervantes and Plebanek [62], we know that the products $\prod_{i=1}^n K_i$ and $\prod_{j=1}^k L_j$ are not homeomorphic. It follows that the corresponding spaces of continuous functions are not isometric. In [65], Michalak proved, in particular, that if all the compact lines K_i, L_j are additionally separable, then the Banach spaces $C(\prod_{i=1}^n K_i)$ and $C(\prod_{j=1}^k L_j)$ are not isomorphic. Thus, it remains to check what happens for nonseparable compact lines.

Due to the limitations of our methods, we were able to obtain results only for compact lines of uncountable character. Nonseparable linearly ordered spaces of countable character can be quite unusual, such as Suslin lines (which are consistently nonexistent in ZFC) or Aronszajn lines (see [68]).

Our approach is inspired by Semadeni's article [80], which contains a proof that the space $C([0,\omega_1])$ of continuous functions on the set of ordinals not greater than the first uncountable ordinal, is not isomorphic to each of its finite powers $\big(C([0,\omega_1])\big)^n$. This was one of the first examples of a Banach space that was not isomorphic to its square, addressing another long-standing problem from the book of Banach [9]. Building on top of this result, Candido in [20] defined the Semadeni derivative $\mathcal{S}(X) = X^{\mathfrak{s}}/X$ of a Banach space X, where

$$X^{\mathfrak{s}} = \{x^{**} \in X^{**} : x^{**} \text{ is a weak}^* \text{ sequentially continuous functional}\}.$$

At the end of the Semadeni's paper [80] there is a suggestion by Pełczyński to use a very similar notion, replacing weak* sequential continuity with continuity on weak* separable subspaces.

In this chapter, we denote by κX the space of functionals in X^{**} that are weak* continuous on subspaces of density κ, and define the κ-Semadeni-Pełczyński derivative of X (or briefly the SP derivative) as $\mathcal{SP}_\kappa(X) = \kappa X/X$.

The properties of the Semadeni-Pełczyński derivative are the subject of section 4.3. Most of them were proven for the Semadeni derivative by Candido [**20**], but several proofs presented here are substantially different. Later, we introduce the Semadeni-Pełczyński dimension (SP dimension), which measures how many iterations of the derivative are required to obtain the trivial space.

Next, we proceed with some easier results, including the description of the SP derivative of $C(K)$ for a compact line K (Theorem 4.4.3). It turns out that the κ-SP dimension is nontrivial only for compact lines of character greater than κ. As a consequence, the character of the compact line K is an isomorphic invariant of the space $C(K)$. We also present (Theorem 4.4.8) an alternative proof of the results of Galego [**41**] about the isomorphic classification of spaces $C(2^\theta \times [0, \lambda^+])$ for cardinal numbers κ, λ.

The main result of this chapter is a partial answer to Question 1, presented in section 4.5. In Theorem 4.5.2, we prove that if K is a finite product of compact lines, then the number of factors of a given uncountable topological character is an isomorphic invariant of the space $C(K)$. In particular, if compact lines K_1, K_2, L all have an uncountable character, then there are no linear surjections from $C(L)$ onto $C(K_1 \times K_2)$, nor can $C(K_1 \times K_2)$ be isomorphic to a subspace of $C(L)$.

We finish this chapter with section 4.6, where we present a collection of thoughts broadly related to the subject. We prove partial results related to Michalak's paper [**65**] on isomorphisms of spaces on continuous functions on products of separable compact lines (Corollary 4.6.4) and recall some facts about Suslin lines.

4.2. Preliminaries

We recall a classical topological concept introduced by Arhangel'skii in [**4**].

Definition 4.2.1. Let F be a topological space and κ be a cardinal number. A function $f\colon F \to \mathbb{R}$ is called *κ-continuous* if its restriction to any subset of F of cardinality at most κ is continuous.

It might be easier to use the following, rather straightforward, characterisation of κ-continuity.

Lemma 4.2.2 ([**4**, Corollary 1]). *A function f on a topological space F is κ-continuous if and only if $f|G$ is continuous for every subspace $G \subseteq F$ of density $\leq \kappa$.*

The notion of κ-continuity is closely related to the functional tightness of a topological space. For any topological space F, let $t_0(F)$ be the least cardinal κ such that every κ-continuous function on F is continuous.

It is not difficult to show that, for any topological space F, the functional tightness $t_0(F)$ does not exceed to the character $\chi(F)$. In particular, we have $t_0([0, \kappa^+)) = \kappa$ for every infinite cardinal κ. In [**56**], Krupski proved, in particular, that functional tightness is preserved under finite products. We will use this fact without explicitly referring to it each time.

Now, let us introduce some definitions useful in characterisation of the space of κ-continuous functions on products of compact lines. In the following, we will use them to calculate the κ-SP derivative of spaces of continuous functions on such products. We write $A \sqcup B$ for the disjoint union of sets A and B.

Definition 4.2.3. Let K be a compact line and κ a cardinal number. A point $k \in K$ is called κ-inaccessible from the left [right] if $k \notin \overline{A}$ for every set $A \subseteq (\leftarrow, k)$ $[A \subseteq (k, \rightarrow)]$ of size $\leq \kappa$.

We denote

$$K_L = \{k \in K : \chi(k, (\leftarrow, k]) > \kappa\},$$
$$K_R = \{k \in K : \chi(k, [k, \rightarrow)) > \kappa\},$$
$$K^{\uparrow} = K_L \sqcup K_R,$$

so that K_L and K_R are the sets of $\kappa-$inaccessible points of K from the left or right, respectively. The set of all κ-inaccessible points of K is denoted $K^{\uparrow}$ (including duplicates, if a point is κ-inaccessible from both sides). Note that the sets $K_L, K_R, K^{\uparrow}$ all depend on the parameter κ.

The methods developed by Semadeni in [80] have so far been used primarily for ordinal intervals, with the main focus on the space $[0, \omega_1]$. The following classical fact motivates the application of results obtained for ordinal intervals to compact lines of uncountable character.

Fact 4.2.4. *Let κ be an infinite cardinal number. If K is a compact line and $k \in K^{\uparrow}$ is a point of character $> \kappa$, then we can find a topological copy of $[0, \kappa^+]$ inside K such that k corresponds to the point κ^+.*

Let κ be any infinite cardinal and K a compact line. Due to cofinality reasons, any continuous function $[0, \kappa^+) \rightarrow \mathbb{R}$ must be eventually constant. It follows that $\beta[0, \kappa^+) = [0, \kappa^+]$.

Note that for $k \in K_L$ $[k \in K_R]$, the space $(\leftarrow, k)$ $[(k, \rightarrow)]$ is a pseudocompact locally compact space. By Fact 4.2.4, for any point $k \in K_L$ $[k \in K_R]$ we have $\beta(\leftarrow, k) = (\leftarrow, k]$ $[\beta(k, \rightarrow) = [k, \rightarrow)]$. Using Theorems 2.2.1, 2.2.2, it follows that a similar property holds for products of compact lines.

Corollary 4.2.5. *Consider compact lines $K_1, \ldots, K_n$ and points $k_i \in K_i^{\uparrow}$ for a given cardinal number κ. Let*

$$I_i = \begin{cases} (\leftarrow, k_i) & \text{for } k_i \in K_{iL}, \\ (k_i, \rightarrow) & \text{for } k_i \in K_{iR}. \end{cases}$$

Then $\beta \prod_{i=1}^n I_i = \prod_{i=1}^n \beta I_i = \prod_{i=1}^n \overline{I_i}^{K_i}$.

4.3. New objects and their basic properties

4.3.1. Properties of Semadeni-Pełczyński derivative. First, we define the SP derivative. Then, we prove its properties, namely that it is preserved by embeddings and continuous surjections, and that it commutes with c_0-sums.

Definition 4.3.1. Let X be any Banach space. For an infinite cardinal number κ put

$$\kappa X = \{x^{**} \in X^{**} : \forall A \subseteq X^* \ |A| \leq \kappa \ \exists x \in X \ x^{**}|A = x|A\}.$$

The space κX is a closed subspace of X^{**} containing X. We define κ-Semadeni-Pełczyński derivative by $\mathcal{SP}_\kappa(X) = \kappa X/X$.

There is some additional context to this definition. Recall that a topological space F is realcompact if it can be topologically embedded as a closed subspace in the product $\mathbb{R}^\Gamma$ for some set Γ. The space ωX has already been considered in the literature (see [**83**, Section 2.4]), as it is the realcompactification of the Banach space X considered with the weak topology. The notation ωX was introduced by Corson [**30**] (who was actually writing $\aleph_0 X$).

It turns out that for any compact space K satisfying some natural properties, we can describe elements of the space $\kappa C(K)$ as κ-continuous functions on K. The following theorem is a modified version of [**76**, Proposition, page 29].

Theorem 4.3.2. *Suppose that K is a compact space such that every measure $\mu \in M_1^+(K)$ has a separable set of full measure. Then $\varphi \in \kappa C(K)$ if and only if φ is represented by a κ-continuous function on K.*

Proof. Consider any $\varphi \in \kappa C(K)$ and define a function $g_\varphi \colon K \to \mathbb{R}$ by $g_\varphi(k) = \varphi(\delta_k)$, for all $k \in K$. By the definition of $\kappa C(K)$, the function g_φ is κ-continuous.

It remains to show that $\varphi(\mu) = \int_K g_\varphi \, \mathrm{d}\mu$ for all probability measures $\mu \in M(K)$. By the assumption, for every such μ, there is a closed set $F \subseteq K$ of full measure which is the closure of a countable set D. The desired formula follows from Lemma 4.2.2.

Conversely, for a κ-continuous function $f \colon K \to \mathbb{R}$, we define $\varphi(\mu) = \int_K f \, \mathrm{d}\mu$. We can see that φ is well-defined as μ has a separable set of full measure. The functional φ is an element of $\kappa C(K)$, since the union of supports of κ many measures is a set of density κ on which, by Lemma 4.2.2, f is continuous. △

Note that the assumption of Theorem 4.3.2 can be relaxed — it is enough to assume that every probability measure on K has a subspace of density κ of full measure. Recall that if K is a compact line, then every measure $\mu \in M_1^+(K)$ has a separable support; see [**63**, page 86]. This observation easily implies the following.

Lemma 4.3.3. *If $K = K_1 \times \ldots \times K_n$ is a finite product of compact lines K_i, then every measure $\mu \in M_1^+(K)$ admits a separable set of full measure.*

For the Semadeni derivative, the following properties were proven by Candido; see [**20**, Theorem 3.1, Lemma 3.10 and Theorem 1.2]. Here, we show that the Semadeni-Pełczyński derivative behaves very similarly to the Semadeni derivative and present slightly different proofs of these facts. We begin by showing that the SP derivative behaves well with respect to isomorphic embeddings and continuous linear surjections.

Lemma 4.3.4. *For Banach spaces X, Y, if $X \hookrightarrow Y$, then $\mathcal{SP}_\kappa(X) \hookrightarrow \mathcal{SP}_\kappa(Y)$ and if $X \twoheadrightarrow Y$, then $\mathcal{SP}_\kappa(X) \twoheadrightarrow \mathcal{SP}_\kappa(Y)$.*

Proof. Consider a linear operator $T \colon X \to Y$. We can lift T to the dual operators $T^* \colon Y^* \to X^*$ and $T^{**} \colon X^{**} \to Y^{**}$. Recall that X can be seen as a subspace of X^{**}.

It is a standard to check that $T^{**}x = Tx$ for $x \in X$ and if for some $x^{**} \in X^{**}$ we have $T^{**}x^{**} \in Y$, then in fact $x^{**} \in X$.

First, we prove that for $\varphi \in \kappa X$ we have $T^{**}\varphi \in \kappa Y$. Fix some element $\varphi \in \kappa X$ and let $A \subseteq Y^*$ be any set of cardinality at most κ. Put $B = T^*[A] \subseteq X^*$. Since the cardinality of B is at most κ, there exists an element $x \in X$ satisfying $x|B = \varphi|B$. It follows that $T^{**}x|A = T^{**}\varphi|A$ and $T^{**}\varphi \in \kappa Y$.

Define an operator $\widehat{T}\colon \kappa X \to \mathcal{SP}_\kappa(Y)$ by $\widehat{T}(x^{**}) = [T^{**}x^{**}]$, its norm is bounded by $\|T\|$. Observe that $X \subseteq \ker \widehat{T}$ since for $x \in X$ we have $T^{**}x \in Y$. By Lemma 2.3.1, we obtain an injective operator $S\colon \kappa X/\ker \widehat{T} \to \mathcal{SP}_\kappa(Y)$ of the same norm as $\widehat{T}$.

If T was a surjection, then S is also a surjection, and $\kappa X/\ker \widehat{T}$ is a quotient of $\mathcal{SP}_\kappa(X)$. Therefore, $\mathcal{SP}_\kappa(X) \twoheadrightarrow \mathcal{SP}_\kappa(Y)$.

Now assume that T is an embedding. If $\widehat{T}(x^{**}) = 0$, then $T^{**}x^{**} \in Y$, which implies $x^{**} = x$ for some $x \in X$. It follows that $\ker \widehat{T} = X$ and $S\colon \mathcal{SP}_\kappa(X) \to \mathcal{SP}_\kappa(Y)$.

We need to check whether S^{-1} is bounded. Assume towards a contradiction that there exists $\varphi \in \kappa X$ such that $\|[\varphi]\| = 1$ and $\|\widehat{T}\varphi\| < \|T^{-1}\|/2$. Then we can pick $y \in Y$ such that $\|T^{**}\varphi - y\| < \|T^{-1}\|/2$. Note that $\operatorname{dist}(T^{**}\varphi, T^{**}[X]) \geq \|T^{-1}\|$, as otherwise there is some $x \in X$ such that $\|T^{**}(\varphi - x)\| < \|T^{-1}\|$ and $\|\varphi - x\| < 1$. From the triangle inequality it follows that

$$\operatorname{dist}(T^{**}[X], y) \geq \operatorname{dist}(T^{**}\varphi, T^{**}[X]) - \|T^{**}\varphi - y\| > \|T^{-1}\|/2.$$

By the Hahn-Banach theorem, there is an element $y^* \in Y^*$ of norm one such that $y^*(y) > \|T^{-1}\|/2$ and $y^*|T^{**}[X] = 0$. This leads to a contradiction, as

$$\|T^{-1}\|/2 < |y^*(y)| = |\varphi(T^*y^*) - y^*(y)| = |(T^{**}\varphi - y)(y^*)| \leq \|T^{**}\varphi - y\| < \|T^{-1}\|/2.$$

△

To show that the SP derivative commutes with c_0-sums, we will need the following lemma.

LEMMA 4.3.5. *If $\{X_i : i \in I\}$ is a family of Banach spaces, then for every infinite cardinal number κ we have*

$$\kappa(c_0(I, X_i)) = c_0(I, \kappa X_i).$$

PROOF. Consider an element $(\varphi_i)_{i\in I} \in c_0(I, \kappa X_i)$ with only one non-zero coordinate, say $\varphi_{i_0} \neq 0$ and $\varphi_j = 0$ for $j \in I \setminus \{i_0\}$. Since every functional from $c_0(I, X_i)^*$ acts on $(\varphi_i)_{i\in I}$ in the same way as some functional from $X_{i_0}^*$, we have that $(\varphi_i)_{i\in I} \in \kappa(c_0(I, X_i))$ and

$$\|(\varphi_i)_{i\in I}\|_{c_0(I,\kappa X_i)} = \|(\varphi_i)_{i\in I}\|_{\kappa(c_0(I,X_i))}.$$

It follows that the closed linear span of such vectors must span an isometric copy of $c_0(I, \kappa X_i)$ in $\kappa(c_0(I, X_i))$.

Take any $(\varphi_i)_{i\in I} \in \kappa(c_0(I, X_i)) \subseteq \ell_\infty(I, X_i^{**})$. First, notice that for every $i \in I$ we have $\varphi_i \in \kappa X_i$, as each functional from X_i^* can be seen as an element of $c_0(I, X_i)^*$. Moreover, for every $\varepsilon > 0$, only finitely many elements φ_i can have norm greater than ε. Otherwise, we could find a countable sequence of functionals on which no element of $c_0(I, X_i)$ could match the values of $(\varphi_i)_{i\in I}$, thus $(\varphi_i)_{i\in I} \in c_0(I, \kappa X_i)$. △

Theorem 4.3.6. *If $\{X_i : i \in I\}$ is a family of Banach spaces, then for every cardinal number κ we have*

$$\mathcal{SP}_\kappa(c_0(I, X_i)) = c_0(I, \mathcal{SP}_\kappa(X_i)).$$

Proof. If follows from Lemmas 4.3.5 and 2.3.1. △

4.3.2. Semadeni-Pełczyński dimension. We can define the SP dimension by iterating the SP derivative. Later, we will use it to show the main results of this chapter. For a Banach space X denote $\mathcal{SP}^{(1)}_\kappa(X) = \mathcal{SP}_\kappa(X)$ and $\mathcal{SP}^{(n+1)}_\kappa(X) = \mathcal{SP}_\kappa(\mathcal{SP}^{(n)}_\kappa(X))$.

Definition 4.3.7. We define κ-Semadeni-Pełczyński dimension of a Banach space X by the following conditions

- $sp_\kappa(X) = -1$ if $X = \{0\}$,
- $sp_\kappa(X) = n$ if $\mathcal{SP}^{(n+1)}_\kappa(X) = \{0\}$ and $\mathcal{SP}^{(n)}_\kappa(X) \neq \{0\}$,
- $sp_\kappa(X) = \infty$ if for all $n \in \omega$ we have $\mathcal{SP}^{(n)}_\kappa(X) \neq \{0\}$.

In the following, we show that the κ-SP dimension satisfies some expected properties. In particular, we check that it is preserved under isomorphic embeddings and linear surjections.

Proposition 4.3.8. *If X, Y are Banach spaces and $X \hookrightarrow Y$ or $Y \twoheadrightarrow X$, then $sp_\kappa(X) \leq sp_\kappa(Y)$.*

Proof. If $X \hookrightarrow Y$, then by Lemma 4.3.4, we have $\mathcal{SP}^{(n)}_\kappa(X) \hookrightarrow \mathcal{SP}^{(n)}_\kappa(Y)$ for every $n \in \omega$. It follows that if $sp_\kappa(Y) \leq n$, then also $sp_\kappa(X) \leq n$.

If $Y \twoheadrightarrow X$, then again by Lemma 4.3.4, we have $\mathcal{SP}^{(n)}_\kappa(Y) \twoheadrightarrow \mathcal{SP}^{(n)}_\kappa(X)$ for every $n \in \omega$. It follows that if $sp_\kappa(X) \geq n$, then also $sp_\kappa(Y) \geq n$. △

The class of Banach spaces for which the κ-Semadeni-Pełczyński dimension is equal to zero is rather wide. In fact, it includes all realcompact Banach spaces (in the weak topology) and, by Theorem 4.3.2, all spaces $C(K)$ where every measure on the compact space K has a separable support and $t_0(K) \leq \kappa$. In the sequel, we will also use the fact that this dimension behaves well under c_0-sums.

Proposition 4.3.9. *Consider a nonempty family of Banach spaces $\{X_i : i \in I\}$ and $n \in \omega \cup \{-1\}$. We have the following.*

(1) If for all $i \in I$ we have $sp_\kappa(X_i) \leq n$, then we have $sp_\kappa(c_0(I, X_i)) \leq n$.
(2) If there is $i \in I$ such that $sp_\kappa(X_i) \geq n$, then we have $sp_\kappa(c_0(I, X_i)) \geq n$.

Proof. (1) If $sp_\kappa(X_i) \leq n$ for all $i \in I$, then by Theorem 4.3.6 and Lemma 4.3.4, we have

$$\mathcal{SP}^{(n)}_\kappa(c_0(I, X_i)) = c_0(I, \mathcal{SP}^{(n)}_\kappa(X_i)) = \{0\},$$

so $sp_\kappa(c_0(I, X_i)) \leq n$. The proof for the case of (2) is analogous. △

4.4. First applications

As a warm-up, we calculate the Semadeni-Pełczyński derivative of the space $C(K)$ for a compact line K (Theorem 4.4.3). For a cardinal θ, we denote by 2^θ the Cantor cube of appropriate size (not the cardinal exponentiation). In the latter part of this section, we use the SP derivative to obtain a partial isomorphic characterisation of spaces

$C(2^\theta \times [0,\lambda^+])$ (Theorem 4.4.8). This result was inspired by Galego, who in [41] provided (consistently) a complete isomorphic characterisation of these spaces using different methods.

4.4.1. Semadeni-Pełczyński derivative and compact lines. We describe the space of κ-continuous functions on a compact line K as the space of continuous functions on some other compact line $\mathbb{K}$. This new line can be constructed using the κ-inaccessible points of K (see Definition 4.2.3).

Definition 4.4.1. For a compact line K define its κ-continuous completion as

$$\mathbb{K} = K_L \times \{-1\} \cup K \times \{0\} \cup K_R \times \{1\},$$

considered with the lexicographic order and the order topology. It is easy to see that for any $\kappa \geq \omega$, $\mathbb{K}$ is a compact line of character $\chi(\mathbb{K}) = \chi(K)$. By π_K we denote the projection from $\mathbb{K}$ to the first coordinate.

It turns out that the space $\kappa C(K)$ is isometric to $C(\mathbb{K})$.

Lemma 4.4.2. *For any compact line K, there is an isometric isomorphism $T\colon \kappa C(K) \to C(\mathbb{K})$ that satisfies $T[C(K)] = \{f \circ \pi_K : f \in C(K)\}$*

Proof. Due to Theorem 4.3.2, we can identify the space $\kappa C(K)$ with the space of real-valued κ-continuous functions on K.

Define an operator $S\colon C(\mathbb{K}) \to \kappa C(K)$ by $Sf(k) = f(k,0)$.

Claim. Sf is a κ-continuous function.

Proof. For every subset $A \subseteq K$ of cardinality κ, any point $k \in K_L$ $[k \in K_R]$ is not in the closure of the set $A \cap (\leftarrow, k)$ $[A \cap (k, \rightarrow)]$. Hence, Sf is κ-continuous. ▲

We can also see that S is linear and has norm 1. Moreover, since $K \times \{0\}$ is dense in $\mathbb{K}$, the operator S is injective.

Claim. The operator S is onto and its inverse S^{-1} has norm 1.

Proof. Fix any κ-continuous function $g \in \kappa C(K)$. We show that g extends to a continuous function $\widetilde{g} \in C(\mathbb{K})$. By Corollary 2.3.3, it suffices to show that $\operatorname{osc}_K g(\Bbbk) = 0$ for every point $\Bbbk \in \mathbb{K}$.

Consider any point $\Bbbk = (k,b) \in \mathbb{K}$ and assume that $\operatorname{osc}_K g(\Bbbk) > 0$. If $b = 0$, then either $k \notin K^\uparrow$, $k \in K_L \cap K_R$ or $k \in K_L \triangle K_R$. In all these cases, we have $\chi(\Bbbk) \leq \kappa$, thus $\operatorname{osc}_K g(\Bbbk) = 0$, which contradicts our assumption.

Now, without loss of generality, assume that $b = -1$ (case $b = 1$ is symmetric). Then we can find two sequences of points $(x_n)_{n\in\omega}, (y_n)_{n\in\omega}$ in K such that for all $n \in \omega$ we have

$$x_n < y_n < x_{n+1} < k \text{ and } |g(x_n) - g(y_n)| > \operatorname{osc}_K g(\Bbbk).$$

By compactness and κ-inaccessibility of $\Bbbk$, both sequences $(x_n)_{n\in\omega}, (y_n)_{n\in\omega}$ converge to some point $x \in K$ below k, contradicting κ-continuity of g.

It should be clear that $S\widetilde{g} = g$ and $\|\widetilde{g}\| = \|g\|$. Note that if g is continuous, then $\widetilde{g} = g \circ \pi_K$. ▲

We may choose S^{-1} as the operator T from the statement. △

It is widely known that if $\mathbb{S}$ is a double arrow space, then

$$C(\mathbb{S})/\{f\circ\pi_{[0,1]} : f\in C([0,1])\}\cong c_0(\mathfrak{c}).$$

A similar operation can be performed on any $C(K)$ space where K is a separable compact line (as they all look very similar to $\mathbb{S}$, see 2.2.6). This topic was studied in [**14**, Section 3] and [**65**], where Michalak called such quotient operators increment-derivative operators and obtained very interesting results by studying their properties. The following lemma is an analogue of this phenomenon.

Lemma 4.4.3. *For any compact line K, we have*

$$\mathcal{SP}_\kappa(C(K))\cong c_0(K^\uparrow).$$

Proof. By Lemma 4.4.2, there is an isometric isomorphism $T\colon \kappa C(K)\to C(\mathbb{K})$, which allows us to identify $\kappa C(K)$ with $C(\mathbb{K})$ and $C(K)$ with $T[C(K)]=\{f\circ\pi_K : f\in C(K)\}$. Put $S\colon C(\mathbb{K})\to c_0(K^\uparrow)$ given for $g\in C(\mathbb{K})$ by

$$Sg(k)=\begin{cases}[g(k,0)-g(k,-1)]/2 & \text{for } k\in K_L,\\ [g(k,1)-g(k,0)]/2 & \text{for } k\in K_R.\end{cases}$$

It should be clear that S is a norm-one continuous linear operator. First, we check that $Sg\in c_0(K^\uparrow)$.

Take $g\in C(\mathbb{K})$ and assume that for some $\varepsilon>0$ there exists an infinite set $A\subseteq K^\uparrow$ such that $|Sg(a)|>\varepsilon$ for $a\in A$. Then A has an accumulation point in $\mathbb{K}$, at which the oscillation of g cannot be 0. It follows that g is not ω-continuous — and therefore not κ-continuous for $\kappa\geq\omega$.

We can see that for any $g\in C(\mathbb{K})$ we have $\|Sg\|=\frac{1}{2}\mathrm{osc}\,(g|K\times\{0\})$, so $\ker S=C(K)$. By Proposition 2.3.2, it follows that $\|[g]\|=\|Sg\|$, where $[g]$ denotes an element of

$$C(\mathbb{K})/T[C(K)]=\mathcal{SP}_\kappa(C(K)).$$

Thus, the image of S is closed in $c_0(K^\uparrow)$.

Now, let us show that the image of S contains a dense subset of $c_0(K^\uparrow)$, namely all elements with finite support, denoted $c_{00}(K^\uparrow)$. Consider the family of intervals

$$\mathcal{I}=\{((k,-1),\to)\subseteq\mathbb{K} : k\in K_L\}\cup\{[(k,1),\to)\subseteq\mathbb{K} : k\in K_R\}.$$

It is standard to check that any element of $c_{00}(K^\uparrow)$ can be represented as $S(\sum_{i=0}^m a_i\chi_{I_i})$ for some $a_i\in\mathbb{R}$ and $I_i\in\mathcal{I}$.

Wrapping up everything so far, Lemma 2.3.1 gives us a linear isometry

$$\widehat{S}\colon \mathcal{SP}_\kappa(C(K))\to c_0(K^\uparrow).$$

△

In general, the character of the topological space K is not an isomorphic invariant of the space $C(K)$ (the author is aware of an example by Koszmider, which was perhaps not published). However, by Theorem 4.4.3, the situation is different for compact lines.

Corollary 4.4.4. *If K,L are compact lines and $C(K)\simeq C(L)$, then $\chi(K)=\chi(L)$.*

4.4.2. Semadeni-Pełczyński derivative of $C(2^\theta\times[0,\lambda^+])$. Let us recall a few classical notions from the theory of large cardinal numbers. Consider an ultrafilter $\mathcal{U}$ on some set A. Then $\mathcal{U}$ is κ-complete for some cardinal κ if and only if there is no partition $A = \bigcup_{\alpha<\kappa} X_\alpha$ of A into κ many disjoint sets such that $X_\alpha \notin \mathcal{U}$ for all α.

A cardinal number κ is measurable if there exists a κ-complete nonprincipal ultrafilter on κ (in other words, a two-valued κ-additive measure). We denote the least uncountable measurable cardinal by $\mathfrak{M}$ (if it exists). Every measurable cardinal is inaccessible, so their existence is not provable in ZFC. For more details, see [45, Chapter 10].

Our interest in measurable cardinals stems from the following result by Talagrand. Note that in this context, by 2^θ we mean the Cantor cube of size θ.

THEOREM 4.4.5 (Talagrand [83]). *If $\theta < \mathfrak{M}$, then the space $C(2^\theta)$ is realcompact (i.e. $\omega C(2^\theta) = C(2^\theta)$).*

In fact, under the same assumption on θ, Talagrand's result gives that $\kappa C(2^\theta) = C(2^\theta)$ for every cardinal $\kappa \geq \omega$. A simpler proof of Theorem 4.4.5 than the original can be found in [75]. It is also a known fact that $t_0(2^\theta) = \omega$ (see, for example, [75, Theorem 3]). We now proceed to an analogue of Lemma 4.3.2.

LEMMA 4.4.6. *Consider infinite cardinal numbers κ, λ, θ. Let $K = 2^\theta \times [0, \lambda^+]$. If $\theta < \mathfrak{M}$, then $\varphi \in \kappa C(K)$ if and only if φ is represented by a κ-continuous function on K.*

PROOF. Consider any $\varphi \in \kappa C(K)$ and define a function $g_\varphi \colon K \to \mathbb{R}$ by $g_\varphi(k) = \varphi(\delta_k)$, for all $k \in K$. By the definition of $\kappa C(K)$, the function g_φ is κ-continuous. Thus, it remains to check that

$$\varphi(\mu) = \int_K g_\varphi \, \mathrm{d}\mu \tag{4.4.1}$$

for every measure $\mu \in M(K)$.

For any $\alpha < \lambda^+$, if we write $K_\alpha = 2^\theta \times \{\alpha\}$, then the space $C(K_\alpha)$ embeds into $C(K)$. It follows from Theorem 4.4.5 that (1) holds for measures supported in K_α. Note that, since $[0, \lambda^+]$ is scattered, measures concentrated on $\bigcup_{\alpha \in I} K_\alpha$ for a finite set $I \subseteq \lambda^+$ form a norm-dense set in $M(K)$, from which we can conclude (1) in the general case.

For the converse, let g be a real-valued κ-continuous function on K. The support of every measure $\mu \in M(K)$ is contained in a countable union of K_α's, so for κ many measures, g is continuous on their support. It follows that φ given by (1) defines an element of $\kappa C(K)$. △

We can now calculate the SP derivative of the space $C(2^\theta \times [0, \lambda^+])$.

LEMMA 4.4.7. *Consider infinite cardinal numbers θ, λ. If $\theta < \mathfrak{M}$, then*

$$\mathcal{SP}_\lambda(C(2^\theta \times [0, \lambda^+])) \cong C(2^\theta)$$

and

$$\mathcal{SP}_\kappa(C(2^\theta \times [0, \lambda^+])) = 0,$$

for every cardinal $\kappa > \lambda$.

PROOF. Consider any $\varphi \in \kappa C(2^\theta \times [0, \lambda^+])$ and define a function $g_\varphi \colon 2^\theta \times [0, \lambda^+] \to \mathbb{R}$ by $g_\varphi(x, \alpha) = \varphi(\delta_{(x,\alpha)})$, for all $x \in 2^\theta, \alpha < \lambda^+$. By Lemma 4.4.6, the function g_φ is

κ-continuous. We know that $t_0(2^\theta \times [0,\lambda^+]) = \lambda^+$, so for $\kappa > \lambda$ it follows that g_φ is continuous. Thus, we have $\mathcal{SP}_\kappa(C(2^\theta \times [0,\lambda^+])) = 0$. From this point on, assume $\kappa = \lambda$.

We know that $t_0(2^\theta \times [0,\lambda^+)) = \lambda$, so the function $g^0_\varphi = g_\varphi | 2^\theta \times [0,\lambda^+)$ is continuous. By Corollary 4.2.5, we have

$$\beta\left[2^\theta \times [0,\lambda^+)\right] = 2^\theta \times [0,\lambda^+],$$

so let h_φ be the unique continuous extension of g^0_φ to $2^\theta \times [0,\lambda^+]$.

Define an operator $S\colon \lambda C(2^\theta \times [0,\lambda^+]) \to C(2^\theta)$ by the formula

$$S\varphi(x) = \left[g_\varphi(x,\lambda^+) - h_\varphi(x,\lambda^+)\right]/2,$$

for $x \in 2^\theta$. It is immediate that S is continuous and linear. Moreover, by Lemma 4.4.6, we have $\ker S = C(2^\theta \times [0,\lambda^+])$, as $\varphi \in \ker S$ if and only if $g_\varphi = h_\varphi$, so when g_φ is continuous.

By Proposition 2.3.2, we have $\|[\varphi]\| = \|S\varphi\|$ for $[\varphi] \in \mathcal{SP}_\lambda(C(2^\theta \times [0,\lambda^+]))$. We can also see that S is a surjection, as for every $f \in C(2^\theta)$ if we put

$$\widetilde{f}(x,\alpha) = \begin{cases} 2f(x) & \alpha = \lambda^+, \\ 0 & \alpha < \lambda^+, \end{cases}$$

then $S\widetilde{f} = f$. By Lemma 2.3.1, there is a linear isometry between $\mathcal{SP}_\lambda(C(2^\theta \times [0,\lambda^+]))$ and $C(2^\theta)$. △

THEOREM 4.4.8. *For any infinite cardinal numbers $\theta,\theta',\lambda,\lambda'$, if θ and θ' are below the first measurable cardinal, then the spaces $C(2^\theta \times [0,\lambda^+])$ and $C(2^{\theta'} \times [0,\lambda'^+])$ are isomorphic if and only if $\lambda = \lambda'$ and $\theta = \theta'$.*

PROOF. If $C(2^\theta \times [0,\lambda^+]) \simeq C(2^{\theta'} \times [0,\lambda'^+])$, then due to Lemma 4.3.4, we also have

$$\mathcal{SP}_\kappa\Big(C(2^\theta \times [0,\lambda^+])\Big) \simeq \mathcal{SP}_\kappa\Big(C(2^{\theta'} \times [0,\lambda'^+])\Big)$$

for every cardinal $\kappa \geq \omega$. Without loss of generality $\lambda \geq \lambda'$. Now we have two options; in both of them we apply Lemma 4.4.7 with $\theta = \lambda$ to obtain the result. Either

- $\lambda > \lambda'$, then $C(2^\theta) \simeq 0$, which is a contradiction, or
- $\lambda = \lambda'$, then $C(2^\theta) \simeq C(2^{\theta'})$, which is possible only for $\theta = \theta'$.

△

4.5. Calculating Semadeni-Pełczyński dimension

It will be convenient to introduce some notation and properly define classes of objects of interest.

DEFINITION 4.5.1. For $n \in \omega$ define a class $\mathcal{C}^n_\kappa$ of compact spaces K satisfying the following

- $K = \prod_{i=1}^m K_i$ for compact lines $K_1,\ldots,K_m$ and
- $\chi(K_i) > \kappa$ if and only if $i \leq n$.

In other words, $\mathcal{C}^n_\kappa$ is the class of all finite products of compact lines with exactly n factors of character greater than κ (maybe up to the permutation of axes). For technical reasons, let $\mathcal{C}^{-1}_\kappa$ be the singleton of the empty set.

If $K \in \mathcal{C}_\kappa^n$ for some n, then denote $\mathbb{K} = \prod_{i=1}^m \mathbb{K}_i$, where each $\mathbb{K}_i$ is the κ-continuous completion of K_i; see Definition 4.4.1. Let $\pi_K\colon \mathbb{K} \to K$ denote the projection given by $\pi_K((k_i, b_i)_{i=1}^m) = (k_i)_{i=1}^m$. For $i \leq m$, we will also write $K(i) = \prod_{j=1, j\neq i}^m K_j$ and $\mathbb{K}(i) = \prod_{j=1,j\neq i}^m \mathbb{K}_j$.

The main result of this section is the following theorem, which will be proven later, after all the necessary lemmas.

THEOREM 4.5.2. *Consider any compact space $K \in \mathcal{C}_\kappa^n$ for some $n \in \omega$. Then*

$$sp_\kappa(C(K)) = n.$$

Note that if $K \in \mathcal{C}_\kappa^n$ and $L \in \mathcal{C}_\kappa^m$, then $K \times L \in \mathcal{C}_\kappa^{n+m}$. Using Proposition 4.3.8, we can easily deduce the following.

COROLLARY 4.5.3. *Let $K \in \mathcal{C}_\kappa^n, L \in \mathcal{C}_\kappa^m$ for $n > m$. Then*

$$C(K) \not\hookrightarrow C(L) \text{ and } C(L) \not\twoheadrightarrow C(K).$$

In particular, for compact lines of uncountable character $K_1, \ldots, K_n, L_1, \ldots, L_m$, if $n > m$, then $C(\prod_{i=1}^n K_i) \not\hookrightarrow C(\prod_{j=1}^m L_j)$ and $C(\prod_{j=1}^m L_j) \not\twoheadrightarrow C(\prod_{i=1}^n K_i)$.

To prove Theorem 4.5.2, we need a few lemmas about the structure of $\kappa C(K)$ and $\mathcal{SP}_\kappa(C(K))$. We begin with one technical lemma about the extension of functions to κ-continuous completion. Recall that all necessary notation was introduced in Definition 4.5.1.

LEMMA 4.5.4. *Consider a product of compact lines $K = K_1 \times \ldots \times K_n$. Then every κ-continuous function $g\colon K \to \mathbb{R}$ has a unique extension to a continuous function $\widetilde{g}\colon \mathbb{K} \to \mathbb{R}$.*

PROOF. Fix any κ-continuous function $g\colon K \to \mathbb{R}$. By Corollary 2.3.3, it suffices to show that $\operatorname{osc}_K g(\Bbbk) = 0$ for every point $\Bbbk \in \mathbb{K}$. Take any point $\Bbbk = (k_i, b_i)_{i\leq n} \in \mathbb{K}$ and assume that $\operatorname{osc}_K g(\Bbbk) > 0$. Denote by $A_\Bbbk$ the set of coordinates $i \leq n$ such that $k_i \in K_i^\uparrow$ and by $B_\Bbbk$ the rest of the coordinates. Without loss of generality, assume that $k_i \in K_{iL}$ for every $i \in A_\Bbbk$.

Since every point on a compact line has a descending neighbourhood base, we can construct two sequences of points $(x_\alpha)_{\alpha<\kappa}, (y_\alpha)_{\alpha<\kappa}$ in K such that:

- $x_\alpha(i) < y_\beta(i)$ for $i \in A_\Bbbk$ and $\alpha \leq \beta < \kappa$,
- $y_\alpha(i) < x_\beta(i)$ for $i \in A_\Bbbk$ and $\alpha < \beta < \kappa$,
- $(x_\alpha(j))_{\alpha<\kappa}, (y_\alpha(j))_{\alpha<\kappa}$ converge both to $x_j = k_j$ for $j \in B_\Bbbk$.

Then, for every $i \in A_\Bbbk$, the sequences $(x_\alpha(i))_{\alpha<\kappa}, (y_\alpha(i))_{\alpha<\kappa}$ have a common supremum $x_i \in K_i$ below k_i. It follows that the function g is not continuous on the set $\{x_\alpha, y_\alpha : \alpha < \kappa\} \cup \{(x_i)_{i\leq n}\}$. △

Now we can proceed to the product version of Lemma 4.4.2.

LEMMA 4.5.5. *For any compact lines $K_1, \ldots, K_n$, there is an isometric isomorphism $T\colon \kappa C(K) \to C(\mathbb{K})$ that satisfies $T[C(K)] = \{f \circ \pi_K : f \in C(K)\}$.*

PROOF. Due to Theorem 4.3.2, we can identify the space $\kappa C(K)$ with the space of real-valued κ-continuous functions on K. Let us denote $\widetilde{K} = \prod_{i=1}^m (K_i \times \{0\}) \subseteq \mathbb{K}$.

Define an operator $S\colon C(\mathbb{K}) \to \kappa C(K)$ by $Sf = f|\widetilde{K}$.

Claim. Sf is a κ-continuous function.

Proof. Let $A \subseteq K$ be a subset of cardinality κ. For every coordinate $i \leq n$, any point $k \in K_{iL}$ $\left[k \in K_{iR}\right]$ is not in the closure of the set $\pi_i[A] \cap (\leftarrow, k)$ $\left[\pi_i[A] \cap (k, \rightarrow)\right]$, from which it follows that Sf is κ-continuous. ▲

We can also see that S is linear, has norm 1 and since $\widetilde{K}$ is dense in $\mathbb{K}$, the operator S is injective.

Claim. The operator S is onto and its inverse S^{-1} has norm 1.

Proof. Fix any κ-continuous function $g \in \kappa C(K)$. By Lemma 4.5.4, there is a continuous extension $\widetilde{g}$ of g on $\mathbb{K}$.

It should be clear that $S\widetilde{g} = g$ and $\|\widetilde{g}\| = \|g\|$. Note that if g is continuous, then $\widetilde{g} = g \circ \pi_K$. ▲

Thus, we can choose S^{-1} as the operator T from the statement. △

After establishing a characterisation of $\kappa C(K)$, we proceed to what is arguably the most difficult result of this chapter, from which Theorem 4.5.2 follows easily.

Lemma 4.5.6. *Let $K \in \mathcal{C}^n_\kappa$ for some $n \in \omega$. Then the space $\mathcal{SP}_\kappa(C(K))$ is isomorphic to a c_0-sum of spaces from the class $\mathcal{C}^{n-1}_\kappa$.*

Proof. By Lemma 4.5.5, there exists an isometry $T\colon \kappa C(K) \to C(\mathbb{K})$ that allows us to identify $\kappa C(K)$ with $C(\mathbb{K})$ and $C(K)$ with $T[C(K)] = \{f \circ \pi_K : f \in C(K)\}$. We denote $\widetilde{K} = \prod_{i=1}^m (K_i \times \{0\}) \subseteq \mathbb{K}$.

Let us introduce one more piece of notation. For any $(k, b) \in \mathbb{K}_i$, let

$$J(k,b) = \prod_{j=1}^{i-1} \mathbb{K}_j \times (k,b) \times \prod_{j=i+1}^{m} \mathbb{K}_j.$$

Now define an operator $S\colon C(\mathbb{K}) \to \prod_{i=1}^n c_0\Big(K_i^{\uparrow}, C(\mathbb{K}(i))\Big)$, for $g \in C(\mathbb{K})$, by

$$Sg(i)(k) = \begin{cases} \left[g|J_i(k,0) - g|J_i(k,-1)\right]/2 & \text{for } k \in K_{iL}, \\ \left[g|J_i(k,1) - g|J_i(k,0)\right]/2 & \text{for } k \in K_{iR}. \end{cases}$$

It is standard to check that S is a continuous linear operator. We will prove that S is well-defined and $\ker S = C(K)$.

Fix $i \leq n$ and any function $g \in C(\mathbb{K})$. Suppose that for some $\varepsilon > 0$, there is a countable infinite set $A \subseteq K_i^{\uparrow}$ such that $\|Sg(i)(a)\| > \varepsilon$ for $a \in A$. Then A has an accumulation point in K_i at which the oscillation of g cannot be 0, so g is not ω-continuous (and also κ-continuous for $\kappa \geq \omega$).

Take any $g \in C(\mathbb{K})$ and let

$$H = \{(k_i)_{i \leq n} \in K : \exists i \leq n \; k_i \in K_i^{\uparrow}\}$$

be the set of points in K with at least one κ-inaccessible coordinate. It should be clear that

$$\operatorname{osc} g|\widetilde{K} = \sup_{k \in H} \max_{\Bbbk_1, \Bbbk_2 \in \mathbb{K}} \{f(\Bbbk_1) - f(\Bbbk_2) : \pi_K(\Bbbk_1) = \pi_K(\Bbbk_2) = k\},$$

as outside the copy of H in $\widetilde{K}$, the function $g|\widetilde{K}$ is continuous. Consider any $k \in H$. For some $d \leq 2n$ we can find a sequence of points $\Bbbk_{j\leq d}$ in $\mathbb{K}$ such that, for each $j \leq d$, the value $|f(\Bbbk_j) - f(\Bbbk_{j+1})|/2$ is, up to a sign, obtained by Sg and the oscillation of $g|\widetilde{K}$ at k is equal to $f(\Bbbk_1) - f(\Bbbk_d)$. It follows that $\|Sg\| \geq \frac{1}{4n}\operatorname{osc} g|\widetilde{K}$, so $\ker S$ contains only those functions $g \in C(\mathbb{K})$ for which the oscillation of $g|\widetilde{K}$ is 0. It is also clear that for any $g \in C(K)$, we have $Sg = 0$, thus $\ker S = C(K)$.

By Proposition 2.3.2, if for $g \in C(\mathbb{K})$ we consider $[g]$ as an element of

$$C(\mathbb{K})/T[C(K)] \cong \mathcal{SP}_\kappa(C(K)),$$

then $\|[g]\| = \frac{1}{2}\operatorname{osc} g|\widetilde{K}$. We have already shown that

$$\|Sg\| \geq \frac{1}{4n}\operatorname{osc} g|\widetilde{K} = \frac{1}{2n}\|[g]\|,$$

which implies that the image of S is closed in $c_0(K^\uparrow, C(\mathbb{K}(i)))$.

Now we will show that the image of S is dense in its codomain. By linearity, it is enough to show that for each $i \leq n$, the image of S is dense in $c_0(K_i^\uparrow, C(\mathbb{K}(i)))$. Our goal is to show that all elements with only finitely many non-zero coordinates, denoted $c_{00}(K_i^\uparrow, C(\mathbb{K}(i)))$, lie in the image of S. For $k \in K_L$ consider a clopen rectangle

$$P_k^i = \{(k_i, b_i)_{i\leq n} \in \mathbb{K} : (k_i, b_i) > (k, -1)\}$$

and for $k \in K_R$ a clopen rectangle

$$R_k^i = \{(k_i, b_i)_{i\leq n} \in \mathbb{K} : (k_i, b_i) > (k, 0)\},$$

then by $\mathcal{I}_i$ denote the family of all these rectangles, i.e.

$$\mathcal{I}_i = \{P_k^i : k \in K_L\} \cup \{R_k^i : k \in K_R\}.$$

It is standard to verify that any element of $c_{00}(K_i^\uparrow, C(\mathbb{K}(i)))$ can be represented as $S(\sum_{j=0}^m f_j \chi_{I_j})$, for some $I_j \in \mathcal{I}_i$ and $f_j \in C(\mathbb{K}(i))$.

Wrapping up everything so far, Lemma 2.3.1 gives us an isomorphism

$$\widehat{S} \colon \mathcal{SP}_\kappa(C(K)) \to \prod_{i=1}^{n} c_0(K_i^\uparrow, C(\mathbb{K}(i))).$$

△

We now have all the tools necessary to calculate the SP dimension for spaces $C(K)$, where K is a finite product of compact lines.

Proof. (of Theorem 4.5.2) We prove the statement by induction on n.

By Theorem 4.3.2, for $K \in \mathcal{C}_\kappa^0$, we have $\kappa C(K) = C(K)$, so $sp_\kappa(C(K)) = 0$.

Now assume that for some $n \geq 1$ and every $K \in \mathcal{C}_\kappa^{n-1}$ we have $sp_\kappa(C(K)) = n - 1$. Consider any space $K \in \mathcal{C}_\kappa^n$. Then there are some compact lines $K_1, \ldots, K_m$ such that $K = \prod_{i=1}^m K_i$, and $\chi(K_i) > \kappa$ if and only if $i \leq n$.

By Lemma 4.5.6, we have $\mathcal{SP}_\kappa(C(K)) \simeq c_0(I, C(L_i))$ for some set I and spaces $L_i \in \mathcal{C}_\kappa^{n-1}$. From Proposition 4.3.9 it follows that

$$sp_\kappa\left[c_0(I, C(L_i))\right] = \sup\{sp_\kappa\left(C(L_i)\right) : i \in I\}.$$

By the induction hypothesis $sp_\kappa(C(L_i)) = n-1$ and thus

$$sp_\kappa(C(K)) = 1 + sp_\kappa(\mathcal{SP}_\kappa(C(K))) = 1 + \sup\{sp_\kappa(C(L_i)) : i \in I\} = 1 + n - 1 = n.$$

△

4.6. Remarks and problems

In this section, we present a collection of scattered thoughts, remarks and questions broadly related to the subject of the chapter. All results follow from already known facts, so the proofs should be straightforward and may get quite sketchy. Moreover, some of the remarks made here might be known in the broader community.

4.6.1. Remarks on the double arrow space. Recall that by $\mathbb{S}$ we denote the double arrow space. In what follows, we present a few remarks on isomorphisms between spaces of continuous functions on $\mathbb{S}$ and its products.

Proposition 4.6.1. $C(\mathbb{S}) \simeq C(\mathbb{S} \times [0,\omega])$.

Proof. The space $\mathbb{S}$ contains nontrivial convergent sequences, which implies that the space $C(\mathbb{S})$ is isomorphic to its hyperspaces. Denote $I_n = [(1/(n+1),1),(1/n,0)] \subseteq \mathbb{S}$ and let φ_n be the homeomorphism between $\mathbb{S}$ and I_n. Then the formula

$$Tf = ((f|I_n) \circ \varphi_n)_{n\in\omega}$$

defines an isometry between a hyperplane $C'(\mathbb{S}) = \{f \in C(\mathbb{S}) : f(0,1) = 0\}$ and the space $c_0(\omega, C(\mathbb{S}))$. It follows that

$$C(\mathbb{S}) \simeq C'(\mathbb{S}) \cong c_0(C(\mathbb{S})) \simeq C(\mathbb{S} \times [0,\omega]).$$

△

By Bessaga and Pełczyński's characterisation of isomorphisms of spaces of continuous functions on countable compacta [12], it follows that $C([0,\omega]) \not\simeq C([0,\alpha])$ for any ordinal $\omega^\omega \leq \alpha < \omega_1$. This raises the following question.

Question 1. *Is the space $C(\mathbb{S})$ isomorphic to $C(\mathbb{S} \times [0,\alpha])$ for $\omega^\omega \leq \alpha < \omega_1$?*

However, it is interesting to note that Proposition 4.6.1 does not hold if $[0,\omega]$ is replaced by any nonscattered compact space. Note that the proof of the following result is similar to that of Lemma 4.5.6.

Proposition 4.6.2. *If K is a compact space such that $C(K)$ does not embed into $c_0(\mathfrak{c})$, then $C(\mathbb{S}) \not\simeq C(\mathbb{S} \times K)$.*

Proof. Assume that $T\colon C(\mathbb{S}) \to C(\mathbb{S} \times K)$ is an isomorphism. There is a natural copy of $C([0,1])$ embedded in $C(\mathbb{S})$ (of functions constant on the second coordinate). Let $Y = T[C([0,1])]$; this is a separable subspace of $C(\mathbb{S} \times K)$. It is well known that the quotient $C(\mathbb{S})/C([0,1])$ is isomorphic to $c_0(\mathfrak{c})$; see [14, Section 3].

Let us show that there exists an isomorphic copy of $C(K)$ inside $C(\mathbb{S}\times K)/Y$. Denote by D a countable dense subset of Y. For each $f \in C(\mathbb{S} \times K)$, define the section function $f_s(k) = f(s,k)$ for $s \in \mathbb{S}, k \in K$ and put

$$A_x = \{f \in D : f_{(x,0)} \neq f_{(x,1)}\}$$

for $x \in [0,1]$.

Since D is countable, we can fix some $x \in [0,1]$ such that $A_x = \emptyset$. Then for all $f \in Y$, we have $f_{(x,0)} = f_{(x,1)}$. Define

$$\widetilde{f}(s,k) = \chi_{[(x,1),\to)}(s) \cdot f(k)$$

for $f \in C(K)$, $s \in \mathbb{S}$ and $k \in K$. Then the subspace $C_x = \{\widetilde{f} : f \in C(K)\}$ defines an isomorphic copy of $C(K)$ inside $C(\mathbb{S} \times K)/Y$, as for every $f \in C_x$ we have $\operatorname{dist}(f, Y) \geq \|f\|/2$. This contradicts the assumption that $C(K) \not\hookrightarrow c_0(\mathfrak{c})$. $\triangle$

Let K be any compact space. Recall that $C(K)$ is an Asplund space if and only if K is scattered, and that every closed subspace of an Asplund space is Asplund. It follows from Proposition 4.6.2 that for any nonscattered compact space K we have $C(\mathbb{S}) \not\simeq C(\mathbb{S} \times K)$, since $c_0(\mathfrak{c})$ is an Asplund space.

In particular, K can be any product of nonmetrizable separable compact lines or the unit interval. Thus, we have obtained an easy way to see some of the results of Michalak from [**65**].

Corollary 4.6.3. $C(\mathbb{S}) \not\simeq C(\mathbb{S} \times [0,1])$.

Corollary 4.6.4. *If K is any product of nonmetrizable separable compact lines, then $C(\mathbb{S}) \not\simeq C(\mathbb{S} \times K)$.*

A closer inspection of the proof of Proposition 4.6.2 shows that, instead of $\mathbb{S}$, we can put any separable compact line of weight greater than ω to obtain a similar result.

4.6.2. Remarks on the Suslin lines. Using classical results, we can prove one simple observation about the isomorphic structure of spaces of continuous functions on the products of Suslin lines. The first thing to note is the following classical result by Kurepa.

Theorem 4.6.5 (Kurepa [**57**]). *The square of a Suslin line is not ccc.*

More details on chain conditions can be found in the survey by Todorčević [**84**]. It is worth noting that the ccc property of a compact space translates to an isomorphic property of its space of continuous functions.

Theorem 4.6.6 (Rosenthal [**78**]). *A compact space K is ccc if and only if every weakly compact subset of $C(K)$ is separable.*

Combining the theorems of Kurepa and Rosenthal, we easily get the following.

Corollary 4.6.7. *Let $\mathcal{S}$ be a Suslin line. Then $C(\mathcal{S}) \not\simeq C(\mathcal{S}^n)$.*

It is worth mentioning that the above reasoning does not need to hold for the product of two distinct Suslin lines, as such a product can be ccc, according to the results of Jensen (as claimed by, e.g., Rudin in [**79**]).

CHAPTER 5

Banach-Mazur distance

5.1. Introduction

Given two isomorphic Banach spaces X and Y, the Banach–Mazur distance $d_{\mathrm{BM}}(X,Y)$ between them is defined as the infimum of the distortions $\|T\|\cdot\|T^{-1}\|$ taken over all isomorphisms $T\colon X\to Y$. Our goal is to calculate the values of the Banach–Mazur distance between certain $C(K)$-spaces and to understand the meaning of these numbers.

Recall that if we consider two compacta K and L, the inequality $d_{\mathrm{BM}}\big(C(K),C(L)\big)<2$ implies that K and L are homeomorphic and, consequently, that $C(K)$ and $C(L)$ are isometric. This result was independently proved by Amir [**3**] and Cambern [**18**]. The threshold 2 is sharp — see Cohen and Chu [**26**] for a discussion of this phenomenon.

We divide this chapter into two main parts: the case where both compact spaces are countable (section 5.3) and where they are not (section 5.4). The countable case is conceptually simpler, but the bounds are much tighter, thus the methods involved tend to be more complicated and require computer assistance.

In their classical paper [**12**] on the isomorphic classification of Banach spaces of the form $C(K)$ for countable compact spaces K, Bessaga and Pełczyński asked whether it is possible to calculate the exact Banach–Mazur distance between these spaces. Below, we consider the space $[0,\omega]$, the simplest infinite compactum consisting of a convergent sequence together with its limit; clearly, $C[0,\omega]$ represents the classical Banach space of convergent sequences. Recall that $C(K)$ is isomorphic to $C[0,\omega]$ if and only if K is a scattered compact space of finite height.

There are few pairs of compact spaces K and L for which the Banach–Mazur distance $d_{\mathrm{BM}}\big(C(K),C(L)\big)$ has been determined. A remarkable exception is provided by the following clean result.

THEOREM 5.1.1. *The formula*

$$d_{\mathrm{BM}}\big(C([0,\omega]^m),C([0,\omega])\big)=m+\sqrt{(m-1)(m+3)},$$

holds for every $m\geq 1$.

Here, the upper bound was given by Candido and Galego [**21**, Corollary 1.3 and Theorem 1.4(b)], while Malec and Piasecki [**59**] recently obtained the lower estimate. In a recent preprint on arXiv, Cuth, Havelka, Rondoš and Sari [**32**] generalised this result and showed several results on the positive version of the Banach–Mazur distance.

Our main contribution to this topic is the introduction of a relatively simple idea that yields lower bounds for $d_{\mathrm{BM}}\big(C(K),C(L)\big)$ for some pairs of K,L — this is described in Lemma 5.3.2. It is a refinement of methods developed by Gordon [**43**] and Gergont and Piasecki [**42**]. As an application, we obtain substantially shorter arguments that lead to their results. Actually, our method gives estimates for the lower bound of Theorem 5.1.1 in a more general setting; see Theorems 5.3.3 and 5.3.4 for details.

The rest of section 5.3 is devoted to the study of $d_{\mathrm{BM}}\big(C(K), C[0,\omega]\big)$ where $K = [0,\omega] \times k$ for a natural number k. We use the convention that $k = \{0, 1, \dots, k-1\}$, so that the space K consists of k copies of a convergent sequence. At first glance, this case may seem rather innocent. Moreover, Gordon [**43**] proved that

$$d_{\mathrm{BM}}\big(C([0,\omega] \times 2), C[0,\omega]\big) = 3.$$

However, when passing from two to three copies of $[0,\omega]$, the problem seems much harder. Gergont and Piasecki [**42**] proved that

$$3.23 < d_{\mathrm{BM}}\big(C([0,\omega] \times 3), C[0,\omega]\big) < 3.89$$

using an involved argument and computer-assisted calculations. Using Lemma 5.3.2 again, we show that the lower bound here is at least 3.47. However, our method requires solving several systems of linear inequalities, for which computer assistance again proves indispensable. The code used for these computations is stored on GitHub at

`https://github.com/ememak/Bounds-for-Banach-Mazur-distance`.

In section 5.3.4 we also give a slightly improved upper bound of around 3.875, which may, in fact, be optimal.

In section 5.4, we are mostly interested in the Banach–Mazur distance between the classical Banach spaces $L_\infty[0,1]$ and ℓ_∞. These spaces are isomorphic, as demonstrated by Pełczyński [**71**] by these breath-taking lines:

$$L_\infty \simeq \ell_\infty \oplus A \cong \ell_\infty \oplus \ell_\infty \oplus A \simeq \ell_\infty \oplus L_\infty, \tag{5.1.1}$$

$$\ell_\infty \simeq L_\infty \oplus B \cong L_\infty \oplus L_\infty \oplus B \simeq L_\infty \oplus \ell_\infty. \tag{5.1.2}$$

Here, $L_\infty = L_\infty[0,1]$ and A, B denote suitable Banach spaces. Note that the space ℓ_∞ is isometric to $C(\beta\omega)$, the space of continuous functions on the Stone–Čech compactification of the discrete space ω of natural numbers; see Semadeni [**81**, Chapter IV]. In turn, $L_\infty[0,1]$ can be isometrically represented as the space $C(K)$, where K is the Stone space of the measure algebra of the Lebesgue measure; here again Semadeni [**81**] and Fremlin [**39**] serve as standard references.

It seems that little is known about the Banach–Mazur distance of these spaces; see e.g., [**33**, page 131]. For instance, if we follow directly the Pełczyński decomposition method (and equip direct sums with the max-norm) then we get only

$$d_{\mathrm{BM}}(L_\infty[0,1], \ell_\infty \oplus L_\infty[0,1]) \le 9, \quad d_{\mathrm{BM}}(\ell_\infty, \ell_\infty \oplus L_\infty[0,1]) \le 9,$$

so $d_{\mathrm{BM}}(L_\infty[0,1], \ell_\infty) \le 81$, since the Banach–Mazur distance is multiplicative.

Our first result, Theorem 5.4.1, is quite general: if K is a zero-dimensional compact space without isolated points and L is any compactification of the set of natural numbers, then $d_{\mathrm{BM}}\big(C(K), C(L)\big) \ge 3 + 2\sqrt{2}$. This result is of particular interest, as it connects the topological properties of the spaces K, L with the Banach–Mazur distance between the corresponding spaces $C(K)$ and $C(L)$.

Our method is developed in the next section, where we prove that $d_{\mathrm{BM}}(L_\infty[0,1],\ell_\infty) > 7.41$. The exact form of the constant is somewhat overwhelming; see (5.4.3). However, it is worth noting that the appearance of algebraic numbers in such estimates is unavoidable.

The following upper bound is established in Section 5.4.3:

$$d_{\mathrm{BM}}\big(L_\infty[0,1],\ell_\infty\big) \le (3+\sqrt{2})^2 < 19.49.$$

This bound is obtained by a detailed analysis of the decomposition method and does not seem optimal. It is unclear whether this bound can be significantly improved by further studying the decomposition method.

We also apply our methods to the separable case, obtaining some bounds for the Banach–Mazur distance between the spaces $C(2^\omega)$ and $C(L)$, where L is the Pełczyński compactum consisting of the Cantor space 2^ω and a dense set of isolated points.

The gap between the lower and upper bounds obtained here remains quite large. Although the precise numerical values of these constants are not crucial from the broader perspective of the isomorphic theory of Banach spaces, obtaining meaningful lower bounds for Banach–Mazur distances seems to require a deeper understanding of the structure of isomorphisms between the spaces in question.

It is also worth mentioning that ℓ_∞ and $L_\infty[0,1]$ may not be isomorphic in certain models of set theory without the axiom of choice; see Väth [85]. One can ask whether it is consistent with ZF that, for example, $100 < d_{\mathrm{BM}}(\ell_\infty, L_\infty[0,1]) < \infty$ and what are the consequences of this fact?

5.2. Preliminaries

An operator $T\colon X \to Y$ between Banach spaces is said to be *norm-increasing* if $\|x\| \le \|Tx\|$ for every $x \in X$. Note that any isomorphism T can be made norm-increasing by multiplying it by the constant $\|T^{-1}\|$. In this chapter, we usually consider a norm-increasing isomorphism T and write $t = \|T\|$ for the norm of T.

Remark 5.2.1. Given two isomorphic Banach spaces X and Y, their Banach–Mazur distance $d_{\mathrm{BM}}(X,Y)$ equals the infimum of $\|T\|$, taken over all norm-increasing isomorphisms T from X onto Y.

Throughout, K and L denote compact spaces. Suppose that $T\colon C(K) \to C(L)$ is a norm-increasing isomorphism. For each $y \in L$ we denote by ν_y the signed measure on K defined for $g \in C(K)$ by $\nu_y(g) = Tg(y)$; in other words, $\nu_y = T^*\delta_y$. Our starting point can be described as follows: when seeking information on the value of $\|T\|$, it suffices to study the supremum of the norms of the measures $\nu_y \in C(K)^*$. In this setting, we note the following.

Lemma 5.2.2. *Measures ν_y for $y \in L$ form a 1-norming subset of $M(K)$. Moreover, for every $h \in C(L)$, there exists $\varphi \in C(K)$ such that $\nu_y(\varphi) = h(y)$ for every $y \in L$.*

Proof. If $g \in C(K)$ and $\|g\| = 1$, then $\|Tg\| \ge \|g\| = 1$, so there is $y \in L$ such that $|\nu_y(g)| = |Tg(y)| \ge 1$. This shows that $\{\nu_y : y \in L\}$ is a 1-norming set.

For any $h \in C(L)$, there is $\varphi \in C(K)$ satisfying $T\varphi = h$. Then $\nu_y(\varphi) = T\varphi(y) = h(y)$ for every $y \in L$, as required. △

Later, it will be convenient to collect the following facts concerning the variation of a signed measure.

LEMMA 5.2.3. *Let K be any compact space and $\mu_i, \mu \in M(K)$.*

(a) If $\mu_i \to \mu$ in the w^* topology, then $|\mu|(C) \leq \liminf_i |\mu_i|(C)$ for every clopen set $C \subseteq K$.

(b) If $\|\mu\| \leq t$, h is a norm-one measurable function and $B \subseteq A$ are two measurable sets, then $|\mu(h\chi_A)| \leq c$ implies $|\mu(h\chi_B)| \leq (t+c)/2$.

PROOF. For (a), take any $\varepsilon > 0$ and a continuous function $g\colon C \to [-1,1]$ such that $\mu(g) > |\mu|(C) - \varepsilon$. Then

$$|\mu|(C) < \mu(g) + \varepsilon = \lim_i \mu_i(g) + \varepsilon \leq \liminf_i |\mu_i|(|g|) + \varepsilon \leq \liminf_i |\mu_i|(C) + \varepsilon.$$

For part (b) note that

$$-c \leq \mu(h\chi_A) = \mu(h\chi_B) + \mu(h\chi_{A\setminus B}) \leq c, \text{ and}$$
$$-t \leq \mu(h\chi_B) - \mu(h\chi_{A\setminus B}) \leq t,$$

so $-t - c \leq 2\mu(h\chi_B) \leq t + c$, as required. △

The lemma below is technical, but will be the key tool for establishing our lower bounds. The statement is general, as it will be applied later not only to regular Borel measures on a compact space, but also to finitely additive measures on $\mathrm{Bor}[0,1]$ (the norm $\|\cdot\|$ discussed here is the supremum norm).

LEMMA 5.2.4. *Let ν be a finitely additive signed measure on an algebra $\mathcal{A}$ of subsets of some space K. Suppose also that we are given $A \in \mathcal{A}$ and $\mathcal{A}$-measurable functions φ, ψ on K.*

If $r, t, \varepsilon > 0$ are some constants such that:

(i) $|\nu|(K) \leq t$,

(ii) $\varphi(x) \geq r - \varepsilon$ for every $x \in A$,

(iii) $\big\|\sigma_1 \cdot \varphi + \sigma_2 \cdot \psi\big\| \leq r$ for all $\sigma_1, \sigma_2 \in \{-1, 0, 1\}$,

then $|\nu|(K) \geq 2|\nu(A)| - \nu(\varphi) + (1/r) \cdot |\nu(\psi)| - (2t/r)\varepsilon$.

PROOF. We can suppose that the values of $\nu(A)$ and $\nu(\psi)$ are positive (consider $-\nu$ or $-\psi$ otherwise). Denote

$$\varphi_0 = \varphi \cdot \chi_A, \varphi_1 = \varphi - \varphi_0, \quad \psi_0 = \psi \cdot \chi_A, \psi_1 = \psi - \psi_0.$$

Note that, by (iii), we have $\|\psi_0\| \leq \varepsilon$. Therefore

$$\nu(\psi_1) = \nu(\psi_0 + \psi_1) - \nu(\psi_0) \geq 1 - t\varepsilon. \tag{5.2.1}$$

Similarly, since $\|\varphi_0 - r \cdot \chi_A\| \leq \varepsilon$, we obtain $\big|\nu(\varphi_0) - r\nu(A)\big| \leq t\varepsilon$ and hence

$$\nu(\varphi_0) \geq r\nu(A) - t\varepsilon.$$

It follows that

$$-\nu(\varphi_1) = \nu(\varphi_0) - \nu(\varphi) \geq r\nu(A) - t\varepsilon - \nu(\varphi). \tag{5.2.2}$$

Using (5.2.1) and (5.2.2) we arrive at

$$|\nu|(K \setminus A) \geq (1/r) \cdot \nu(\psi_1 - \varphi_1) \geq \nu(A) - (t\varepsilon)/r - \nu(\varphi) + (1/r) \cdot \nu(\psi_1).$$

Since $\nu(\psi) \geq \nu(\psi_1) - t\varepsilon$, we finally get

$$|\nu|(K) = |\nu|(A) + |\nu|(K \setminus A) \geq 2\nu(A) - \nu(\varphi) + (1/r) \cdot \nu(\psi) - (2t/r)\varepsilon,$$

so the proof is complete. △

Note that, for any clopen set $C \subseteq K$, $A \subseteq C$ and functions φ, ψ supported in C, we can use Lemma 5.2.4 to estimate the value of $|\nu|(C)$.

5.3. When K is countable

5.3.1. Basic tool. We consider here any scattered compact space K of finite height, a norm-increasing isomorphism $T: C(K) \to C[0, \omega]$ and the associated measures

$$\nu_i = T^*\delta_i, \nu = T^*\delta_\omega \in M(K)$$

for $i \in \omega$.

Note that $\nu_i \to \nu$ in the w^* topology of $M(K)$.

Lemma 5.3.1. *If $T: C(K) \to C[0, \omega]$ is a norm-increasing isomorphism, then $|\nu|(K) \geq 1$.*

Proof. By Lemma 5.2.2, there is $\varphi \in C(K)$ such that $\nu_i(\varphi) = 1$ for every $i \leq \omega$. Then $\|\varphi\| \leq 1$ and $|\nu|(K) \geq \nu(\varphi) = 1$. △

In the proof below, as well as elsewhere, we use the asymptotic symbol $\lesssim$ in the following sense: $a \lesssim b$ means that the real-valued functions a and b defined for $\varepsilon > 0$ satisfy $\lim_{\varepsilon \to 0^+} a(\varepsilon) \leq \lim_{\varepsilon \to 0^+} b(\varepsilon)$.

Lemma 5.3.2. *Suppose that $x \in K^{(1)}$, $C \subseteq K$ is a clopen set containing x and $f \in C(K)$ satisfies*

(i) $t \geq s = \|Tf\| = \sup_i |\nu_i(f)| > 1$;

(ii) $f(x) = 1 = \|f\chi_C\|$.

Then

$$t \geq 2\frac{s - |\nu(f)|}{s - 1} - |\nu(f\chi_C)| + |\nu|(K \setminus C).$$

Proof. We first fix a sequence of isolated points $x_n \in C$ converging to x. Write $e_n \in C(K)$ for the characteristic function of $\{x_n\}$.

Fix $\varepsilon > 0$ and consider the functions

$$g_n = \frac{1}{s + \varepsilon} \cdot f + \left(1 - \frac{1 - \varepsilon}{s + \varepsilon}\right) \cdot e_n.$$

Note that $f(x_n) > 1 - \varepsilon$ for large n and then $\|g_n\| \geq g_n(x_n) \geq 1$.

As $\nu_i(f) \to \nu(f)$, there is i_0 such that for every $i \geq i_0$ we have $|\nu_i(f) - \nu(f)| < \varepsilon$. Then fix N such that for every $n \geq N$ and every $i < i_0$ we have $|\nu_i(e_n)| < \delta$, where δ will be specified in a while.

Claim. For every $n \geq N$ there is $i = i(n) \geq i_0$ such that $|\nu_i(g_n)| \geq \|g_n\|$.

Proof. Observe that for every $i < i_0$ and $n \geq N$,

$$|\nu_i(g_n)| < \frac{s}{s + \varepsilon} + \left(1 - \frac{1 - \varepsilon}{s + \varepsilon}\right) \cdot \delta < 1$$

whenever δ is small enough. In other words, we have checked that the initial measures cannot norm g_n for large n. ▲

We can now perform the following approximate calculations:

$$1 \leq |\nu_{i(n)}(g_n)| \lesssim \frac{|\nu(f)|}{s} + |\nu_{i(n)}(e_n)|(1 - 1/s), \text{ so}$$

$$|\nu_{i(n)}(e_n)| \gtrsim \frac{s - |\nu(f)|}{s - 1} (> 0).$$

Note that a measure of finite variation may have only finitely many large atoms; hence the sequence $i(n)$ is unbounded. By Lemma 5.2.3(a) applied for $K \setminus C$, there is a norm-one function h supported outside of C such that $\nu_i(h) > |\nu|(K \setminus C) - \varepsilon$ for i large enough.

Now, we can use Lemma 5.2.4 with $r = 1, A = \{x_n\}, \varphi = f\chi_C$ and $\psi = h$ to obtain

$$|\nu_{i(n)}|(C) \gtrsim 2\frac{s - |\nu(f)|}{s - 1} - |\nu(f\chi_C)| + |\nu|(K \setminus C).$$

Since we started from an arbitrary $\varepsilon > 0$, the asymptotic formula above ends the proof. △

5.3.2. When $K^{(2)} \neq \emptyset$. We apply Lemma 5.3.2 to estimate the Banach–Mazur distance between $C[0, \omega]$ and spaces of the form $C(K)$, where K has nonempty higher derivatives.

THEOREM 5.3.3. *Let K be a compact space such that $K^{(2)} \neq \emptyset$. Then*

$$d_{\text{BM}}\big(C(K), C[0, \omega]\big) \geq 2 + \sqrt{5}.$$

PROOF. Let $T\colon C(K) \to C[0, \omega]$ be a norm-increasing isomorphism and let ν_i be the corresponding measures on K (see section 5.2). Writing $t = \|T\|$, we shall prove that $t = \sup_i |\nu_i|(K) \geq 2 + \sqrt{5}$.

Since K is necessarily scattered, we may fix an isolated point $z \in K^{(2)}$ and a sequence $y_m \in K^{(1)}$ convergent to z. Write $\theta = \nu(\{z\})$; without loss of generality, we can assume that $\theta \geq 0$. Choose a clopen set $A_0 \subseteq K$ such that $A_0 \cap K^{(2)} = \{z\}$ and $|\nu|(A_0) \approx \theta$.

CLAIM. There is a nonempty clopen set $A_1 \subseteq A_0$ such that

$$\nu(A_1) \approx 0 \text{ and } |\nu_i(A_1)| \lesssim \frac{t + \theta}{2} \text{ for every } i.$$

PROOF. To prove the claim, fix pairwise disjoint clopen sets $C_n \subseteq A_0$ such that $C_n \cap K^{(1)} = \{y_n\}$ whenever $y_n \in A_0$.

For any $\varepsilon > 0$ there is i_0 such that $|\nu_i(A_0) - \nu(A_0)| \leq \varepsilon$ for every $i \geq i_0$. Then we can choose A_1 among the sets C_n such that $|\nu_i(A_1)| < \varepsilon$ for every $i < i_0$ and $|\nu|(A_1) < \varepsilon$. By Lemma 5.2.3(b), we have $|\nu_i(A_1)| \leq (t + \theta + \varepsilon)/2$ for every $i \geq i_0$. ▲

We now apply Lemma 5.3.2 for $f = \chi_{A_1}$ with $s \approx (t + \theta)/2$: since $\nu(A_1) \approx 0$, we get

$$t \gtrsim 2\frac{(t + \theta)/2}{(t + \theta)/2 - 1} + |\nu|(K \setminus A_1).$$

Here $|\nu|(K \setminus A_1) \gtrsim \max(1, \theta)$ and therefore

$$t \geq 2\frac{t+\theta}{t+\theta-2} + \max(1,\theta) = 2 + \frac{4}{t+\theta-2} + \max(1,\theta).$$

If $\theta \leq 1$, then

$$t \geq 2 + \frac{4}{t-1} + 1 \text{ that is } t^2 - 4t - 1 \geq 0 \text{ implying } t \geq 2 + \sqrt{5}.$$

Otherwise, we have $\theta > 1$ and

$$t \geq 2\frac{t+\theta}{t+\theta-2} + \theta.$$

WolframAlpha says that $t \geq 2 + \sqrt{\theta^2 + 4} > 2 + \sqrt{5}$, and the proof is complete. △

Extending the argument above we prove the following general result which, in particular, gives the lower bound needed for Theorem 5.1.1.

Theorem 5.3.4. *Let K be a compact space such that $K^{(m)} \neq \emptyset$ for $m \geq 2$. Then*

$$d_{\mathrm{BM}}\big(C(K), C[0,\omega]\big) \geq m + \sqrt{(m-1)(m+3)}.$$

Proof. We follow here the notation from the beginning of the previous proof. Take $z \in K^{(m)}$ and suppose, as before, that $\theta = \nu(\{z\}) \geq 0$ and A_0 is chosen. We first extend the claim from the previous proof.

Claim. There are clopen sets $A_0 \supseteq A_1 \supseteq A_2 \supseteq A_{m-1}$, where $A_{m-1} \cap K^{(1)} \neq \emptyset$ and indices $i_0 < i_1 < \ldots$ such that

— $|\nu|(A_1) \approx 0$ and $|\nu_i(A_1)| \approx 0$ for $i < i_0$;
— $|\nu_i(A_1)| \lesssim (t+\theta)/2$ for every $i_0 \leq i < i_1$;
— $|\nu_i(A_1)| \approx 0$ for every $i \geq i_1$;
— $|\nu_i(A_2)| \lesssim (t-1)/2$ for every $i_1 \leq i < i_2$;
— ...

Proof. Indeed, we choose A_1 as before, but now we can assume that $z_n \in A_1 \cap K^{(m-1)}$ so z_1 is a limit of points $y_n \in K^{(m-2)}$. We find the corresponding sets $C_n \subseteq A_1$ and set A_2 to be one of them. The only difference is that now, since $\|\nu\| \geq 1$ and $|\nu|(B) \geq 1$ on some set disjoint from A_1, we have $|\nu_i|(A_1) \lesssim t-1$ for large i. In this context, Lemma 5.2.3(b) gives $|\nu_i(A_2)| \lesssim (t-1)/2$. We can continue in this manner until we reach the first derivative of K. ▲

We then consider a function which is a convex combination of the form

$$f = p \cdot \chi_{A_1} + \frac{1-p}{m-2} \cdot \sum_{j=2}^{m-1} \chi_{A_j}.$$

We have $\|f\| = 1$ and we want to choose $p \in (0,1)$ to minimize $\|Tf\|$. This p is determined by the equation

$$p \cdot \frac{t+\theta}{2} = \frac{1-p}{m-2} \cdot \frac{t-1}{2}.$$

Then

$$p = \frac{t-1}{(t+\theta)(m-2)+t-1}, \text{ so}$$

$$\|Tf\| \lesssim s(\theta) := \frac{t-1}{(t+\theta)(m-2)+t-1} \cdot \frac{t+\theta}{2}.$$

Consider first the case $\theta \leq 1$: Lemma 5.3.2 gives

$$t \geq \frac{2s(\theta)}{s(\theta)-1} + 1,$$

and, for $m \geq 2, \theta \geq 0$, this implies[1]

$$t \geq (1/2)\Big(\sqrt{4m^2+4m(\theta+1)+\theta^2-6\theta-7}+2m-\theta+1\Big) := h(\theta).$$

We claim that the function $h(\theta)$ on the right hand side is decreasing on $[0,1]$. Indeed, the derivative

$$h'(\theta) = \frac{1}{2}\left(-1+\frac{(-3+2m+\theta)}{\sqrt{-7+4m^2-6\theta+\theta^2+4m(1+\theta)}}\right),$$

is negative — if we suppose that $h'(\theta) \geq 0$, then

$$(-3+2m+\theta)^2 \geq -7+4m^2-6\theta+\theta^2+4m(1+\theta)$$

so $16-16m \geq 0$, which is a contradiction. We thus conclude

$$t \geq h(1) = (1/2)\Big(\sqrt{4m^2+8m-12}+2m\Big) = m+\sqrt{(m-1)(m+3)}.$$

Suppose now that $\theta \geq 1$; then

$$t \geq \frac{2s(\theta)}{s(\theta)-1} + \theta.$$

Analysis using Mathematica (see the file "Derivatives.nb" in the project files) shows that $s'(\theta) \geq 0$ for $t \geq 3$. Hence, the function $s(\theta)$ is increasing, so

$$t \geq \frac{2s(1)}{s(1)-1} + 1,$$

giving the same estimate. △

The previous result may be slightly generalised. Consider a space L such that $L^{(2)} = \emptyset \neq L^{(1)}$. Then we can identify L with $[0,\omega] \times \{0,1,\ldots,k-1\}$ for some k. Then one can re-examine the proof of 5.3.4: we have to deal with a finite number of sequences of converging measures $\nu_i \to \nu$ but the essence will be the same.

Corollary 5.3.5. *Let K be a compact space such that $K^{(m)} \neq \emptyset$. Then*

$$d_{\mathrm{BM}}\big(C(K), C(L)\big) \geq m+\sqrt{(m-1)(m+3)}$$

for every compact space L with an empty second derivative.

It also seems worth noting what Theorem 5.3.3 and Corollary 5.3.5 mean for K of infinite height.

Corollary 5.3.6. *If $K^{(\omega)} \neq \emptyset$ and $L^{(2)} = \emptyset$, then $d_{\mathrm{BM}}\big(C(K), C(L)\big) = \infty$.*

5.3.3. When $K^{(2)} = \emptyset$. We compare here two compacta

$$K = [0,\omega] \times k \text{ and } L = [0,\omega],$$

[1]using WolframAlpha

where k ($=\{0,1,\dots k-1\}$) is a fixed natural number $k \geq 2$. We again fix a norm-increasing isomorphism $T\colon C(K) \to C(L)$ and set $t = \|T\|$.

We now decompose the limit measure ν as

$$\text{(D)} \qquad \nu = \sum_{m<k} \theta_m \delta_{(\omega,m)} + \nu',$$

where ν' vanishes at all endpoints (ω, m). Recall that $\nu_i \to \nu$ in the *weak** topology of $C(K)$. Lemma 5.3.1 implies that

$$|\nu'|(K) + \sum_{m<k} |\theta_m| \geq 1.$$

Let us first explain why we can in fact assume that $\theta_m \geq 0$ for every $m < k$.

LEMMA 5.3.7. *Let $T\colon C(K) \to C(L)$ be a norm-increasing isomorphism and $\sigma \in C(K)$ be a function such that $\sigma^2 = 1$. Then $\widehat{T}$ defined as $\widehat{T}(g) = T(g \cdot \sigma)$ for $g \in C(K)$ is a norm-increasing isomorphism of the same norm.*

PROOF. Clearly, for every $f \in C(K)$ we have $\|\sigma \cdot f\| = \|f\|$ so

$$\|f\| = \|\sigma \cdot f\| \leq \|T(\sigma \cdot f)\| = \|\widehat{T}f\|, \text{ and}$$
$$\|\widehat{T}f\| = \|T(\sigma \cdot f)\| \leq \|T\|\|\sigma \cdot f\| = \|T\|\|f\|.$$

The operator $\widehat{T}$ is surjective: for any $h \in C(L)$ there is $f \in C(K)$ such that $Tf = h$; then $\widehat{T}(\sigma \cdot f) = T(\sigma \cdot \sigma \cdot f) = Tf = h$. △

COROLLARY 5.3.8. *Let $T\colon C(K) \to C(L)$ be a norm-increasing isomorphism and let $\sigma \in C(K)$ be a function such that $\sigma(n,m) = \operatorname{sgn}(\theta_m)$ for every $n \leq \omega$ and $m < k$.*

Then $\widehat{T}\colon C(K) \to C(L)$ defined as $\widehat{T}f = T(\sigma \cdot f)$ for $f \in C(K)$ is also a norm-increasing isomorphism with the same norm. Moreover, if we decompose the measure $\widehat{\nu} = \widehat{T}^*\delta_\omega$ as in (D), then $\widehat{\theta}_m \geq 0$ for every $m < k$.

PROOF. Since $\sigma \cdot \sigma = 1$, by Lemma 5.3.7 it is enough to note that $\widehat{\nu}$ restricted to a given level $(\omega+1) \times \{m\}$ is equal to $\operatorname{sgn}(\theta_m)\nu$. △

It will be convenient to use the following notation. For any $n \in \omega$ and $m < k$ we write

$$A_m(n) = [n, \omega] \times \{m\}.$$

COROLLARY 5.3.9. *If $\theta_m = 0$ for some $m < k$, then $t \geq 2 + \sqrt{3}$.*

PROOF. Suppose, for instance, that $\theta_0 = 0$. Then $|\nu'|(A_0(n)) \approx 0$ for n large enough, hence $\nu(A_0(n)) \approx 0$. We apply Lemma 5.3.2 with $C = A_0(n)$ and $s = t$ to obtain $t \gtrsim 2t/(t-1) + 1$, so $t \geq 2 + \sqrt{3}$. △

COROLLARY 5.3.10. *If $k \geq 2$ and $I \subseteq k$ is any doubleton then*

$$t \geq \frac{2t}{t-1} + |\nu'|(K) + \sum_{m \notin I} \theta_m.$$

PROOF. Suppose, for instance, that $I = \{0,1\}$ and $\theta_0 \leq \theta_1$. Consider the function

$$f = \chi_{A_0(n)} - \big(\theta_0/\theta_1\big)\chi_{A_1(n)},$$

where n is large enough. We have $\nu(f) \approx 0$, so Lemma 5.3.2, applied with $C = A_0(n) \cup A_1(n)$, gives the declared formula. △

The following was proved by Gergont and Piasecki [42, Theorem 3.2]:

THEOREM 5.3.11. *Given any $k \geq 2$,*

$$d_{\mathrm{BM}}\big(C([0,\omega] \times k), C[0,\omega]\big) \geq \frac{\sqrt{3k^2 - 2k + 1} + 2k - 1}{k}.$$

PROOF. Write $a = |\nu'|(K)$ and $b = \sum_{m<k} \theta_m$ for simplicity. Let $c(I) = \sum_{m \in I} \theta_m$ for any $I \in [k]^2$. Note that

$$\sum_{I \in [k]^2} c(I) = (k-1)b,$$

so there is I such that

$$c(I) \leq \frac{k-1}{\binom{k}{2}} b = \frac{2b}{k}.$$

We apply Corollary 5.3.10 for such I: we have $a + b \geq 1$ and hence

$$t \geq \frac{2t}{t-1} + a + \frac{k-2}{k} b \geq \frac{2t}{t-1} + \frac{k-2}{k},$$

and the assertion follows by solving the related inequality. △

Theorem 5.3.11 says, in particular, that

$$d_{\mathrm{BM}}\big(C([0,\omega] \times 2), C[0,\omega]\big) \geq 3,$$

which is the optimal bound; see Gordon [43].

The next lemma will be crucial for the next section.

LEMMA 5.3.12. *Denote $d_m = \chi_{\{(\omega,m)\}}$ for $m < k$ and let c be a constant such that $t/2 < c \leq t$. Further let J be a set of those $m < k$ for which there is $i(m)$ such that $|\nu_{i(m)}(d_m)| > c$. Then*

$$|\nu'|(K) + \frac{t-c-1}{c} \sum_{m \in J} \theta_m + \sum_{m \notin J} \theta_m \geq 1.$$

PROOF. Note that the mapping $J \ni m \mapsto i(m)$ is injective since $c > t/2$. We choose a function $\varphi \in C(K)$ such that

$$\nu_{i(m)}(\varphi) = \begin{cases} -\mathrm{sgn}(\nu_{i(m)}(d_m)) & \text{for } m \in J, \\ 1 & \text{for } i \notin \{i(m) : m \in J\}. \end{cases}$$

Note that $\|\varphi\| \leq 1$ and write $\varphi_m = \varphi(\omega, m)$ for simplicity.

CLAIM. For every $m < k$ we have $\varphi_m \leq \frac{t-c-1}{c}$

PROOF. Suppose that $\varphi_m \geq 0$. If $\nu_{i(m)}(d_m) > c$, then

$$-1 = \nu_{i(m)}(\varphi) = \nu_{i(m)}(d_m)\varphi_m + \nu_{i(m)}(\varphi - d_m) \geq \varphi_m c - (t-c), \text{ so } \varphi_m \leq \frac{t-c-1}{c}.$$

If $\nu_{i(m)}(d_m) < -c$, then

$$1 = \nu_{i(m)}(\varphi) = \nu_{i(m)}(d_m)\varphi_m + \nu_{i(m)}(\varphi - d_m) \leq \varphi_m(-c) + t - c,$$

so, again, $\varphi_m \leq (t-c-1)/c$. ▲

Recall that $\theta_m \geq 0$ for every $m < k$. Applying the claim above and writing $I = \{m < k : \varphi_m \geq 0\}$, we conclude that

$$1 = \nu(\varphi) = \nu'(\varphi) + \sum_{m<k} \varphi_m\theta_m \leq |\nu'|(K) + \sum_{m\in I} \varphi_m\theta_m \leq$$
$$\leq |\nu'|(K) + \sum_{m\in I\cap J} \varphi_m\theta_m + \sum_{m\in I\setminus J} \varphi_m\theta_m \leq$$
$$\leq |\nu'|(K) + \frac{t-c-1}{c} \sum_{m\in I\cap J} \theta_m + \sum_{m\in I\setminus J} \theta_m \leq |\nu'|(K) + \frac{t-c-1}{c}\sum_{m\in J}\theta_m + \sum_{m\notin J}\theta_m,$$

and we are done. △

5.3.4. The mysterious case $k = 3$. The result of Gergont and Piasecki reproduced here as Theorem 5.3.11 states that

$$d_{\mathrm{BM}}\big(C([0,\omega]\times 3), C[0,\omega]\big) \geq \frac{\sqrt{3\cdot 3^2 - 2\cdot 3 + 1} + 2\cdot 3 - 1}{k} = \frac{\sqrt{22}+5}{3} \approx 3.23.$$

We outline here a method for proving that the distance in question is actually greater than 3.47 and include an analysis of a result from [42] that provides its upper bound.

5.3.4.1. *Lower bound.* Following the notation of the previous section, we additionally assume that $0 \leq \theta_0 \leq \theta_1 \leq \theta_2$. Write $a = |\nu'|(K)$, $b = \theta_0 + \theta_1 + \theta_2$; note that $\theta_2 \geq b/3$.

The main idea is to apply Lemma 5.3.2 to several functions $f \in C(K)$ and use Lemma 5.3.12 for a certain value $c > t/2$ to formulate a system of linear inequalities in variables $\theta_0, \theta_1, \theta_2, a \geq 0$ with t as a parameter. Lemma 5.3.12 defines a set $J \subseteq \{0,1,2\}$ — depending on its structure, we get four different cases. We ask the following question: *what is the maximal value of t for which none of these systems of inequalities has a solution?* This argument will produce a certain lower bound for t.

Recall that a system of linear inequalities in four variables defines a polytope in $\mathbb{R}^4$, which is nonempty if and only if it has a vertex. Such a vertex is uniquely determined by four linearly independent equations related to those inequalities. Hence, a manual analysis is, in principle, possible; however, in our situation, the computations are too involved, and we had to rely on computational assistance.

Let us briefly explain the origin of these inequalities. For every n and $m < 3$ we write $A_m(n) = [n,\omega] \times \{m\}$. Note that, given $\varepsilon > 0$, there is n_0 such that for every $m < 3$ we have $|\nu'|(A_m(n_0)) < \varepsilon$. Then there is i_0 such that for every $i \geq i_0$

$$|\nu_i(A_m(n_0)) - \theta_m| < \varepsilon.$$

Consider the function

$$f = \chi_{A_0(n_0)} - \big(\theta_0/\theta_1\big)\chi_{A_1(n_0)},$$

where n_0 is large enough — a norm-one function for which $\nu'(f) \approx 0$ and $\nu(f) \approx 0$. Hence, Lemma 5.3.2 applied for $C = A_0(n_0)$ and $s = t$ gives

$$t \gtrsim \frac{2t}{t-1} - \theta_0 + \theta_1 + \theta_2 + a.$$

This will give a suitable bound if $\theta_2 + a$ is large enough.

Then consider

$$f = \chi_{A_0(n_0)},$$

which will give the estimate

$$t \geq 2\frac{t-\theta_0}{t-1} - \theta_0 + \theta_1 + \theta_2 + a,$$

a suitable one whenever θ_0 is small.

Other inequalities refer to the set $J \subseteq \{0, 1, 2\}$ defined in 5.3.12; suppose, for instance, that $0 \notin J$.

We choose $n_1 \geq n_0$ so that

$$|\nu_i|(A_m(n_1) \setminus \{(\omega, 0)\}) < \varepsilon \text{ for every } i < i_0 \text{ and } m < 3.$$

Applying Lemma 5.3.2 to $f = \chi_{A_0(n_1)}$ with $s \lesssim c$ (see Lemma 5.2.3(b); we obtain

$$t \geq 2\frac{c-\theta_0}{c-1} - \theta_0 + \theta_1 + \theta_2 + a.$$

We also need an inequality suitable for the intermediate case when the values θ_i's are neither large nor small. Suppose, for instance, that $1 \notin J$, which means that $|\nu_i(A_1(n_1))| \lesssim c$. Consider

$$h = \chi_{A_1(n_1)} - (1/2)\chi_{A_0(n_1)} - (1/2)\chi_{A_2(n_1)},$$

and note that

$$\|h\| = \left\|(1/2)\chi_{A_1(n_1)} + (1/2)\big(\chi_{A_1(n_1)} - \chi_{A_0(n_1)} - \chi_{A_2(n_1)}\big)\right\| \lesssim c/2 + t/2,$$

then we get the following

$$t \geq 2\frac{t/2 + c/2 - (\theta_1 - (\theta_0 + \theta_2)/2)}{t/2 + c/2 - 1} - \theta_1 + \theta_0 + \theta_2 + a.$$

We can also apply similar functions for different axes.

Finally, the constraint of another type comes directly from Lemma 5.3.12. For instance, if $J = \{0, 1, 2\}$, then

$$\frac{t-c-1}{c}(\theta_0 + \theta_1 + \theta_2) + a \geq 1.$$

Precise systems of inequalities are written in Appendix A. We approximate the minimal value of t for which these systems have a solution in the following manner. First, we fix the value of c to be a certain fraction of t (such as $(t+1/2)/2$ or $(t+1)/2$). If c depends linearly on t, the corresponding systems exhibit a notable property — monotonicity: if a solution exists for some t_0, then there is also one for any $t > t_0$. This monotonicity allows us to apply a simple binary search algorithm.

From the known results, the systems have no solutions for $t = 3$, while they do for $t = 5$. Our binary search reveals that $c \approx (t+0.29)/2$ works best and that our systems of inequalities have no solutions for $t \leq 3.4704$, allowing us to state the following theorem.

Theorem 5.3.13. $d_{\mathrm{BM}}\big(C([0,\omega] \times 3), C[0,\omega]\big) \geq 3.4704.$

The computations were performed in Mathematica; the corresponding code is available on GitHub (see the file "Binsearch model.nb" in the project files).

5.3.4.2. *Upper bound.* Gergont and Piasecki in [42, Section 3] introduced a very natural class of isomorphisms between $C(K)$ and $C(L)$ for $K = [0,\omega] \times 3$ and $L = [0,\omega]$. We briefly outline their construction here, mainly to provide an explicit value of the parameter t that seems optimal within this class.

It is worth noting that just before the original article on this topic [54] was completed, the authors were informed that Marek Cuth had independently obtained the same result.

Theorem 5.3.14.

$$d_{\mathrm{BM}}\big(C([0,\omega]\times 3), C[0,\omega]\big) \le \frac{4+\sqrt[3]{73-6\sqrt{87}}+\sqrt[3]{73+6\sqrt{87}}}{3} \approx 3.87512\ldots$$

Proof. We define two matrices depending on a parameter t with $3 \le t \le 4$, which will be the norm of the isomorphism we construct. Put

$$M = \begin{pmatrix} t-2 & -1 & -1 \\ 0 & t/2 & -t/2 \\ \frac{t-2}{t} & -\frac{t^2-5t+2}{4} & -\frac{t^2-5t+2}{4} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{2t}{t+1} & 0 & 0 \\ 0 & \frac{t^2-t+2}{2t} & 0 \\ 0 & 0 & \frac{t^2-t+2}{2t} \end{pmatrix},$$

denote the last row of M by M_3 and let $M' = \begin{pmatrix} M_3 \\ M_3 \\ M_3 \end{pmatrix}$ be a 3x3 matrix with each row equal to M_3. Given $f \in C(K)$, define

$$\begin{pmatrix} Tf(1) \\ Tf(2) \\ Tf(\omega) \end{pmatrix} = M \cdot \begin{pmatrix} f(\omega,0) \\ f(\omega,1) \\ f(\omega,2) \end{pmatrix}$$

and

$$\begin{pmatrix} Tf(3m) \\ Tf(3m+1) \\ Tf(3m+2) \end{pmatrix} = C \cdot \begin{pmatrix} f(m,0) \\ f(m,1) \\ f(m,2) \end{pmatrix} + (M'-C) \cdot \begin{pmatrix} f(\omega,0) \\ f(\omega,1) \\ f(\omega,2) \end{pmatrix},$$

where $m \in [1,\omega)$. We use here a slightly modified notation compared with [42, Section 3], but T still belongs to the same class of isomorphisms (even in a simplified form).

It is straightforward, though somewhat tedious (or best verified by a computer), to check that

$$M^{-1} = \begin{pmatrix} \frac{t(t^2-5t+2)}{t^4-7t^3+12t^2-8t+8} & 0 & -\frac{4t}{t^4-7t^3+12t^2-8t+8} \\ 2/(t^3-5t^2+2t-4) & 1/t & -(2t)/(t^3-5t^2+2t-4) \\ 2/(t^3-5t^2+2t-4) & -1/t & -(2t)/(t^3-5t^2+2t-4) \end{pmatrix}.$$

Next, define $S\colon C([1,\omega]) \to C([1,\omega]\times 3)$ as

$$\begin{pmatrix} Sg(\omega,0) \\ Sg(\omega,1) \\ Sg(\omega,2) \end{pmatrix} = M^{-1}\cdot \begin{pmatrix} g(1) \\ g(2) \\ g(\omega) \end{pmatrix}$$

and

$$\begin{pmatrix} Sg(m,0) \\ Sg(m,1) \\ Sg(m,2) \end{pmatrix} = C^{-1}\cdot \left(\begin{pmatrix} g(3m) \\ g(3m+1) \\ g(3m+2) \end{pmatrix} - (M'-C)\cdot M^{-1}\cdot \begin{pmatrix} g(1) \\ g(2) \\ g(\omega) \end{pmatrix} \right),$$

for $g \in C([1,\omega])$ and $m \in \omega$. Then S is an inverse of T.

Now the point is that if

$$t = \frac{4 + \sqrt[3]{73 - 6\sqrt{87}} + \sqrt[3]{73 + 6\sqrt{87}}}{3},$$

then we have $\|T\| = t$ and $\|S\| = 1$. △

The analysis showing that the isomorphism constructed above is indeed optimal within the class defined by Gergont and Piasecki lies beyond the scope of this chapter.

5.4. When K is nonscattered

Recall that both ℓ_∞ and $L_\infty[0,1]$ are isometrically isomorphic to spaces of continuous functions. The space ℓ_∞ is isometric to $C(\beta\omega)$, while $L_\infty[0,1]$ can be isometrically represented as the space $C(K)$, where K is the Stone space of the measure algebra of the Lebesgue measure. By the measure algebra, we mean here the quotient algebra $\mathrm{Bor}[0,1]/\mathcal{N}$, where $\mathcal{N}$ is the σ-ideal of Lebesgue null sets. It is well-known that K is a nonseparable extremally disconnected compact space without isolated points.

Note that if we identify $L_\infty[0,1]$ with $C(K)$ as above, the Riesz representation theorem gives $L_\infty^*[0,1] = M(K)$. In what follows, we use a more direct description of the dual space $L_\infty^*[0,1]$, as established by Yosida and Hewitt [88, Theorem 2.3]: every continuous functional on $L_\infty[0,1]$ can be uniquely represented as a finitely additive signed measure ν on $\mathrm{Bor}[0,1]$ that vanishes on $\mathcal{N}$. Furthermore, the norm of the functional is equal to the total variation of the corresponding measure, $\|\nu\| = |\nu|([0,1])$.

5.4.1. First lower bound. Let L be any compactification of ω; in other words, L is a compact space that has a countable dense set of isolated points (which we identify with ω).

Theorem 5.4.1. *If K is a compact zero-dimensional space without isolated points, then*

$$d_{\mathrm{BM}}\big(C(K), C(L)\big) \ge 3 + 2\sqrt{2}\ (> 5.82).$$

Proof. Consider a norm-increasing isomorphism $T\colon C(K) \to C(L)$ and write $\nu_i = T^*\delta_i$ for $i \in \omega \subseteq L$.

Let Φ be a family of functions $\varphi = \varphi(I,\sigma) \in C(K)$, where

— the set $I \subseteq \omega$ is finite,
— $\sigma\colon I \to \{-1, 1\}$,
— $T\varphi(I,\sigma)(i) = \sigma(i)$ for $i \in I$, while

— $T\varphi(I,\sigma)(i) = 0$ whenever $i \in \omega \setminus I$.

Note that the last condition implies that $T\varphi(I,\sigma)(x) = 0$ for $x \in L \setminus \omega$. Let $r = \sup\{\|\varphi\| : \varphi \in \Phi\}$ and fix $\varepsilon > 0$. Since T is norm-increasing, $0 < r \le 1$, and we shall see in a while that $r < 1$ must hold.

We choose $\varphi = \varphi(I,\sigma) \in \Phi$ such that $\|\varphi\| > r - \varepsilon$ and an open set $U \subseteq K$ such that $\varphi|U > r - \varepsilon$ (the negative case will be symmetric). Using the fact that K has no isolated points, we may pick a sequence of pairwise disjoint nonempty clopen sets $A_k \subseteq U$. We then consider the functions

$$g_k = \frac{1}{1+\varepsilon} \cdot \varphi + \frac{1-r+2\varepsilon}{1+\varepsilon} \cdot \chi_{A_k}.$$

The coefficients are chosen so that $\|g_k\| > 1$. On the other hand, there is k such that the value $|\nu_i(A_k)|$ is arbitrarily small for every $i \in I$. This means that for a suitable choice of k, we have $|\nu_i(g_k)| < 1$ whenever $i \in I$. For such k there must be $j \in \omega \setminus I$ satisfying $|\nu_j(g_k)| \ge 1$. Since $\nu_j(\varphi) = 0$, we conclude that

$$|\nu_j(A_k)| \ge \frac{1+\varepsilon}{1-r+2\varepsilon}. \tag{5.4.1}$$

We apply Lemma 5.2.4 with $\nu = \nu_j$, $\psi = \varphi(\{j\}, e_j)$ and $A = A_k$ (where $e_j(i) = 1$ if $i = j$ and 0 otherwise). Using (5.4.1), it follows that

$$t \ge |\nu|(K) \ge 2 \cdot \frac{1+\varepsilon}{1-r+2\varepsilon} + 1/r - 3(t/r)\varepsilon.$$

As $\varepsilon > 0$ can be arbitrarily small,

$$t \ge \frac{2}{1-r} + \frac{1}{r} := \xi(r).$$

Now it remains to find the minimal value of the function $\xi(\cdot)$ defined on right hand side on $(0,1)$: we have

$$\xi'(r) = 2/(1-r)^2 - 1/r^2$$

so $\xi'(r) = 0$ for $r = \sqrt{2} - 1$. Hence,

$$t \ge \xi(\sqrt{2}-1) = 3 + 2\sqrt{2},$$

and we are done. △

5.4.2. Second lower bound. We expand here the method of the previous section to the specific case of a norm-increasing isomorphism $T\colon L_\infty[0,1] \to \ell_\infty$. Again, our task is to find a lower bound for $t = \|T\|$.

We adapt the beginning of the proof of Theorem 5.4.1 to the present setting. Write $\nu_n = T^*\delta_n$ for $n \in \omega \subseteq \beta\omega$. Here, $\delta_n \in \ell_\infty^*$ is given by $\delta_n(x) = x(n)$. Then $\nu_n \in L_\infty[0,1]^*$ so it can be treated as a finitely additive measure on $\mathrm{Bor}[0,1]$ that vanishes on Lebesgue-null sets, with total variation bounded by t.

Further, let $\varphi_n \in L_\infty[0,1]$ satisfy $T\varphi_n = e_n$, where e_n is the usual 'unit' vector in ℓ_∞. Let $\mathcal{F}$ be a family of functions $\varphi = \varphi(I,\sigma) \in L_\infty$ defined by

$$\varphi(I,\sigma) = \sum_{i\in I} \sigma(i)\varphi_i,$$

where I is finite and $\sigma\colon I \to \{-1,1\}$. Then, again, let $r = \sup\{\|\varphi\|\varphi \in \mathcal{F}\}$.

The new ingredient arises from the following observation.

LEMMA 5.4.2. *The series $\sum_n |\varphi_n|$ converges almost everywhere to a function bounded by r and, consequently, $\sum_n a_n\varphi_n$ converges almost everywhere for every bounded sequence of a_n.*

PROOF. Indeed, for every finite set $I \subseteq \omega$, by choosing the signs appropriately, we have

$$\sum_{i\in I} |\varphi_i(x)| = \varphi(I,\sigma)(x) \le \|\varphi(I,\sigma)\| \le r$$

for almost all $x \in [0,1]$. △

At this stage, fix $\varepsilon > 0$ and choose $\varphi = \varphi(I,\sigma) \in \mathcal{F}$ such that $\|\varphi\| > r - \varepsilon$. Then there is a Borel set B such that $\varphi|B > r - \varepsilon$ (the negative case will be symmetric).

For every nonnegligible Borel set $A \subseteq B$ we first consider the function

$$g_A = \frac{1}{1+\varepsilon}\big(\varphi + (1 - r + 2\varepsilon)\cdot \chi_A\big).$$

Let $a = \sup_n |\nu_n(A)| \ge 1$. As in the proof of Theorem 5.4.1, using Lemma 5.2.4 for these sets A, φ and $\psi = \varphi(\{j\}, e_j)$ (for appropriate $j \notin J$), we get

$$t \gtrsim 2a + \frac{1}{r} \text{ and } a \ge \frac{1}{1-r}.$$

Moreover, the bound $t \ge \frac{2}{1-r} + \frac{1}{r}$ already gives $t > 8$ whenever $r \le 1/6$, so for the purposes of our final theorem, we can safely assume that $r \ge 1/6$. We now fix θ (see (5.4.2) for such a choice):

$$\theta = \frac{1-r}{2ar+1},$$

consider the function

$$\Phi_A = \sum_{n\in\omega\setminus I} \nu_n(A)\varphi_n,$$

and examine

$$f_A = \frac{1}{1+\varepsilon}\big(\varphi + (1 - r + 6\varepsilon)\cdot \chi_A - \theta\cdot\Phi_A\big).$$

LEMMA 5.4.3. *For every measure–positive set $A \subset B$ we have $\|f_A\| > 1$. Moreover, there exists a set A for which $|\nu_i(f_A)| < 1$ for every $i \in I$.*

PROOF. First, observe that $a\theta \le 3$ whenever $r \ge 1/6$ (as assumed). Recall that the function $\sum_{n\in\omega\setminus I}\varphi_n$ is bounded by ε on A. Hence $\theta\Phi_A$ is bounded by 3ε on A. Thus, for $x \in A$ we have

$$f_A(x) \ge \frac{1}{1+\varepsilon}\big(r - \varepsilon + 1 - r + 6\varepsilon - \theta\Phi_A(x)\big) \ge \frac{1+2\varepsilon}{1+\varepsilon},$$

so $\|f_A\| > 1$.

Note that to verify the second statement it suffices to check that for every $\varepsilon' > 0$ there is a set $A \subseteq B$ such that

$$|\nu_i(A)| < \varepsilon' \text{ and } \big|\nu_i(\Phi_A)\big| < \varepsilon',$$

for every $i \in I$. This follows from the fact that the measure $\mu = \sum_{i\in I}|\nu_i|$ has finite total variation, together with the observation that $\Phi_{A_1} + \Phi_{A_2} = \Phi_{A_1\cup A_2}$ whenever $A_1 \cap A_2 = \emptyset$.

It follows that if we divide A into a sufficiently large number of pieces A_k, then $\mu(\Phi_{A_k})$ can be arbitrarily small. △

It follows that for A as in Lemma 5.4.3 there exists $n \in \omega \setminus I$ such that $|\nu_n(f_A)| \geq 1$. Since $\nu_n(\varphi) = 0$, this implies

$$\Big|(1-r)\nu_n(A) - \theta\nu_n\big(\Phi_A\big)\Big| \gtrsim 1.$$

Let $\Psi = \sum_{k\neq n} \nu_k(A)\varphi_k$; then $\nu_n(\Phi_A) = \nu_n(A) + \nu_n(\Psi)$, so

$$v = \Big|(1-r-\theta)\nu_n(A) - \theta\nu_n(\Psi)\Big| \gtrsim 1.$$

Here, it is convenient to stop keeping precise control of certain quantities. We write $v \gtrsim 1$ to mean that for every $\eta > 0$ one can ensure $v > 1-\eta$ by an appropriate choice of the parameters involved in the construction (such as ε, the set A, the index n, etc.). This convention will be used throughout the sequel.

Recall that $\theta = \frac{1-r}{2ar+1}$. It is straightforward to check that

$$v = \frac{ar(1-r)}{2ar+1}\left|2\nu_n(A) - \frac{\nu_n(\Psi)}{ar}\right| \gtrsim 1,$$

therefore, by the triangle inequality, we obtain

$$2|\nu_n(A)| + \frac{|\nu_n(\Psi)|}{ar} \gtrsim \frac{2ar+1}{ar(1-r)}. \tag{5.4.2}$$

Theorem 5.4.4.

$$d_{\mathrm{BM}}(L_\infty[0,1], \ell_\infty) > 7.41.$$

Proof. We continue the analysis begun in this section and use the notation introduced above.

We know that $t \gtrsim 2a + 1/r$ - this bound is effective for larger values of a. A new lower bound is obtained by applying (5.4.2) together with Lemma 5.2.4 to A, φ and $\psi = \varphi_n \pm (1/a)\Psi$, where the sign is chosen so that $\nu_n(\psi) = \nu_n(\varphi_n) + (1/a)\Psi$. By Lemma 5.4.2, the function ψ satisfies the assumption 5.2.4(iii). Hence

$$t \geq 2|\nu_n(A)| + \frac{|\nu_n(\Psi)|}{ar} + 1/r \gtrsim \frac{2ar+1}{ar(1-r)} + 1/r,$$

and this estimate is better for smaller a (e.g. $t \geq 8$ if $a = 2$ and $r = 1/2$). We find the critical value of a by solving

$$\frac{2ar+1}{ar(1-r)} + 1/r = 2a + 1/r,$$

which gives

$$a = \frac{r + \sqrt{(2-r)r}}{2r(1-r)}.$$

Finally, t is bounded from below by the minimum of the function

$$\xi(r) = 2\frac{r + \sqrt{(2-r)r}}{2r(1-r)} + 1/r, \quad 0 < r < 1.$$

According to WolframAlpha, we have $\xi(r) > 7.41$ (note that $\xi(1/2) = 4 + 2\sqrt{3}$). △

Let us record the exact value for the function ξ used above:

$$\min_{r\in(0,1)} \xi(r) = \frac{14 + \sqrt[3]{3554 - 66\sqrt{33}} + \sqrt[3]{3554 + 66\sqrt{33}}}{6}. \tag{5.4.3}$$

5.4.3. Upper bound. Let us briefly examine Pełczyński's argument for $L_\infty[0,1] \simeq \ell_\infty$, outlined in the introduction by (5.1.1) and (5.1.2).

Recall that a subspace of a Banach space is 1-complemented if there exists a norm-one projection onto this subspace. A Banach space is said to be 1-injective if it is 1-complemented in every superspace.

Consider a 1-complemented subspace $Y \subseteq X$; let $P\colon X \to Y$ be a norm-one projection onto Y and let A be the kernel of P. Then there is an isomorphism between X and $Y \oplus A$ with distortion at most 3, given by the formula $Tx = (3Px, 3/2(x - Px))$ for $x \in X$. Indeed, we have $\|T\| \le 3$ and it is straightforward to verify that T is norm-increasing. The constant 3 is optimal in general: take $Y = \mathbb{R}$ as a subspace of the Banach space c of convergent sequences and use the fact that $d_{\mathrm{BM}}(c, c_0) = 3$, which is a special case of a result due to Gordon [**43**].

Therefore, (5.1.1) and (5.1.2) give only

$$d_{\mathrm{BM}}\big(L_\infty[0,1], L_\infty[0,1] \oplus \ell_\infty\big) \le 9, \quad d_{\mathrm{BM}}\big(\ell_\infty, L_\infty[0,1] \oplus \ell_\infty\big) \le 9,$$

and the resulting bound is rather poor: $d_{\mathrm{BM}}\big(\ell_\infty, L_\infty[0,1]\big) \le 81$.

Hence, to get a better estimate, we need to compose the isomorphisms given by the decomposition method in a more sophisticated way. We prove the following general result.

Theorem 5.4.5 (On the norms in the decomposition method). *Assume that X, Y are Banach spaces such that*

(i) there are 1-complemented subspaces $X' \subseteq X$, $Y' \subseteq Y$ with X isometric to Y' and Y isometric to X',

(ii) X is isometric to $X \oplus X$, and Y is isometric to $Y \oplus Y$.

Then $d_{\mathrm{BM}}(X, Y) \le (3 + \sqrt{2})^2$.

Proof. Let P, R be norm-one projections, $P\colon X \to X'$ and $R\colon Y \to Y'$. Recall that $I_X - P$, $I_Y - R$ are projections onto the complements of X', Y'; denote these complements by E, F, respectively, so that $X \simeq X' \oplus E, Y \simeq Y' \oplus F$.

Then we give names to the isometries whose existence is declared by the assumptions:

$$\theta\colon X \to Y', \quad \eta\colon Y \to X',$$

$$\varphi = (\varphi_1, \varphi_2)\colon X \to X \oplus X, \quad \psi = (\psi_1, \psi_2)\colon Y \to Y \oplus Y.$$

Note that $1/(1 + \sqrt{2}) = \sqrt{2} - 1$; these constants become important below.

Using the Pełczyński decomposition method, we obtain the following three lines of isomorphisms, proving that $X \simeq Y$

$$X \simeq X' \oplus E \cong Y \oplus E \cong Y \oplus Y \oplus E,$$

$$Y \oplus Y \oplus E \cong Y \oplus X' \oplus E \simeq Y \oplus X \simeq Y' \oplus F \oplus X \cong X \oplus F \oplus X,$$

$$X \oplus F \oplus X \cong X \oplus F \cong Y' \oplus F \simeq Y.$$

Let us now write these three isomorphisms by explicit formulas:

$$\begin{aligned}
&T\colon X \to Y \oplus Y \oplus E, && Tx = \Big(\psi_1\eta^{-1}Px,\ (\sqrt{2}-1)\psi_2\eta^{-1}Px,\ x - Px\Big),\\
&S\colon Y \oplus Y \oplus E \to X \oplus X \oplus F, && S(y_1,y_2,e) = \Big(\theta^{-1}Ry_1,\ \eta y_2 + e,\ y_1 - Ry_1\Big),\\
&U\colon X \oplus X \oplus F \to Y, && U(x_1,x_2,f) = \theta\varphi^{-1}\Big((1+\sqrt{2})x_1,\ x_2\Big) + f,
\end{aligned}$$

For example, to define T we first map x to the pair $(Px, x - Px)$, then use the isometry η^{-1} to view $Px \in X'$ as an element of Y and finally we split $\eta^{-1}Px$ using ψ. It is straightforward to calculate the inverses of these operators:

$$\begin{aligned}
&T^{-1}\colon Y \oplus Y \oplus E \to X, && T^{-1}(y_1,y_2,e) = \eta\psi^{-1}\Big(y_1,\ (1+\sqrt{2})y_2\Big) + e,\\
&S^{-1}\colon X \oplus X \oplus F \to Y \oplus Y \oplus E, && S^{-1}(x_1,x_2,f) = \Big(\theta x_1 + f,\ \eta^{-1}Px_2,\ x_2 - Px_2\Big),\\
&U^{-1}\colon Y \to X \oplus X \oplus F, && U^{-1}y = \Big((\sqrt{2}-1)\varphi_1\theta^{-1}Ry,\ \varphi_2\theta^{-1}Ry,\ y - Ry\Big).
\end{aligned}$$

Now, we can compose them to obtain UST (we include the intermediate steps to make the calculations easier to verify)

$$STx = \Big(\theta^{-1}R\psi_1\eta^{-1}Px,\ (\sqrt{2}-1)\eta\psi_2\eta^{-1}Px + x - Px,\ \psi_1\eta^{-1}Px - R\psi_1\eta^{-1}Px\Big)$$

$$\begin{aligned}
USTx = \theta\varphi^{-1}\Big(&(1+\sqrt{2})\theta^{-1}R\psi_1\eta^{-1}Px,\ (\sqrt{2}-1)\eta\psi_2\eta^{-1}Px + x - Px\Big)\\
&+ \psi_1\eta^{-1}Px - R\psi_1\eta^{-1}Px.
\end{aligned}$$

and its inverse $T^{-1}S^{-1}U^{-1}$

$$S^{-1}U^{-1}y = \Big((\sqrt{2}-1)\theta\varphi_1\theta^{-1}Ry + y - Ry,\ \eta^{-1}P\varphi_2\theta^{-1}Ry,\ \varphi_2\theta^{-1}Ry - P\varphi_2\theta^{-1}Ry\Big),$$

$$\begin{aligned}
T^{-1}S^{-1}U^{-1}y = \eta\psi^{-1}\Big(&(\sqrt{2}-1)\theta\varphi_1\theta^{-1}Ry + y - Ry,\ (1+\sqrt{2})\eta^{-1}P\varphi_2\theta^{-1}Ry\Big)\\
&+ \varphi_2\theta^{-1}Ry - P\varphi_2\theta^{-1}Ry.
\end{aligned}$$

Recall that all direct sums of Banach spaces are equipped with the max-norm and that operators $\varphi, \psi, \eta, \theta$ have norm 1, and so do their inverses. It follows that

$$\|UST\| \le 3 + \sqrt{2}, \quad \|(UST)^{-1}\| \le 3 + \sqrt{2},$$

so the distortion of this isomorphism is bounded by $(3+\sqrt{2})^2$. △

The author is aware that Tomasz Kania [48] was able to obtain a slightly better estimate by adjusting the constants in the argument above more effectively.

5.4.4. Conclusions. Let us apply the results of this section to obtain some concrete bounds for the Banach–Mazur distance.

For $L_\infty[0,1]$ and ℓ_∞, we have

Corollary 5.4.6. $7.41 < d_{\mathrm{BM}}\big(\ell_\infty, L_\infty[0,1]\big) \le (3+\sqrt{2})^2 < 19.49$.

Proof. The lower bound follows from Theorem 5.4.4. We only need to verify the assumptions of Theorem 5.4.5. It is well known that both $L_\infty[0,1]$ and ℓ_∞ are isometric to their squares. Moreover, both $L_\infty[0,1]$ and ℓ_∞ are 1-injective; see, e.g., [33, 2.5] or [2, section 4.3].

We can isometrically embed ℓ_∞ as a subspace X of $L_\infty[0,1]$: take a sequence of pairwise disjoint open intervals $A_n \subseteq [0,1]$ and send $x \in \ell_\infty$ to $\sum_{n\in\omega} x(n)\chi_{A_n}$. By 1-injectivity, there is a norm-one projection from $L_\infty[0,1]$ onto X; in this case, such a projection can be effectively defined.

Finally, it is a classical fact that $L_\infty[0,1]$ has a weak*-separable dual ball and every such space embeds isometrically into ℓ_∞. Again, by 1-injectivity, the embedded copy of $L_\infty[0,1]$ must be 1-complemented in ℓ_∞. △

We can also use our results to obtain bounds on the Banach–Mazur distances for certain separable Banach spaces. Recall that for every compact metric space K, there is a unique (up to a homeomorphism) compactification L of ω such that $L\backslash\omega$ is homeomorphic to K; see [13, Proposition 4.3.]. For the Cantor set 2^ω, this space L is sometimes (as in [35, 69]) called the Pełczyński compactum due to the classical article from 1965 [72]. The author would like to thank Benjamin Vejnar for pointing out these facts. Note that L has to be homeomorphic to $L \times \{0,1\}$.

Corollary 5.4.7. *Let L be the Pełczyński compactum. Then*

$$3 + 2\sqrt{2} \leq d_{\mathrm{BM}}\big(C(L), C(2^\omega)\big) \leq (3+\sqrt{2})^2,$$

Proof. The lower bound follows from Theorem 5.4.1, whose assumptions are clearly satisfied.

The assumption that L is homeomorphic to $L \times \{0,1\}$ implies that $C(L)$ is isometric to $C(L) \oplus C(L)$. A similar statement holds for the Cantor set: $C(2^\omega) \cong C(2^\omega) \oplus C(2^\omega)$.

Since L is metrizable, the homeomorphic copy of 2^ω is a retract of L. It follows that there exists an extension operator of norm 1 (see [73, Section 2]) and thus a 1-complemented copy of $C(2^\omega)$ in $C(L)$. It is also a classical fact that every zero-dimensional separable metric space can be embedded into 2^ω (see, e.g. [49, Theorem 7.8]). Using the same arguments, $C(2^\omega)$ contains a 1-complemented copy of $C(L)$. Now, Theorem 5.4.5 yields the upper bound. △

The most interesting case, without a doubt, is the pair of spaces $C[0,1]$ and $C(2^\omega)$. The proof that these spaces are isomorphic is based on a version of the decomposition method, using, in particular, the fact that $C(2^\omega)$ is isomorphic to its c_0-sum (see [2, Theorem 2.2.3] for details). Pełczyński [73, page 73], in his closing remarks, mentioned the following estimate:

$$d_{\mathrm{BM}}\big(C[0,1], C(2^\omega)\big) \leq 12. \tag{5.4.4}$$

We do not know how to verify Pełczyński's conjecture, nor have we been able to find any relevant discussion in the literature. To the best of our knowledge, the status of (5.4.4) therefore remains unclear. Using methods analogous to the proof of Theorem 5.4.5, we can obtain an upper bound for $d_{\mathrm{BM}}\big(C[0,1], C(2^\omega)\big)$, but it will be considerably larger than 12.

APPENDIX A

Four linear problems

In this appendix, we list the linear systems arising from the analysis outlined in Section 5.3.4. Recall that

(i) we can choose any $c \geq t/2$ (however, c close to t makes some cases trivial but others give weak bounds);

(ii) the problem is to determine the maximal value of t for which none of these systems of inequalities has a solution

Recall also that the code used to compute the corresponding values of t was implemented in MATHEMATICA and is available on GitHub (see the file "Binsearch model.nb" in the project files).

A.1. Case: $J = \{0, 1, 2\}$.

$$t \geq \frac{2t}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.1.1}$$

$$t \geq 2\frac{t-\theta_0}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.1.2}$$

$$\frac{t-c-1}{c}(\theta_0 + \theta_1 + \theta_2) + a \geq 1 \tag{A.1.3}$$

$$0 \leq \theta_0 \leq \theta_1 \leq \theta_2, a \geq 0 \tag{A.1.4}$$

A.2. Case: $0 \notin J$.

$$t \geq \frac{2t}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.2.1}$$

$$t \geq 2\frac{c-\theta_0}{c-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.2.2}$$

$$t \geq 2\frac{t/2 + c/2 - (\theta_0 - (\theta_1 + \theta_2)/2)}{t/2 + c/2 - 1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.2.3}$$

$$\theta_0 + \theta_1 + \theta_2 + a \geq 1 \tag{A.2.4}$$

$$0 \leq \theta_0 \leq \theta_1 \leq \theta_2, a \geq 0 \tag{A.2.5}$$

A.3. Case: $0 \in J, 1 \notin J$.

$$t \geq \frac{2t}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.3.1}$$

$$t \geq 2\frac{t-\theta_0}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.3.2}$$

$$t \geq 2\frac{c-\theta_1}{c-1} - \theta_1 + \theta_0 + \theta_2 + a \tag{A.3.3}$$

$$t \geq 2\frac{t/2 + c/2 - (\theta_1 - (\theta_0 + \theta_2)/2)}{t/2 + c/2 - 1} - \theta_1 + \theta_0 + \theta_2 + a \tag{A.3.4}$$

$$\frac{t-c-1}{c}\theta_0 + \theta_1 + \theta_2 + a \geq 1 \tag{A.3.5}$$

$$0 \leq \theta_0 \leq \theta_1 \leq \theta_2, a \geq 0 \tag{A.3.6}$$

A.4. Case: $0, 1 \in J, 2 \notin J$.

$$t \geq \frac{2t}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.4.1}$$

$$t \geq 2\frac{t-\theta_0}{t-1} - \theta_0 + \theta_1 + \theta_2 + a \tag{A.4.2}$$

$$t \geq 2\frac{c-\theta_2}{c-1} - \theta_2 + \theta_0 + \theta_1 + a \tag{A.4.3}$$

$$t \geq 2\frac{t/2 + c/2 - (\theta_2 - (\theta_0 + \theta_1)/2)}{t/2 + c/2 - 1} - \theta_2 + \theta_0 + \theta_1 + a \tag{A.4.4}$$

$$\frac{t-c-1}{c}\big(\theta_0 + \theta_1\big) + \theta_2 + a \geq 1 \tag{A.4.5}$$

$$0 \leq \theta_0 \leq \theta_1 \leq \theta_2, a \geq 0 \tag{A.4.6}$$

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