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Asymptotyczne zachowanie ekstremalnej pozycji w gałązkowym spacerze losowym

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Doctoral thesis

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Asymptotic behavior of the extremal position in a branching random walk

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Streszczenie

Procesy gałązkowe stanowią dynamicznie rozwijający się obszar teorii prawdopodobieństwa. Początkowo wykorzystywane do opisu prostych procesów narodzin i śmierci, znalazły następnie szerokie zastosowania w fizyce oraz biologii, w tym w modelowaniu reakcji nuklearnych czy dynamiki epidemii.

Klasyczny gałązkowy spacer losowy rozpoczyna się od pojedynczej cząstki umieszczonej w punkcie początkowym. W chwili 1 cząstka ta znika, a jej miejsce zajmuje losowa liczba nowych cząstek, z których każda przyjmuje losową pozycję. W kolejnych chwilach każda cząstka niezależnie powtarza ten proces - jest zastępowana przez losową liczbę potomków, których pozycje powstają w wyniku losowego przesunięcia pozycji rodzica. Proces ten jest kontynuowany w nieskończoność. W tej rozprawie rozważane są dwie modyfikacje gałązkowych spacerów losowych: modele wielotypowe oraz modele z zaburzeniami.

Model wielotypowy stanowi naturalne uogólnienie konstrukcji klasycznej, umożliwiając analizę cząstek należących do odmiennych klas, które determinują zarówno rozkład pozycji, jak i liczebność potomstwa. Pozwala to na opis zjawisk o większym stopniu złożoności, takich jak dynamika populacji komórek o zróżnicowanych fenotypach. Uwzględnienie wielu typów prowadzi do możliwości interakcji między nimi, co może skutkować zaskakującymi zachowaniami, takimi jak rozprzestrzenianie się populacji w znacząco wyższym tempie, niż ma to miejsce w modelu z którymkolwiek z typów rozważanym osobno.

Model z zaburzeniami wprowadza dodatkową losowość do mechanizmu ustalania pozycji cząstki, co zwiększa elastyczność w opisie procesów stochastycznych. Długoterminowe własności takiego modelu mogą wykazywać istotne różnice względem klasycznego przypadku.

Analiza asymptotycznego zachowania ekstremalnej pozycji stanowi od wielu lat przedmiot intensywnych badań w kontekście klasycznych gałązkowych spacerów losowych, ponieważ dostarcza zasadniczych informacji o długoterminowej dynamice procesu i umożliwia pogłębione zrozumienie mechanizmów występujących w biologii, fizyce czy epidemiologii. Głównym celem niniejszej rozprawy jest opisanie tego zachowania w obu wymienionych wyżej modyfikacjach w możliwie najbardziej ogólnym ujęciu.

Abstract

Branching processes are a rapidly developing area of probability theory. Initially introduced to describe simple birth-and-death dynamics, they have subsequently found numerous applications in physics and biology, including models of nuclear reactions and epidemic spread.

The classical branching random walk starts with a single particle located at the origin. At time 1, this particle dies and is replaced by a random number of randomly placed offspring. At the next time step, every particle repeats the process and is again replaced by a random number of descendants whose positions are determined by a random displacement of the parent's position. The process is then iterated indefinitely. In this thesis, we investigate two modifications of branching random walks: multi-type models and perturbed models.

The multi-type branching random walk is a natural generalization of the standard model, allowing for particles belonging to distinct classes. These classes determine both the offspring distribution and the displacement law of each particle. This framework enables the description of more complex phenomena, such as the dynamics of cell populations with different phenotypes. The interaction between different types may lead to surprising results, including propagation at significantly higher speeds than in any of the corresponding single-type models.

The perturbed branching random walk introduces an additional source of randomness in determining particle positions, providing greater flexibility in modeling stochastic systems. The long-term behavior of such models can differ significantly from that of the classical setting.

The asymptotic behavior of the maximal position has been a central topic of research in the context of classical branching random walks, as it gives fundamental insights into the long-term dynamics of the process and allows for a deeper understanding of complex mechanisms arising in biology, physics, and epidemiology. The principal objective of this thesis is to provide as general as possible a description of this behavior in both models mentioned above.

Podziękowania

W pierwszej kolejności chcę szczególnie podziękować mojemu promotorowi, prof. Dariuszowi Buraczewskiemu, który wspierał mnie przez cały okres pisania tej pracy, a także wcześniej – podczas studiów magisterskich. Dziękuję również prof. Ewie Damek, która skierowała mnie ku teorii prawdopodobieństwa, gdy byłem jeszcze studentem.

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Dziękuję mojej rodzinie, zwłaszcza mamie i dziadkom, którzy wierzyli we mnie zawsze i na których pomoc nieustannie mogłem liczyć.

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Contents

1 Introduction

Branching processes are fundamental models in probability theory, with applications in biology, physics, and computer science. In this setting, a population evolves through both reproduction and random movement in space. Two central objects in this theory are the *Branching Brownian Motion* (BBM) and the *Branching Random Walk* (BRW).

We begin this chapter with the introduction of BBM, a continuous-time and continuous-space model and its connection to the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation. From there, we transition to the BRW, which is a discrete analogue of BBM, and we finish this chapter with the introduction of two modifications that are the focus of this thesis, namely multi-type and perturbed BRWs. Our main results describe the asymptotic behavior of the extremal position in both models mentioned above. In Chapters 2 and 3 we present the limit theorems regarding multi-type processes. Chapter 4 is dedicated to the perturbed model, and the results presented there have been published in ESAIM: Probability and Statistics [38].

1.1 Branching Brownian Motion

Branching Brownian Motion is a stochastic process where particles move according to independent Brownian motions and branch at exponential times. Formally, starting with a single particle placed at the origin, it moves according to standard Brownian motion, and after an exponential time with rate 1, splits into two particles, which continue the process independently. Let $N(t)$ be the number of particles in the system at time t, and $X_1(t), X_2(t), \ldots, X_{N(t)}(t)$ their positions. A connection arises between BBM and a nonlinear partial differential equation known as the F-KPP equation:

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u),\\ &u(0,x) = f(x).\end{aligned}$$

This equation was introduced in the 1930s by Fisher [26] and Kolmogorov-Petrovskii-Piskunov [39] to model gene propagation in a population. The equation can be solved for a wide class of functions f, but within the probabilistic context the function of particular interest is

$$f(x) = \mathbb{1}_{[0,\infty)}(x).$$

McKean [42] showed that in this case the F-KPP equation describes the probability that all particles in a BBM remain to the left of position x at time t:

$$u(t,x) = \mathbb{E}\left[\prod_{i=1}^{N(t)} f\left(x - X_i(t)\right)\right] = \mathbb{P}\left(\max_{1\le i\le N(t)} X_i(t) \le x\right).$$

The F-KPP equation admits a traveling wave solution of the form

$$u(t,x) = \phi\left(x - \sqrt{2}t\right).$$

By Bramson [14, 15]and Lalley and Selke [40], $M_t = \max_{i\le N(t)} X_i(t)$ satisfies the law of large numbers,

$$\frac{M_t}{t} \xrightarrow[n\to\infty]{a.s.} \sqrt{2},$$

and with centering $m(t) = \sqrt{2}t - \frac{3}{2^{\frac{3}{2}}}\log t$,

$$\mathbb{P}(M_t - m(t) \le x) \xrightarrow[n\to\infty]{} \mathbb{E}\left[\exp\{-cD_\infty e^{-\sqrt{2}x}\}\right] = \phi(x)$$

where $c > 0$ is a constant and D_∞ is a random variable depending on the branching mechanism. These connections provide additional motivation for studying branching processes and the properties of the maximal position in particular.

1.2 Branching Random Walks

A natural discrete analog of BBM is the Branching Random Walk. BRWs found many applications in physics and biology, including modeling nuclear chain reactions [18] and the spread of epidemics [25]. Formally, a BRW is constructed as follows. The process starts with a single particle placed at 0. Given a point process $\mathcal{Z} = \sum_{k=1}^{N} \delta_{\xi_k}$ on $\mathbb{R}$, where N, denoting the size of the offspring, is a random variable on $\mathbb{N}_0$, the original particle at time 1 dies and gives birth to N particles positioned according to $\mathcal{Z}$. These particles are called the first generation of the process. At time 2, each of these particles reproduces independently and has offspring with positions relative to their parents' position given by an independent copy of $\mathcal{Z}$.

The process continues infinitely. As a result, we obtain a marked tree $(S, \mathbb{T})$, where the tree $\mathbb{T}$ is the set of all particles equipped with the natural tree structure, and S_v is the position of a given particle $v \in \mathbb{T}$.

We write $|v|$ for the generation of v and $m = \mathbb{E}[N]$ for the mean number of offspring. For a BRW with displacements given by ξ, let

$$R_n = \sup_{|v|=n} S_v$$

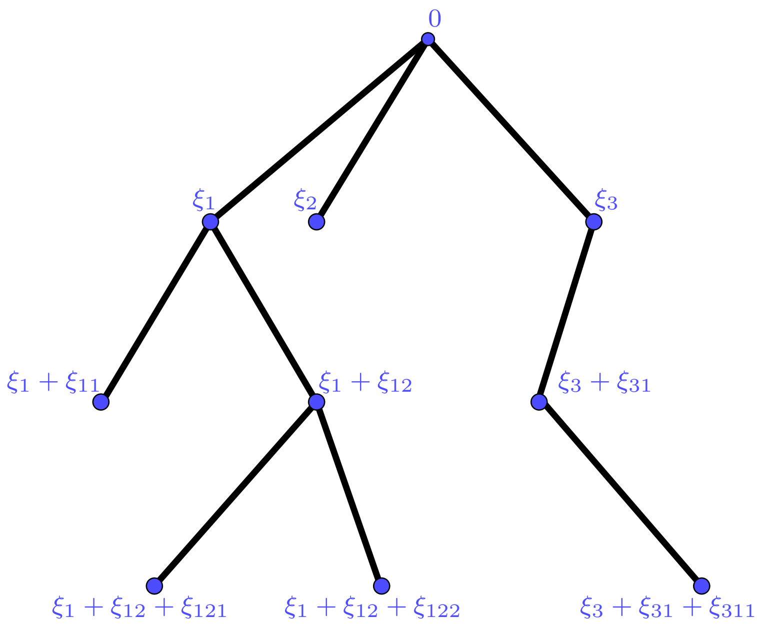

Figure 1: Branching random walk diagram.

denote the position of the most right particle at time n. The asymptotic behavior of R_n is most commonly studied under the following exponential moment assumption:

$$\text{there exists } \theta > 0, \text{ such that } \mathbb{E}\left[\sum_{i=1}^{n} e^{\theta \xi_i}\right] < \infty. \tag{1.1}$$

Within this framework, we can define the log-Laplace transform of $\mathcal{Z}$:

$$\nu(\theta) = \log \mathbb{E}\left[\sum_{i=1}^{N} e^{\theta \xi_i}\right],$$

and the critical parameter:

$$\theta_0 = \inf\left\{\theta > 0 : \nu(\theta) = \theta \nu'(\theta)\right\}, \tag{1.2}$$

where

$$\nu'(\theta) = e^{-\nu(\theta)} \mathbb{E}\left[\sum_{i=1}^{N} \xi_i e^{\theta \xi_i}\right].$$

Note that ν does not have to be differentiable at θ for this quantity to exist, and that in general θ_0 may be infinite.

Under (1.1), Biggins [9] proved in 1976 the law of large numbers for R_n, i.e. $\frac{R_n}{n}$ converges almost surely to $\frac{\nu(\theta_0)}{\theta_0}$.

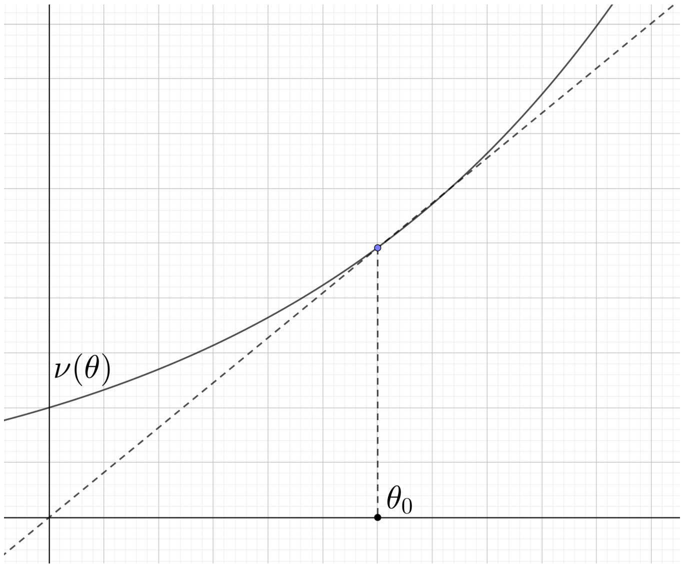

Figure 2: Example plot of ν with θ_0 highlighted.

The corresponding second order limit theorem was proved by Aïdékon [1] in 2013, who showed that $R_n - \frac{\nu(\theta_0)}{\theta_0} n + \frac{3}{2\theta_0} \log n$ converges in distribution to a random shift of the Gumbel distribution. We refer to Shi [43] for an extensive description of recent results on branching random walks with finite exponential moments.

The assumption (1.1) is critical to the linear growth of R_n. Durrett [20] showed in 1983 that if one assumes instead that the displacements have regularly varying tails, R_n grows exponentially fast. More specifically, assume that for some slowly varying L and some $r > 0$, we have

$$\mathbb{P}(\xi > x) \sim L(x)x^{-r} \quad \text{as } x \to \infty \tag{1.3}$$

and

$$\log(-x)\mathbb{P}(\xi \leq x) \to 0, \quad \text{as } x \to -\infty. \tag{1.4}$$

Then

$$\mathbb{P}(R_n \leq a_n x) \xrightarrow[n\to\infty]{} \mathbb{E}[e^{-cWx^{-r}}]$$

where $c > 0$ is a constant, W is a random variable depending on the underlying Galton-Watson process, and $\{a_n\}_{n\in\mathbb{N}}$ satisfies

$$m^n \mathbb{P}(\xi > a_n) \xrightarrow[n\to\infty]{} 1. \tag{1.5}$$

Another model present in the literature considers displacements with semi-exponential tails. Assume that for some slowly varying a, L and $r \in (0,1)$,

$$\mathbb{P}(\xi > x) = a(x) \exp\{-L(x)x^r\}.$$

Then, according to Gantert [28],

$$\frac{R_n}{b_n} \xrightarrow[n\to\infty]{a.s.} (\log m)^{\frac{1}{r}},$$

where b_n satisfies

$$\frac{L(b_n)b_n^r}{n} \xrightarrow[n\to\infty]{} 1. \tag{1.6}$$

This model was further explored in a series of papers by Dyszewski, Gantert, and Höfelsauer in the context of large deviations [22], extremal point process [23] and second-order fluctuations [24].

1.3 Multi-type branching random walk

Multi-type branching random walks extend the ideas of one-type branching processes to a multidimensional setting, which is necessary to model various phenomena, such as cell population dynamics with different phenotypes [34]. Formally, a multi-type branching random walk is constructed analogously to the one-type model. Take a set of types $\mathcal{C} = \{1, 2, \dots, d\}$ and a corresponding family of point processes $\{\mathcal{Z}_{ij}\}_{i,j\in\mathcal{C}}$, where $\mathcal{Z}_{ij} = \sum_{k=1}^{N_{i,j}} \delta_{\xi_k^j}$, and for each $j \in \mathcal{C}$, $\{\xi_k^j\}_{k\in\mathbb{N}}$ are marginally identically distributed. We start with a single particle of any given type i placed at the origin. For each $j \in \mathcal{C}$, this particle gives birth to $N_{i,j}$ children of type j, positioned according to $\mathcal{Z}_{ij}$, and subsequently dies. At time 2, each particle of type j reproduces independently according to copies of $\{\mathcal{Z}_{jk}\}_{k\in\mathcal{C}}$, and subsequently dies. The process continues infinitely.

In this case, the number of offspring depends on the type of parent, but the displacement of a particle depends only on its own type. We write $Z_n = \{Z_n^1, Z_n^2, \dots, Z_n^d\}$ for the d-dimensional Galton-Watson process recording the number of particles of each type in the n-th generation and define the mean matrix $M = (\mathbb{E}[N_{i,j}])_{i,j\in\mathcal{C}}$. Since all entries in M are nonnegative, it has the principal (although possibly not unique) eigenvalue that we denote by ρ. We utilize the one-type notation and define $\sigma(v) = i$ whenever v belongs to type i. Our main point of interest is again the asymptotic behavior of the maximum position R_n.

In the multi-type model, one needs to distinguish between two significantly different regimes. We call the process **irreducible** if a particle of any given type can appear in any line of descent with positive probability and **reducible** otherwise. In terms of the mean matrix M, irreducibility translates to the following statement: for any $i, j \in \mathcal{C}$, there exists $n \in \mathbb{N}$, such that $M^n(i,j) > 0$. The previous results on the multi-type model under the exponential moment assumption go back to

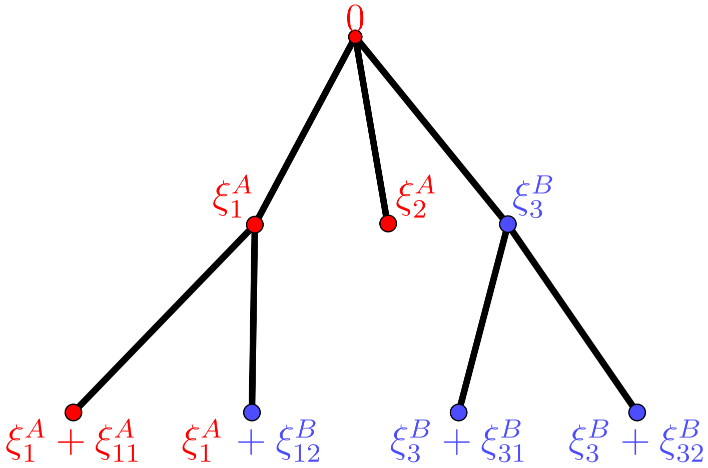

Figure 3: Branching random walk with two types, A and B.

Biggins [9], who showed in 1976 that the irreducible model exhibits linear growth and described the limiting constant. The reducible case proved to be more challenging. Weinberger et al.[44] in 2002 argued that the spreading speed should be the maximum of speeds of the types considered separately, essentially ignoring the interaction between types. A flaw in this argument was identified by Weinberger et al. [45] in 2007, and the correct limiting constant was ultimately described by Biggins [11] in 2012. As it turns out, the interplay between the types can significantly increase the growth speed. This effect was called **anomalous spreading** in Weinberger et al. [45] and makes the study of reducible models particularly appealing.

Bhattacharya, Maulik, Palmowski and Roy [8] in 2019 considered an irreducible model with displacements having regularly varying tails. They showed the convergence of the extremal process to a randomly scaled scale-decorated Poisson point process and, as a result, obtained a limit theorem for the maximum position. In this case, it turns out that the behavior is analogous to the one-type model considered in [20], with the largest eigenvalue of M replacing the mean number of offspring and the heaviest tail dominating the lighter ones.

In this thesis, we aim to provide a corresponding result for the reducible case and describe the complete asymptotics of the extremal position in the previously unstudied multi-type model with semi-exponentially tailed displacements.

1.4 Perturbed branching random walk

The perturbed branching random walk S^* is a modification of a standard BRW S, in which we add a random perturbation to the position of every particle, i.e.

$$S_v^* = S_v + X_v,$$

where $\{X_v\}_{v \in \mathbb{T}}$ are i.i.d. random variables independent of S.

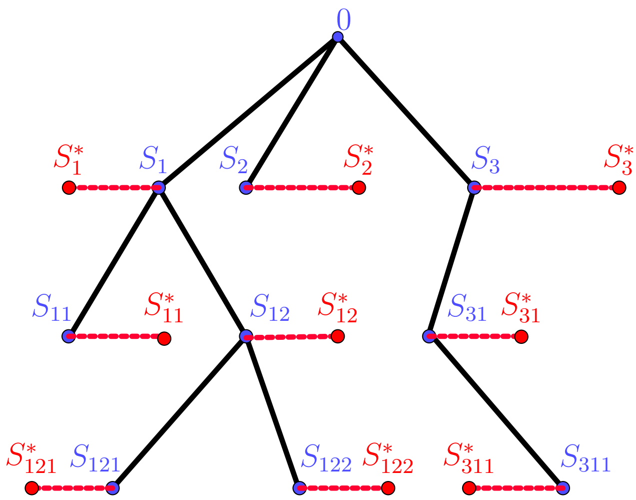

Figure 4: Perturbed branching random walk.

Note that the perturbation added to the position of a vertex $v \in \mathbb{T}$ does not influence the positions of its offspring, which explains that the process is sometimes called *last progeny modified branching random walk*. We are particlarly interested in the model introduced by Bandyopadhyay and Ghosh in [5], where the perturbations have the form

$$X_v(\theta) = \frac{1}{\theta} \log \frac{Y_v}{E_v}$$

for a given positive real number θ, and $\{Y_v\}_{v \in \mathbb{T}}$ which are independent positive random variables with distribution μ, and given $\mathbb{T}$ are independent of $\{E_v\}_{v \in \mathbb{T}}$, which are independent with distribution Exp(1). The model was further studied in the context of large deviations [29], and inhomogeneous time BRW [6]. A more general situation was considered in a recent paper by Ghosh and Mallein[30], where the specific form of X_n was replaced by some exponential bounds on tail behavior and convergence of the extremal process in vague topology was obtained.

The main motivation for considering this model comes from the connection between the supremum of the perturbed BRW $R_n^*(\theta) = \sup_{|v|=n} S_v^*$ and random weighted sums. More precisely, Theorem 3.6 in [5] states, that

$$\theta R_n^*(\theta) \stackrel{d}{=} \log Y_n(\theta) - \log E$$

where $Y_n(\theta) = \sum_{|v|=n} e^{\theta S_v} Y_v$ and E is exponential with parameter 1, independent of $Y_n(\theta)$. The asymptotics of $R_n^*(\theta)$ will be related very closely to the behaviour of $Y_n(\theta)$, which is well described in the literature, see e.g. [13] and [17]. It turns out that properties of $R_n^*(\theta)$ depend strongly on the parameter θ. More precisely, one needs to control its position with respect to the critical parameter θ_0 defined in (1.2).

In [5] branching random walks with such perturbations were studied in the case when μ has finite mean. In particular, the authors proved that

$$\frac{R_n^*(\theta)}{n} \xrightarrow[n\to\infty]{a.s.} \begin{cases} \frac{\nu(\theta)}{\theta} & \theta < \theta_0 \\ \frac{\nu(\theta_0)}{\theta_0} & \theta \geq \theta_0 \end{cases}$$

and identified weak centered asymptotics for $\theta \leq \theta_0$. However, the result for $\theta > \theta_0$ was only obtained for the degenerated perturbations with $\mu = \delta_1$.

Complementing this result and extending the framework beyond the finite mean assumption is another objective of this thesis.

2 Irreducible multi-type branching random walk

In this chapter, we present the results on the irreducible branching random walks. We adopt the notation introduced in the previous chapter and denote

$$\mathbb{P}_i(\cdot) = \mathbb{P}\left(\cdot \mid \text{initial particle is of type } i\right),$$

and $\mathbb{E}_i$ for the expectation with respect to $\mathbb{P}_i$. Whenever the index is omitted, we assume that the initial particle is of type 1. Throughout this chapter, we make the following assumptions on the underlying Galton-Watson process. Firstly, there exists $l \in \mathbb{N}_+$ such that

$$M^l(i,j) > 0 \text{ for all } i,j \in \mathcal{C}. \tag{2.1}$$

This assumption guarantees irreducibility, and through the Perron-Frobenius theorem, it asserts that ρ, the principal eigenvalue of M, is simple. We also assume

$$\rho > 1, \tag{2.2}$$

ensuring that the process survives with positive probability (see [31], Theorem 7.1). Finally, we assume that the Kesten-Stigum condition,

$$\mathbb{E}_i[Z_1^j \log Z_1^j] < \infty, \tag{2.3}$$

holds for all $i,j \in \mathcal{C}$. Under these assumptions, the well known Kesten and Stigum theorem [36] asserts that for any $i \in \mathcal{C}$,

$$\frac{Z_n}{\rho^n} \to Wu \quad \mathbb{P}_i \text{ -a.s.}, \tag{2.4}$$

where W is a non-degenerate random variable and u is the left eigenvector of M. It is a straightforward conclusion that

$$\frac{Z_n}{\rho^n} \cdot v \to W \quad \mathbb{P}_i \text{ -a.s.} \tag{2.5}$$

If we write v for the right eigenvector of M, normalized so that $u \cdot v = 1$, then we also have

$$\mathbb{E}_i[W] = v_i. \tag{2.6}$$

To avoid conditioning on the survival set, we assume $\mathbb{P}(Z_n \to 0) = 0$.

2.1 Displacements with regularly varying tails

Let $F_i(x) = \mathbb{P}(\xi^i \leq x)$. In this section, we assume that the displacements are independent and there exist slowly varying functions $\{L_i\}_{i\in\mathcal{C}}$ and positive constants $\{r_i\}_{i\in\mathcal{C}}$, satisfying

$$\begin{aligned} 1 - F_i(x) \sim L_i(x)x^{-r_i} \quad & \text{as } x \to \infty, \\ \log(-x)F_i(x) \to 0, \quad & \text{as } x \to -\infty. \end{aligned} \tag{2.7}$$

These assumptions are a natural extension of the one-type case considered in [20]. For simplicity, we additionally assume the existence of a unique heaviest tail: There exists $I \in \mathcal{C}$, satisfying $r_I < r_j$ for all $j \neq I$. To simplify the notation, we write $r = r_I$. Our main result is the following theorem.

Theorem 2.1. *Let*

$$\zeta = u_I \sum_{j>0} \rho^{-j} \sum_{l\in\mathcal{C}} \mathbb{P}_I\left(Z_j^l > 0\right).$$

and choose the sequence $\{a_n\}_{n\in\mathbb{N}}$ so that

$$\rho^n \left(1 - F_I(a_n)\right) \xrightarrow[n\to\infty]{} 1. \tag{2.8}$$

Then

$$\mathbb{P}(R_n \leq a_n x) \xrightarrow[n\to\infty]{} \mathbb{E}[e^{-\zeta W x^{-q}}]$$

Remark 2.2. As in the one-type case, the existence of a_n satisfying (2.8) is guaranteed by the result of de Bruijn [16]. If $L^\#$ is the de Bruijn conjugate of L, one can take $a_n = L^\#\left(\rho^{\frac{n}{r}}\right)\rho^{\frac{n}{r}}$. In particular, this guarantees that for any $\varepsilon > 0$,

$$\rho^{\frac{n}{r}(1-\varepsilon)} < a_n < \rho^{\frac{n}{r}(1+\varepsilon)} \tag{2.9}$$

for sufficiently large n.

Remark 2.3. The result partly overlaps with Corollary 3.4 from [8], however there are several differences. We allow the existence of leaves in our tree, and we present a direct argument, in contrast to the result being a conclusion from the convergence of the extremal process. On the other hand, we assume independence of the displacements, as opposed to the more general notion of point processes converging in suitable topology.

Proof of Theorem 2.1. To begin, we present a lemma that characterizes the asymptotic behavior of the total population in an irreducible multi-type Galton-Watson process. For $i \in \mathcal{C}$, let

$$Y_n^i = \left|\bigcup_{k=1}^{n} \{v \in \mathbb{T}_k \ : \ \sigma(v) = i, (\exists w \in \mathbb{T}_n)(w_k = v)\}\right|$$

be the total number of particles of type i that have offspring in the n-th generation.

Lemma 2.4. *Let*

$$\zeta_i = u_i \sum_{j>0} \rho^{-j} \sum_{l\in\mathcal{C}} \mathbb{P}_i \left(Z_j^l > 0\right).$$

Then for all $i \in \mathcal{C}$,

$$\frac{Y_n^i}{\rho^n} \xrightarrow[n\to\infty]{a.s.} \zeta_i W.$$

Proof. Observe

$$Y_n^i = \sum_{j=0}^{n-1} \sum_{l\in\mathcal{C}} \sum_{k=1}^{Z_{n-j}^i} \mathbb{1}_{\{Z_j^l(i,k)>0\}}$$

where for any j and l, $\{Z_j^l(i,k)\}_{k>0}$ are i.i.d. distributed as Z_j^l under $\mathbb{P}_i$, and for $i_1 \neq i_2$, $\{Z_j^l(i_1,k)\}_{k>0}$ are independent of $\{Z_j^l(i_2,k)\}_{k>0}$. Hence

$$\frac{Y_n^i}{\rho^n} = \sum_{j=0}^{n-1} \rho^{-j} \sum_{l\in\mathcal{C}} \frac{Z_{n-j}^i}{\rho^{n-j}} \frac{1}{Z_{n-j}^i} \sum_{k=1}^{Z_{n-j}^i} \mathbb{1}_{\{Z_j^l(i,k)>0\}}$$

Denote $D_{n-j} = \frac{Z_{n-j}^i}{\rho^{n-j}}$ and $E_{n-j}^l = \frac{1}{Z_{n-j}^i} \sum_{k=1}^{Z_{n-j}^i} \mathbb{1}_{\{Z_j^l(i,k)>0\}}$. For any fixed $j > 0$, by the strong law of large numbers, $E_{n-j}^l \xrightarrow[n\to\infty]{a.s.} \mathbb{P}_i(Z_j^l > 0)$, and by (2.4), $D_{n-j} \xrightarrow[n\to\infty]{a.s.} W u_i$.

Now fix $N > 0$. Then for $n > N$

$$\begin{aligned}
\frac{Y_n^i}{\rho^n} &\leq \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l + \sum_{l\in\mathcal{C}} \sup_{k\geq N+1} \{D_k\} \sum_{j=N+1}^{\infty} \rho^{-j} \\
&= \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l + d \frac{\rho^{-N}}{\rho - 1} \sup_{k\geq N+1} \{D_k\}
\end{aligned}$$

So $\mathbb{P}$-almost surely

$$\limsup_n \frac{Y_n^i}{\rho^n} \leq W u_i \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} \mathbb{P}_i(Z_j^l > 0) + d \frac{\rho^{-N}}{\rho - 1} \sup_{k\geq N+1} \{D_k\}$$

Letting $N \to \infty$ we get

$$\limsup_n \frac{Y_n^i}{\rho^n} \leq \zeta_i W \quad \mathbb{P} - \text{a.s.}$$

For the bound from below, we note that

$$\frac{Y_n^i}{\rho^n} \geq \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l$$

so taking $\liminf_n$ and then letting $N \to \infty$ we get

$$\liminf_n \frac{Y_n^i}{\rho^n} \geq \zeta_i W \quad \text{a.s.}$$

which concludes the proof of the lemma. □

Now let

$$M_n = \max\{\xi_{v_k} \ : \ v \in \mathbb{T}_n, k \leq n\}$$

Since up to the n-th generation, for any $i \in \mathcal{C}$, there are Y_n^i displacements with distribution F_i, and all the displacements are independent of each other, we have

$$\mathbb{P}\left(M_n \leq a_n x\right) = \mathbb{E}\left[\prod_{i\in\mathcal{C}} F_i(a_n x)^{Y_n^i}\right]$$

Note that

$$F_i(a_n x)^{Y_n^i} = \exp\left\{\frac{Y_n^i}{\rho^n}\rho^n \log F_i(a_n x)\right\} \tag{2.10}$$

Now, a_n was chosen so that $\rho^n(1 - F_I(a_n)) \xrightarrow[n\to\infty]{} 1$. Furthermore, $1 - F_i(z) \sim L_i(z)z^{-r_i}$ as $z \to \infty$, where L_i is slowly varying. Then, using the fact that for z close to 0, $\log(1 + z) \sim z$, we have

$$\rho^n \log F_I(a_n x) \sim -\rho^n(1 - F_I(a_n x)) = -\rho^n(1 - F_I(a_n))\frac{(1 - F_I(a_n x))}{(1 - F_I(a_n))} \xrightarrow[n\to\infty]{} -x^{-r}. \tag{2.11}$$

Applying Lemma 2.4 and (2.11) in (2.10) yields

$$F_I(a_n x)^{Y_n^I} \xrightarrow[n\to\infty]{a.s.} \exp\left\{ -\zeta W x^{-r}\right\}.$$

Similarly, for $i \neq I$,

$$\begin{aligned}\rho^n \log F_i(a_n x) &\sim \rho^n(1 - F_i(a_n x)) \\ &= \rho^n(1 - F_I(a_n x))\frac{(1 - F_i(a_n x))}{(1 - F_I(a_n x))} \sim x^{-r}(a_n x)^{r-r_i}\frac{L_i(a_n x)}{L_I(a_n x)}\end{aligned} \tag{2.12}$$

which converges to 0, because $r < r_i$ and L_i's are slowly varying. Again, by Lemma 2.4 $\rho^{-n}Y_n^i$ has a finite limit, so (2.12) yields $F_i(a_n x)^{Y_n^i} \xrightarrow[n\to\infty]{\mathbb{P}-a.s.} 1$ for $i \neq I$. Hence,

$$\prod_{i\in\mathcal{C}} F_i(a_n x)^{Y_n^i} \xrightarrow[n\to\infty]{a.s.} \exp\left\{ -\zeta W x^{-r}\right\}$$

and using the dominated convergence theorem, we have shown

$$\mathbb{P}\left(M_n \leq a_n x\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left[\exp\left\{ -\zeta W x^{-r}\right\}\right] \tag{2.13}$$

To finish the proof, we need to show that $\mathbb{P}\left(M_n \leq a_n x\right) \sim \mathbb{P}\left(R_n \leq a_n x\right)$ as $n \to \infty$. Observe that for any $\varepsilon > 0$

$$\mathbb{P}\left(R_n > a_n x\right) \leq \mathbb{P}\left(M_n > a_n(x-\varepsilon)\right) + \mathbb{P}\left(R_n > a_n x, M_n \leq a_n(x-\varepsilon)\right)$$

and

$$\mathbb{P}(R_n > a_n x) \geq \mathbb{P}(M_n > a_n(x+\varepsilon)) - \mathbb{P}(R_n \leq a_n x, M_n > a_n(x+\varepsilon)).$$

Hence, it suffices to show

$$\mathbb{P}\left(R_n > a_n x, M_n \leq a_n(x-\varepsilon)\right) \xrightarrow[n\to\infty]{} 0 \tag{2.14}$$

and

$$\mathbb{P}(R_n \leq a_n x, M_n \geq a_n(x+\varepsilon)) \xrightarrow[n\to\infty]{} 0. \tag{2.15}$$

We start by showing (2.14). First observe

$$\mathbb{P}(R_n > a_n x, M_n \leq a_n(x-\varepsilon)) \leq \mathbb{E}\left[Z_n(a_n x, \infty)\mathbb{1}_{M_n \leq a_n(x-\varepsilon)}\right]$$

where $Z_n(a_n x, \infty)$ is the number particles in the n-th generation, that are positioned above $a_n x$. We now need to introduce some new notation: Denote $F_{i,n}$ for n-th convolution of F_i (the distribution function of a sum of n independent random variables distributed as F_i). Furthermore, for $\vec{n} = (n_1, n_2, \ldots, n_d)$, let

$$F_{\vec{n}}(x) = F_{1,n_1} * F_{2,n_2} * \cdots * F_{d,n_d}(x)$$

For a distribution function F and $x, y \in \mathbb{R}$, let $F^y(x) = F(x) \wedge F(y)$ be the distribution function F trimmed at y. Note that if S_n is a random walk with step distribution F, then

$$F^y_n(x) = \mathbb{P}(S_n < x, \sup_{1\leq k\leq n} S_k - S_{k-1} < y) \tag{2.16}$$

where F^y_n is the n-th convolution of F^y. Now, for a particle in the n-th generation, which had n_i ancestors of type i, with $n = \sum_{i\in\mathcal{C}} n_i$, the probability of it ending up in $(a_n x, \infty)$, while all the displacements on the path are smaller than $a_n(x-\varepsilon)$, is

$$F_{\vec{n}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{n}}^{(a_n(x-\varepsilon))}(a_n x).$$

Let $A_{\vec{n}}$ be the expected number of particles in the n-th generation, that had n_i ancestors of type i for each respective $i \in \mathcal{C}$. Then

$$\mathbb{E}[Z_n(a_n x, \infty)\mathbb{1}_{M_n \leq a_n(x-\varepsilon)}] \leq \sum_{\vec{n}} A_{\vec{n}} \left(F_{\vec{n}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{n}}^{(a_n(x-\varepsilon))}(a_n x) \right) \tag{2.17}$$

Here we want to apply inequality (1) from step 3 of the proof in [20]. It states that for a regularly varying distribution function F with exponent r, all $x, \varepsilon, \delta > 0$ and $s \in (0, r)$, and a constant $C > 0$, we have

$$F_n^{(a_n(x-\varepsilon))}(\infty) - F_n^{(a_n(x-\varepsilon))}(a_n x) \leq C \left(\frac{nC_s}{a_n^s (x-\varepsilon)^s} \right)^{\frac{x(1-\delta)}{(x-\varepsilon)}} \tag{2.18}$$

for all n. where C_s is a constant depending only on s. This is not immediately applicable in our case, as $F_{\vec{n}}$ is a convolution consisting of a number of different distributions. However, the statement can be easily generalized to our case as long as all distributions satisfy the requirements. To see that this is true, first note that the aforementioned result in [20] is based on a more general bound obtained in the proof of Lemma 3 in [19]. To generalize the bound to the case with mixed distributions, note that the proof relies on the observation that for $h > 0$, and $x > y$,

$$F_n^y(\infty) - F_n^y(x) \leq R(h, y)^n \exp(-hx)$$

where

$$R(h, y) = \int_{-\infty}^{y} e^{hu} F^y(du).$$

The conclusion is then the result of the bounds on $R(h, y)$. In the case of mixed distributions, we can obtain a similar inequality. That is, let $R_i(h, y) = \int_{-\infty}^{y} e^{hu} F_i^y(du)$. Then

$$F_{\vec{n}}^y(\infty) - F_{\vec{n}}^y(x) = \int_x^{\infty} e^{-hu} e^{hu} dF_{\vec{n}}^y(u) \leq e^{-hx} \int_{-\infty}^{\infty} e^{hu} dF_{\vec{n}}^y(u) = e^{-hx} \int_{-\infty}^{ny} e^{hu} dF_{\vec{n}}^y(u). \tag{2.19}$$

Now fix any $i \in \mathcal{C}$. Integrating by parts and exchanging integrals, we get

$$\begin{aligned}
\int_{-\infty}^{ny} e^{hu} dF_{\vec{n}}^{y}(u) &= e^{hny} \prod_{j=1}^{d} F_j(y)^{n_j} - \int_{-\infty}^{ny} F_{\vec{n}}^{y}(u) de^{hu} \\
&= e^{hny} \prod_{j=1}^{d} F_j(y)^{n_j} - \int_{-\infty}^{ny} \left[\int_{-\infty}^{\infty} F_{i,n_i}^{y}(u-z) dF_{\vec{n}/n_i}^{y}(z) \right] de^{hu} \\
&= e^{hny} \prod_{j=1}^{d} F_j(y)^{n_j} - \int_{-\infty}^{(n-n_i)y} \left[\int_{-\infty}^{ny} F_{i,n_i}^{y}(u-z) de^{hu} \right] dF_{\vec{n}/n_i}^{y}(z)
\end{aligned} \tag{2.20}$$

where

$$F_{\vec{n}/n_i}^{y}(z) = F_{1,n_1} * \cdots * F_{i-1,n_{i-1}} * F_{i+1,n_{i+1}} * \cdots * F_{d,n_d}(x).$$

Integrating by parts again,

$$\begin{aligned}
&\int_{-\infty}^{(n-n_i)y} \left[\int_{-\infty}^{ny} F_{i,n_i}^{y}(u-z) de^{hu} \right] dF_{\vec{n}/n_i}^{y}(z) \\
&= \int_{-\infty}^{(n-n_i)y} \left[e^{hny} F_{i,n_i}^{y}(ny-z) - \int_{-\infty}^{ny} e^{hu} dF_{i,n_i}^{y}(u-z) \right] dF_{\vec{n}/n_i}^{y}(z) \\
&= \int_{-\infty}^{(n-n_i)y} e^{hny} F_{i,n_i}^{y}(ny-z) dF_{\vec{n}/n_i}^{y}(z) - \int_{-\infty}^{(n-n_i)y} \int_{-\infty}^{ny} e^{hu} dF_{i,n_i}^{y}(u-z) dF_{\vec{n}/n_i}^{y}(z) \\
&= e^{hny} F_{i,n_i}^{y}(n_i y) F_{\vec{n}/n_i}^{y}((n-n_i)y) - \int_{-\infty}^{(n-n_i)y} \int_{-\infty}^{ny} e^{hu} dF_{i,n_i}^{y}(u-z) dF_{\vec{n}/n_i}^{y}(z) \\
&= e^{hny} \prod_{j=1}^{d} F_j(y)^{n_j} - \int_{-\infty}^{(n-n_i)y} e^{hz} \int_{-\infty}^{ny-z} e^{hw} dF_{i,n_i}^{y}(w) dF_{\vec{n}/n_i}^{y}(z)
\end{aligned} \tag{2.21}$$

The last two equalities are justified by the fact that if $x > ny$, then $F_{\vec{n}}^{y}(x) = \prod_{j=1}^{d} F_j(y)^{n_j}$ (see (2.16)). Similarly, the inner integral in the last line only goes up to $n_i y$, as $z \le (n-n_i)y$ and $F_{i,n_i}^{y}(w)$ is constant for $w \ge n_i y$. Using an analogous procedure of integrating by parts, expanding the convolution (this time with respect to F_i^y and F_{i,n_i-1}^{y}), interchanging the integrals, and integrating by parts again, we similarly obtain the following equality.

$$\begin{aligned}
\int_{-\infty}^{n_i y} e^{hw} dF_{i,n_i}^{y}(w) &= \int_{-\infty}^{(n_i-1)y} e^{hz} \int_{-\infty}^{y} e^{hw} dF_i^{y}(w) dF_{i,n_i-1}^{y}(z) \\
&= \int_{-\infty}^{(n_i-1)y} e^{hz} R_i(h,y) dF_{i,n_i-1}^{y}(z)
\end{aligned}$$

Repeating $n_i - 1$ times,

$$\int_{-\infty}^{n_i y} e^{hw} dF_{i,n_i}^{y}(w) = R_i(h,y)^{n_i}.$$

Plugging this into (2.21), we obtain

$$\int_{-\infty}^{(n-n_i)y} \left[\int_{-\infty}^{ny} F_{i,n_i}^{y}(u-z) de^{hu} \right] dF_{\vec{n}/n_i}^{y}(z)$$
$$= e^{hny} \prod_{j=1}^{d} F_j(y)^{n_j} - \int_{-\infty}^{(n-n_i)y} e^{hz} R_i(h,y)^{n_i} dF_{\vec{n}/n_i}^{y}(z)$$

and together with (2.20), this yields

$$\int_{-\infty}^{ny} e^{hu} dF_{\vec{n}}^{y}(u) = \int_{-\infty}^{(n-n_i)y} e^{hz} R_i(h,y)^{n_i} dF_{\vec{n}/n_i}^{y}(z).$$

Iterating the whole procedure $d-1$ times to cycle through all types and applying the result in (2.19), we obtain the following result.

$$F_{\vec{n}}^{y}(\infty) - F_{\vec{n}}(x) \leq \exp(-hx) \prod_{i=1}^{d} R_i(h,y)^{n_i},$$

Letting $R(h,y) = \max_{i \in \mathcal{C}} R_i(h,y)$, we can write

$$F_{\vec{n}}^{y}(\infty) - F_{\vec{n}}(x) \leq \exp(-hx) R(h,y)^n.$$

Applying to $R(h,y)$ the same bounds as in the proof of Lemma 3 in [19], and then the truncation argument from Step 3 of the proof in [20] to adapt the result to regularly varying distributions (and noting that $r = \min_i r_i$, $s < r$ implies $s < r_i$ for all i), we see that (2.19) indeed holds for mixed distributions. Hence,

$$\left(F_{\vec{n}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{n}}^{(a_n(x-\varepsilon))}(a_n x) \right) \leq C \left(\frac{nC_s}{a_n^s (x-\varepsilon)^s} \right)^{\frac{x(1-\delta)}{(x-\varepsilon)}} \tag{2.22}$$

Now and choose δ small enough so that $\theta = \frac{x(1-\delta)}{(x-\varepsilon)} > 1$, and take $p \in (r,s)$ satisfying $\frac{s}{p}\theta > 1$. The for some $C' > 0$ (see Remark 2.2),

$$C \left(\frac{nC_s}{a_n^s (x-\varepsilon)^s} \right)^{\theta} \leq C' \rho^{-\frac{s}{p}\theta n} \left(\frac{nC_s}{(x-\varepsilon)^s} \right)^{\theta}$$

Then for $p' \in (1, \frac{s}{p}\theta)$ and suitable $C'' > 0$, we have for all $n > 0$

$$C' \rho^{-\frac{s}{p}\theta n} \left(\frac{nC_s}{(x-\varepsilon)^s} \right)^\theta \leq C'' \rho^{-np'}$$

Ultimately we get

$$\left(F_{\vec{n}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{n}}^{(a_n(x-\varepsilon))}(a_n x) \right) \leq C'' \rho^{-np'}.$$

Hence,

$$\mathbb{E}[Z_n(a_n x, \infty) \mathbb{1}_{M_n \leq a_n(x-\varepsilon)}] \leq C'' \rho^{-np'} \sum_{\vec{n}} A_{\vec{n}} = C'' \rho^{-np'} \mathbb{E}[|Z_n|] = C'' \rho^{-np'} |M_n Z_0|.$$

Since $\rho^{-n} M^n$ has a finite limit, and $p' > 1$, we get

$$C'' \rho^{-np'} |M_n Z_0| \xrightarrow[n \to \infty]{} 0.$$

Thus, we have proved (2.14).

Denote by η^n one of the nth generation particles that descend from a path on which M_n was attained, by Q_n its position, and by $T(i,n)$ the number of its ancestors of type i, excluding the particle attaining M_n. Note that $\sum_{i=1}^d T(i,n) = n-1$, and let $\vec{T}(n) = (T(1,n), T(2,n), \ldots, T(d,n))$. For a distribution function F and $y \in \mathbb{R}$, let $\bar{F}^y(x) = \frac{F^y(x)}{F(y)}$ and denote $\bar{F}_n^y$ the n-th convolution of $\bar{F}^y$. Note that if S_n is a random walk with step distribution F, then

$$\bar{F}_n^y(x) = \frac{F_n^y(x)}{F(y)^n} = \mathbb{P}(S_n < x | \sup_{1 \leq k \leq n} S_k - S_{k-1} < y).$$

With this notation $Q_n - M_n$ has the following distribution function.

$$\mathbb{P}(Q_n - M_n \leq x) = \mathbb{E}\left[\bar{F}_{\vec{T}(n)}^{M_n}(x) \right],$$

That is, it is distributed as a sum of $T(i,n)$ steps from the distributions F_i, respectively, conditioned on the fact that they are all smaller from M_n. Now,

$$\begin{aligned}\mathbb{P}(R_n \le a_n x, M_n \ge a_n(x+\varepsilon)) &\le \mathbb{P}(Q_n \le a_n x, M_n \ge a_n(x+\varepsilon)) \\ &= \mathbb{P}(Q_n - M_n \le a_n x - M_n, M_n \ge a_n(x+\varepsilon)) \\ &\le \mathbb{P}(Q_n - M_n \le a_n x - a_n(x+\varepsilon), M_n \ge a_n(x+\varepsilon)) \\ &\le \mathbb{P}(Q_n - M_n \le -a_n\varepsilon) \\ &= \mathbb{E}\left[\bar{F}^{M_n}_{\vec{T}(n)}(-a_n\varepsilon)\right] \\ &= \mathbb{E}\left[\bar{F}^{M_n}_{\vec{T}(n)}(-a_n\varepsilon)\mathbb{1}_{\{M_n\le 0\}}\right] + \mathbb{E}\left[\bar{F}^{M_n}_{\vec{T}(n)}(-a_n\varepsilon)\mathbb{1}_{\{M_n>0\}}\right] \\ &\le \mathbb{P}(M_n \le 0) + \mathbb{E}\left[\bar{F}^{0}_{\vec{T}(n)}(-a_n\varepsilon)\right].\end{aligned}$$

Since $\mathbb{P}(M_n \le 0) \to 0$, we only need to take care of the second term. To do this, write

$$\mathbb{E}\left[\bar{F}^{0}_{\vec{T}(n)}(-a_n\varepsilon)\right] = \mathbb{E}\left[\int_{-\infty}^{\infty} \bar{F}^{0}_{\vec{T}(n)/T(1,n)}(-a_n\varepsilon - y)\bar{F}^{0}_{T(1,n)}(dy)\right]. \tag{2.23}$$

Now choose $0 < \delta_1 < \varepsilon$ and split the integral at the point $-a_n\delta_1$. Then

$$\begin{aligned}\mathbb{E}\left[\int_{-\infty}^{-a_n\delta_1} \bar{F}^{0}_{\vec{T}(n)/T(1,n)}(-a_n\varepsilon - y)\bar{F}^{0}_{T(1,n)}(dy)\right] &\le \mathbb{E}\left[\bar{F}^{0}_{T(1,n)}(-a_n\delta_1)\right] \\ &\le \mathbb{E}\left[\bar{F}^{0}_{n}(-a_n\delta_1)\right]\end{aligned} \tag{2.24}$$

and

$$\mathbb{E}\left[\int_{-a_n\delta_1}^{\infty} \bar{F}^{0}_{\vec{T}(n)/T(1,n)}(-a_n\varepsilon - y)\bar{F}^{0}_{T(1,n)}(dy)\right] \le \mathbb{E}\left[\bar{F}^{0}_{\vec{T}(n)/T(1,n)}(-a_n\delta_1)\right]. \tag{2.25}$$

Note that the last expression in (2.25) is of the same form as the term we started with in (2.23), except we exchanged ε for δ_1 and eliminated type 1. Therefore, applying (2.24) and (2.25) d times with $\delta_d < \cdots < \delta_1 < \varepsilon$, we get

$$\mathbb{E}\big[\bar{F}^{0}_{\vec{T}(n)}(-a_n\varepsilon)\big] \le \sum_{i\in\mathcal{C}} \mathbb{E}\left[\bar{F}^{0}_{n}(-a_n\delta_i)\right]$$

To see that $\bar{F}^0_n(-a_n\delta) \to 0$ as $n \to \infty$, we refer to Step 4 of the proof in [20]. We note that the arguments provided there are based only on the condition $\log(x)F(-x) \to 0$ as $x \to \infty$, and the fact that a_n grows exponentially fast, so they are also applicable here. This holds for all $i \in \mathcal{C}$, so by the bounded convergence theorem, the whole expression converges to 0.

This concludes the proof of (2.15), and thus of the theorem. □

2.2 Displacements with semi-exponential tails

In this section we assume that the displacements are independent and admit semi-exponential tails:

$$\mathbb{P}\left(\xi^j \geq t\right) = a_j(t)\exp\{-L_j(t)t^{r_j}\}, \tag{2.26}$$

where L_j, a_j are slowly varying functions such that $\frac{L_j(t)}{t^{1-r_j}}$ are eventually nonincreasing, and $r_j \in (0,1)$. We also assume that they have finite moments. These assumptions are analogous to the one-type model studied by Gantert in [28]. In this section, we show an analogous limit theorem for irreducible multi-type branching random walk.

Our result is as follows.

Theorem 2.5. *Let $r = \min\{r_i : i \in \mathcal{C}\}$, $L(t) = \min\{L_i(t) : r = r_i\}$, and choose $\psi(n)$ to be a positive function satisfying*

$$\frac{L(\psi(n))\psi(n)^r}{n} \to 1. \tag{2.27}$$

Then

$$\frac{R_n}{\psi(n)} \xrightarrow{a.s.} (\log\rho)^{\frac{1}{r}}.$$

Remark 2.6. As in the one type case, the existence of $\psi(n)$ satisfying (2.27) is guaranteed by the result of de Bruijn [16]. Indeed, if $K(x)$ is the de Bruijn conjugate of $x \mapsto L\left(x^{\frac{1}{r}}\right)$, then we can take $\psi(n) = K(n)^{\frac{1}{r}} n^{\frac{1}{r}}$. In particular, this implies that for any $\varepsilon > 0$,

$$n^{\frac{1}{r}(1-\varepsilon)} \leq \psi(n) \leq n^{\frac{1}{r}(1+\varepsilon)}$$

for large enough n.

We will also show the following lemmas, which describe the asymptotic behavior of the underlying multi-type Galton-Watson process.

Lemma 2.7. *Let $|Z_n| = \sum_{r\in\mathcal{C}} Z_n^r$ be the sum of all particles in the n-th generation of the process. Then for any $\varepsilon > 0$ there is $0 < \delta \leq \varepsilon$ satisfying*

$$\mathbb{P}\left(\frac{|Z_n|}{\rho^n} < (1-\varepsilon)^n\right) < (1-\delta)^n$$

for all n large enough.

Lemma 2.8. *There exist $\delta > 0$ and $\beta \in (0,1)$, such that for all $i \in \mathcal{C}$ and all n large enough*

$$\mathbb{P}(Z_n^i < \delta|Z_{n-l}|) \leq \beta^n,$$

where $l \in \mathbb{N}$ is as in (2.1).

Proof of the lemma 2.7. First note that if $\rho(1-\varepsilon) \le 1$, then $\mathbb{P}\left(\frac{|Z_n|}{\rho^n} < (1-\varepsilon)^n\right) = 0$ and the statement is trivial. Assume $\rho(1-\varepsilon) > 1$. The key tool to proving the lemma is the result of Athreya and Vidyashankar ([4], Theorem 2.6), which we state below.

Lemma 2.9. *Additionally to our standing assumptions, assume*

$$\text{there exists } \theta_0 > 0, \text{ such that } \mathbb{E}_i\left[\exp\{\theta_0 Z_1^j\}\right] < \infty \text{ for all } i, j \in \mathcal{C}. \tag{2.28}$$

Then there are constants $C > 0$, $\lambda > 0$ such that for any $\varepsilon > 0$

$$\mathbb{P}\left(\left|\frac{Z_n}{\rho^n} \cdot v - W\right| \ge \varepsilon\right) < C \exp\{-\lambda \left(\varepsilon^2 \rho^n\right)^{\frac{1}{3}}\} \tag{2.29}$$

for all n.

Although its stated in [4] for two types, it is clear from the proof that the same argument holds for an arbitrary number of types. Another important inequality we will use is a straightforward consequence of results from Jones [35] describing the small-value probabilities of W: there exists $\alpha > 0$, such that for small enough $\varepsilon > 0$,

$$\mathbb{P}(W \le \varepsilon) \le \varepsilon^\alpha. \tag{2.30}$$

Since we do not assume the existence of exponential moments (2.28), some additional steps are required to use Lemma 2.9. Consider a trimmed Galton-Watson process $Z_n(L)$ generated by random variables $N_{i,j}(L) = N_{i,j}\mathbb{1}_{\{N_{i,j}<L\}}$ for some $L > 0$ and denote by $M(L)$ its mean matrix, by $\rho(L)$ its largest eigenvalue, and by $v(L)$ its left eigenvector. Since $\rho(L) \to \rho$ as $L \to \infty$, we choose L large enough so that $\rho(L) > (1-\varepsilon)\rho$. Clearly,

$$\mathbb{P}\left(\frac{|Z_n|}{\rho^n} < (1-\varepsilon)^n\right) \le \mathbb{P}\left(\frac{|Z_n(L)|}{\rho^n} < (1-\varepsilon)^n\right).$$

Now choose $\delta > 0$ satisfying

$$(1-\delta)\rho(L) \ge (1-\varepsilon)\rho, \quad (1-\delta)^2\rho(L) > 1$$

so that

$$\mathbb{P}\left(\frac{|Z_n(L)|}{\rho^n} < (1-\varepsilon)^n\right) \le \mathbb{P}\left(\frac{|Z_n(L)|}{\rho(L)^n} < (1-\delta)^n\right)$$

Since $Z_n(L)$ satisfies the same assumptions we make on Z_n in this chapter, then (2.4) holds for some $W(L)$ and (2.30) holds for appropriate choice of constants.

Note that $|Z_n(L)| \geq \frac{1}{||v||_\infty} Z_n(L) \cdot v(L)$, and observe

$$\begin{aligned}
&\mathbb{P}\left(\frac{|Z_n(L)|}{\rho(L)^n} < (1-\delta)^n\right) \leq \mathbb{P}\left(\frac{Z_n(L) \cdot v(L)}{\rho(L)^n} < ||v(L)||_\infty (1-\delta)^n\right) \\
&= \mathbb{P}\left(\frac{Z_n(L) \cdot v(L)}{\rho(L)^n} < ||v(L)||_\infty (1-\delta)^n,\ W(L) \geq \frac{3||v(L)||_\infty}{2}(1-\delta)^n\right) \\
&+ \mathbb{P}\left(\frac{Z_n(L) \cdot v(L)}{\rho(L)^n} < ||v(L)||_\infty (1-\delta)^n,\ W(L) < \frac{3||v(L)||_\infty}{2}(1-\delta)^n\right) \\
&\leq \mathbb{P}\left(\left|\frac{Z_n(L) \cdot v(L)}{\rho(L)^n} - W(L)\right| \geq \frac{||v(L)||_\infty}{2}(1-\delta)^n\right) \\
&+ \mathbb{P}\left(W(L) < \frac{3||v(L)||_\infty}{2}(1-\delta)^n\right)
\end{aligned}$$

By (2.30), the second term is bounded by $\left(\frac{3||v(L)||_\infty}{2}\right)^\alpha (1-\delta)^{\alpha n}$ for $\alpha > 0$ and n large enough. The bound for the first term follows from Lemma 2.9. Since $Z_n(L)$ satisfies (2.28) as well as our standing assumptions on Z_n, we conclude from 2.9, that for appropriate $C > 0$, $\lambda > 0$

$$\mathbb{P}\left(\left|\frac{Z_n(L) \cdot v(L)}{\rho(L)^n} - W(L)\right| \geq \frac{||v(L)||_\infty}{2}(1-\delta)^n\right) \leq C \exp\left\{-\lambda\left((1-\delta)^2 \rho(L)\right)^{\frac{n}{3}}\right\},$$

and for large enough n,

$$C \exp\left\{-\lambda\left((1-\delta)^2 \rho(L)\right)^{\frac{n}{3}}\right\} \leq (1-\delta)^{\alpha n}.$$

Since $\left(1 + \left(\frac{3||v(L)||_\infty}{2}\right)^\alpha\right)(1-\delta)^{\alpha n} < (1-\delta_0)^n$ for some $\delta_0 < \delta$ and large enough n, the lemma is proven. $\square$

Proof of the Lemma 2.8. Fix $i \in \mathcal{C}$ and denote by $Z_l^{r \to i}$ a generic random variable distributed as Z_l^i under $\mathbb{P}_r$. Recall that l is a natural number for which M^l has only strictly positive entries (see assumption (2.1)). Consequently, $q_{r,i} = \mathbb{P}\left(Z_l^{r \to i} = 0\right) < 1$ for all $r \in \mathcal{C}$. Let $q_{\max} = \max\{q_{r,i} : r \in \mathcal{C}\}$ and $q_{\min} = \min\{q_{r,i} : r \in \mathcal{C}\}$. Then

$$\mathbb{P}\left(Z_n^i < \delta|Z_{n-l}|\right) = \mathbb{P}\left(\sum_{r \in \mathcal{C}} \sum_{m=1}^{Z_{n-l}^r} Z_l^{r \to i}(m) < \delta|Z_{n-l}|\right) = \mathbb{E}\left[\Phi(Z_{n-l})\right]$$

where for fixed r, $\{Z_l^{r \to i}(m)\}_{m \geq 1}$ are independent copies of $Z_l^{r \to i}$, and for $r_1 \neq r_2$, $\{Z_l^{r_1 \to i}(m)\}_{m \geq 1}$ are independent of $\{Z_l^{r_2 \to i}(m)\}_{m \geq 1}$, and

$$\Phi(k) = \mathbb{P}\left(\sum_{r \in \mathcal{C}} \sum_{m=1}^{k_r} Z_l^{r \to i}(m) < \delta|k|\right)$$

for $k = (k_1, \dots, k_d) \in \mathbb{N}^d$. For a vector $j = (j_1, \dots, j_d) \in \mathbb{N}^d$ satisfying $j_r \le k_r$ for all $r \in \mathcal{C}$, define

$$A_{(j_1,\dots,j_d)} = \bigcap_{r\in\mathcal{C}} \left\{ \left| \left\{ m \le k_r : Z_l^{r\to i}(m) > 0 \right\} \right| = j_r \right\}.$$

Observe

$$\begin{aligned}\mathbb{P}(A_{(j_1,\dots,j_d)}) &= \prod_{r\in\mathcal{C}} \binom{k_r}{j_r} q_{r,i}^{k_r - j_r} (1-q_{r,i})^{j_r} \le \prod_{r\in\mathcal{C}} \binom{k_r}{j_r} q_{\max}^{k_r-j_r} (1-q_{\min})^{j_r} \\ &= \prod_{r\in\mathcal{C}} \binom{k_r}{j_r} q_{\max}^{k_r} \left(\frac{1-q_{\min}}{q_{\max}} \right)^{j_r} = q_{\max}^{|k|} C^{|j|} \prod_{r\in\mathcal{C}} \binom{k_r}{j_r}\end{aligned}$$

where $C = \frac{1-q_{\min}}{q_{\max}}$. By the generalized Vandermonde's identity, for any $j \in \mathbb{N}_+$

$$\sum_{j_1+\dots+j_d=j} \prod_{r\in\mathcal{C}} \binom{k_r}{j_r} = \binom{|k|}{j}$$

where the sum $\sum_{j_1+\dots+j_d=j}$ goes over all partitions of j, and we put $\binom{n}{m} = 0$ if $m > n$. Hence

$$\begin{aligned}\Phi(k) &= \sum_{j=1}^{\infty} \sum_{j_1+\dots+j_d=j} \mathbb{P}\left(\sum_{r\in\mathcal{C}} \sum_{m=1}^{k_r} Z_l^{r\to i}(m) < \delta|k|,\ A_{j_1,\dots,j_2} \right) \\ &\le \sum_{j=1}^{\infty} \sum_{j_1+\dots+j_d=j} \mathbb{P}\left(j < \delta|k|,\ A_{j_1,\dots,j_2} \right) = \sum_{j=1}^{\lfloor \delta|k| \rfloor} \sum_{j_1+\dots+j_d=j} \mathbb{P}\left(A_{j_1,\dots,j_2} \right) \\ &\le \sum_{j=1}^{\lfloor \delta|k| \rfloor} \sum_{j_1+\dots+j_d=j} q_{\max}^{|k|} C^j \prod_{r\in\mathcal{C}} \binom{k_r}{j_r} = q_{\max}^{|k|} \sum_{j=1}^{\lfloor \delta|k| \rfloor} C^j \binom{|k|}{j} \\ &\le q_{\max}^{|k|} \sum_{j=1}^{\lfloor \delta|k| \rfloor} C^j \frac{|k|^j}{j!} = q_{\max}^{|k|} \sum_{j=1}^{\lfloor \delta|k| \rfloor} \left(\frac{C}{\delta} \right)^j \frac{(\delta|k|)^j}{j!}.\end{aligned}$$

Choosing δ so that $\frac{C}{\delta} > 1$, we have

$$\begin{aligned}q_{\max}^{|k|} \sum_{j=1}^{\lfloor \delta|k| \rfloor} \left(\frac{C}{\delta} \right)^j \frac{(\delta|k|)^j}{j!} &\le q_{\max}^{|k|} \left(\frac{C}{\delta} \right)^{\lfloor \delta|k| \rfloor} \sum_{j=1}^{\lfloor \delta|k| \rfloor} \frac{(\delta|k|)^j}{j!} \le q_{\max}^{|k|} \left(\frac{C}{\delta} \right)^{\delta|k|} e^{\delta|k|} \\ &= \left(q_{\max} \left(\frac{Ce}{\delta} \right)^{\delta} \right)^{|k|}\end{aligned}$$

Since $\left(\frac{Ce}{\delta}\right)^{\delta} \to 1$ as $\delta \to 0$ and $q_{\max} < 1$, choosing δ small enough we have

$$\Phi(k) \leq \beta_0^{|k|}$$

for $\beta_0 < 1$. Hence

$$\mathbb{P}\left(Z_n^i < \delta | Z_{n-l}|\right) = \mathbb{E}[\Phi(Z_{n-l})] \leq \mathbb{E}[\beta_0^{Z_{n-l}}] \leq \beta_0^n + \mathbb{P}(Z_{n-l} < n)$$

Since for any ε satisfying $\rho(1-\varepsilon) > 1$ we have $n < \rho^n(1-\varepsilon)^n$ for large enough n, the bound

$$\mathbb{P}(Z_{n-l} < n) < C_1 \beta_0^n$$

is a straightforward conclusion from Lemma 2.7. Hence

$$\mathbb{P}\left(Z_n^i < \delta | Z_{n-l}|\right) \leq (1 + C_1)\beta_0^n \leq \beta^n$$

for some $\beta < 1$ and all large enough n. □

Proof of the Theorem. We start with the upper bound. Let η be a random variable with the distribution function

$$F(x) = \begin{cases} 1 - \max\left\{a_i(x)\exp\{-L_i(x)x^r\} \ : \ i \in \mathcal{C}, r_i = r\right\} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

By choice of r and L, there exists a constant $c > 0$ such that for all $t > c$ and $i \in \mathcal{C}$

$$\mathbb{P}(\eta \geq t) \geq \mathbb{P}(\xi^i \geq t).$$

Hence, $\eta^c = \eta \mathbb{1}_{\{\eta > c\}} + c\mathbb{1}_{\{\eta \leq c\}}$ dominates stochastically $\xi^{i,c} = \xi^i \mathbb{1}_{\{\xi^i > c\}} + c\mathbb{1}_{\{\xi^i \leq c\}}$ for all $i \in \mathcal{C}$. Since stochastic dominance is preserved under convolution, we have that for any $x > 0$, $n \in \mathbb{N}$ and $v \in \mathbb{T}_n$

$$\mathbb{P}\left(\sum_{k=1}^{n} \eta_k^c \geq x\right) \geq \mathbb{P}\left(\sum_{k=1}^{|v|} \xi_{v_k}^c \geq x\right),$$

where $\{\eta_k^c\}_{k \geq 0}$ are i.i.d. distributed as η^c. Then

$$\mathbb{P}(S_v \geq \psi(n)x) \leq \mathbb{P}\left(\sum_{k=1}^{|v|} \xi_{v_k}^c \geq \psi(n)x\right) \leq \mathbb{P}\left(\sum_{k=1}^{n} \eta_k^c \geq \psi(n)x\right).$$

Note that

$$a_{min}(x)\exp\{-L(x)x^r\} \leq 1 - F(x) \leq a_{max}(x)\exp\{-L(x)x^r\}$$

where

$$a_{min}(x) = \min\{a_i(x) \ : \ i \in \mathcal{C}, r_i = r\},$$
$$a_{max}(x) = \max\{a_i(x) \ : \ i \in \mathcal{C}, r_i = r\}.$$

Since a_{min}, a_{max}, and L are all slowly varying, Theorem 3 along with the remark (see the equation (29)) from [28] asserts that for all $x > 0$

$$\lim \frac{1}{n} \log \mathbb{P}\left(\sum_{k=1}^{n} \eta_k^c \geq \psi(n)x\right) = -x^r. \tag{2.31}$$

In particular, for any $\varepsilon > 0$, there is $\delta > 0$ such that for all large enough n,

$$\mathbb{P}\left(S_v \geq \psi(n)(\log\rho + \varepsilon)^{\frac{1}{r}}\right) \leq \exp\{-n(\log\rho + \delta)\} = \rho^{-n}e^{-n\delta} \tag{2.32}$$

for any $v \in \mathbb{T}_n$. Having this bound, we proceed as in [28]. We have

$$\begin{aligned}
&\mathbb{P}\left(\exists v \in \mathbb{T}_n : S_v \geq \psi(n)(\log\rho + \varepsilon)^{\frac{1}{r}}\right) \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left(\exists v \in \mathbb{T}_n : S_v \geq \psi(n)(\log\rho + \varepsilon)^{\frac{1}{r}} \Big| |Z_n| = k\right)\mathbb{P}(|Z_n| = k) \\
&\leq \sum_{k=1}^{\infty} k\rho^{-n}e^{-n\delta}\mathbb{P}(|Z_n| = k) = \mathbb{E}[|Z_n|]\rho^{-n}e^{-n\delta},
\end{aligned} \tag{2.33}$$

where $|Z_n| = \sum_{i\in\mathcal{C}} Z_n^i$. It is easily verifiable by induction, that

$$\mathbb{E}[Z_n] = M^n\mathbb{E}[Z_0],$$

where M is the mean matrix. Hence, $\mathbb{E}[Z_n]\rho^{-n}$ has a limit, and by linearity so does $\mathbb{E}[|Z_n|]\rho^{-n}$. Applying the Borel-Cantelli lemma to (2.33) and letting $\varepsilon \to 0$ entails the upper bound in Theorem 2.5.

The lower bound requires more delicate approach. For $K > 0$, let

$$\mathbb{T}^K = \{v \in \mathbb{T} : \forall_{k\leq|v|}\xi_{v_k} \geq -K\},$$

and $M_n^K = \max\{\xi_v : |v| = n, v \in \mathbb{T}^K\}$, and denote by ρ_K the Perron-Frobenius eigenvalue of the matrix $\{\mathbb{E}[N_{i,j}]\mathbb{P}(\xi^j > -K)\}_{i,j\in\mathcal{C}}$. Since $\rho_K \to \rho > 1$ as $K \to \infty$, choose K large enough so that $\rho_K > 1$. Note that

$$R_n = \max_{|v|=n} S_v \geq \max_{|v|=n, v\in\mathbb{T}^K} S_v \geq M_n^K - (n-1)K. \tag{2.34}$$

By Remark 2.6, $\frac{n-1}{\psi(n)} \to 0$, hence dividing by $\psi(n)$ and taking limits yields

$$\liminf_{n\to\infty} \frac{R_n}{\psi(n)} \geq \liminf_{n\to\infty} \frac{M_n^K}{\psi(n)}. \tag{2.35}$$

Hence, it suffices to show

$$\liminf_{n\to\infty} \frac{M_n^K}{\psi(n)} \geq (\log \rho_K)^{\frac{1}{r}} \tag{2.36}$$

By the Borell-Cantelli lemma, to show (2.36) it is enough to show that for any $\varepsilon > 0$

$$\sum_{n=0}^{\infty} \mathbb{P}\left(\frac{M_n^K}{\psi(n)} < [\log\{\rho_K(1-\varepsilon)\}]^{\frac{1}{r}}\right) < \infty. \tag{2.37}$$

To that end, take any $\varepsilon > 0$ small enoguh to satisfy $\rho_K(1-\varepsilon) > 1$ and let

$$Z_n^{K,i} = \#\{v \in \mathbb{T}^K : \sigma(v) = i, |v| = n\},$$

and $Z_n^K = \sum_{i\in\mathcal{C}} Z_n^{K,i}$. To simplify the notation, denote

$$b_n = \psi(n)\,[\log\{\rho_K(1-\varepsilon)\}]^{\frac{1}{r}}$$

and let

$$I(n) = \underset{i\in\mathcal{C}, r_i = r}{\arg\max}\, L_i(b_n).$$

Then

$$\begin{aligned}
&\mathbb{P}\left(\frac{M_n^K}{\psi(n)} < [\log\{\rho_K(1-\varepsilon)\}]^{\frac{1}{r}}\right) = \mathbb{E}\left[\prod_{i\in\mathcal{C}} \mathbb{P}\left(\xi^i < b_n\right)^{Z_n^{K,i}}\right] \\
&= \mathbb{E}\left[\prod_{i\in\mathcal{C}} \left(1-\mathbb{P}\left(\xi^i \geq b_n\right)\right)^{Z_n^{K,i}}\right] \leq \mathbb{E}\left[\prod_{i\in\mathcal{C}} \exp\left\{-Z_n^{K,i}\mathbb{P}\left(\xi^i \geq b_n\right)\right\}\right] \\
&\leq \mathbb{E}\left[\exp\left\{-Z_n^{K,I(n)}\mathbb{P}\left(\xi^{I(n)} \geq b_n\right)\right\}\right]
\end{aligned}$$

Using our assumption 2.26, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-Z_n^{K,I(n)}\mathbb{P}\left(\xi^{I(n)} \geq b_n\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-Z_n^{K,I(n)} a_{I(n)}(b_n)\left(\rho_K(1-\varepsilon)\right)^{-L_{I(n)}(b_n)\psi(n)^r}\right\}\right]
\end{aligned}$$

Note that $L_{I(n)}(b_n) = L(b_n)$, hence by choice of L (see (2.27)),

$$\frac{L_{I(n)}(b_n)\,\psi(n)^r}{n} \to 1,$$

and recall that all a_i are slowly varying. Hence we have

$$a_{I(n)}(b_n)\left(\rho_K(1-\varepsilon)\right)^{-L_I(b_n)\psi(n)^r} \geq \left(\rho_K(1-\varepsilon_1)\right)^{-n}$$

for some $\varepsilon_1 \in (0,\varepsilon)$ and all sufficiently large n. Hence, by Borel-Cantelli lemma, (2.37) will follow from the convergence of the series

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\exp\left\{-\frac{Z_n^{K,I}}{(\rho_K(1-\varepsilon_1))^n}\right\}\right].$$

Using the formula

$$\mathbb{E}\left[\exp\left\{-\frac{Z_n^{K,I}}{(\rho_K(1-\varepsilon_1))^n}\right\}\right] \leq \exp\left\{-\frac{\left(1-\frac{\varepsilon_1}{2}\right)^n}{(1-\varepsilon_1)^n}\right\} + \mathbb{P}\left(\frac{Z_n^{K,I}}{\rho_K^n} \leq \left(1-\frac{\varepsilon_1}{2}\right)^n\right)$$

we see that it is sufficient to show

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{Z_n^{K,I}}{\rho_K^n} \leq \left(1-\frac{\varepsilon_1}{2}\right)^n\right) < \infty.$$

and this is a straightforward consequence of applying Lemmas 2.8 and 2.7. □

3 Reducible multi-type branching random walk

In this chapter, we consider the case where the mean matrix M is reducible. We divide $\mathcal{C}$ by the following equivalence relation: $i \sim j$ if there are l_1, l_2 such that $M^{l_1}(i,j) > 0$ and $M^{l_2}(j,i) > 0$. We denote $\mathcal{C}_\sim = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m\}$ and introduce a partial ordering of $\mathcal{C}$ through the following relation: $i \preceq j$ if there exists $n \in \mathbb{N}$ such that $M^n(i,j) > 0$. This induces a partial ordering of $\mathcal{C}_\sim$. We will abuse the notation and write $a \preceq b$ when there exist $i \in \mathcal{C}_a$ and $j \in \mathcal{C}_b$ such that $i \preceq j$, and $a \preceq i$ if $i \in \mathcal{C}_b$ and $a \preceq b$. By renumbering the types, we may and will assume that M is of form

$$\begin{pmatrix} M[1] & M[1,2] & \dots & M[1,m] \\ 0 & M[2] & \dots & M[2,m] \\ \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & M[m] \end{pmatrix}. \tag{3.1}$$

That is, it has cages $\{M[a]\}_{a \in \mathcal{C}_\sim}$ on the diagonal and zeros below. Throughout the chapter, we make the following assumptions.

$$M[a] \text{ is positively regular in the sense of (2.1) for all } a \leq m. \tag{3.2}$$

For any $a \leq m$, denote by $\rho(a)$ the largest eigenvalue of $M[a]$ and assume

$$\rho(1) > 1 \tag{3.3}$$

It is easy to see that the spectrum of M is just a union of spectrums of $M[a]$'s.

It is clear that when the starting particle comes from $\mathcal{C}_a$ the problem is reduced to the analysis of the types of classes following (and including) a, hence without loss of generality we may assume that the starting particle's type belongs to class 1. Since the specific type will be of little significance, we will assume for simplicity that the starting particle is of type 1. Similarly, if some class does not follow the first class, it will never appear in the process, so we assume $1 \preceq a$ for all $a \leq m$. To avoid conditioning on the survival set, we assume that type 1 (or equivalently class 1) survives with probability 1.

Analogously to the previous section, we assume the following Kesten-Stigum condition:

$$\text{for all } a \leq m \text{ and all pairs } i,j \in \mathcal{C}_a,\ \mathbb{E}[N_{i,j} \log N_{i,j}] < \infty. \tag{3.4}$$

By the result of Kesten and Stigum [37], under this assumptions, if

$$\rho_j = \max\{\rho_a \ : \ a \in \mathcal{C}_\sim, a \preceq j\}$$

then for some $k > 0$,

$$\frac{Z_n^j}{n^k \rho_j^n} \xrightarrow{\mathbb{P}-a.s.} W(j), \tag{3.5}$$

and $W(j)$ is positive if u_j^α is positive, where u^α is the left eigenvector of $M[\alpha]$. In other words, the asymptotic number of particles of any given type is driven by the number of particles preceding it. More explicit expressions can be provided for $W(j)$ in certain examples, but a general formula seems difficult to obtain. One can show that the randomness in $W(j)$ is contained in the preceding classes that have the highest number of offspring, that is, if for some classes α and β we have $\rho_\beta < \rho_\alpha = \max\{\rho_\gamma : \gamma \preceq \beta\}$, then $W(\beta) = (W(j))_{j \in \mathcal{C}_\beta}$ is a deterministic linear transformation of $W(\alpha)$. We refer to [37] for a more detailed exploration of the properties of W.

3.1 Displacements with regularly varying tails

Let $F_i(x) = \mathbb{P}(\xi^i \le x)$. In this section, analogously to the irreducible case, we assume that the displacements are independent and there exist slowly varying functions $\{L_i\}_{i\in\mathcal{C}}$ and positive constants $\{r_i\}_{i\in\mathcal{C}}$, satisfying

$$\begin{aligned} 1 - F_i(x) &\sim L_i(x)x^{-r_i} \quad \text{as } x \to \infty, \\ \log(-x)F_i(x) &\to 0, \quad \text{as } x \to -\infty. \end{aligned} \tag{3.6}$$

In contrast to the assumptions of Theorem 2.1, this time we assume that there is a unique pair (α, I) satisfying

$$\rho_\alpha^{\frac{1}{r_I}} = \max\{\rho_a^{\frac{1}{r_i}} \ : \ a \preceq i\}$$

It is perhaps worth noting that unlike in the irreducible model, r_I is not necessarily the minimum of all r_i's, nor is ρ the principal eigenvalue of M. This is due to the fact that the growth speed of a single cage a, just as we have seen in Theorem 2.1, is exponential at the rate $\rho_a^{\frac{1}{r_a}}$, where r_a is the minimal exponent among the types from this cage. In other words, the speed depends on the interplay between the tails of the displacements and the asymptotic expected number of particles. Since in the reducible case the latter may be different for different classes, choosing the "dominant" type, and therefore the correct normalization, requires us to look at both of these quantities.

We denote $\rho = \rho_\alpha$ and $r = r_I$. Furthermore, let $k > 0$ be the constant satisfying

$$\frac{Z_n^I}{n^k \rho^n} \xrightarrow{\mathbb{P}-a.s.} W(I)$$

Our result is as follows.

Theorem 3.1. *Let*

$$\zeta = \sum_{j>0} \rho^{-j} \sum_{l\in\mathcal{C}} \mathbb{P}_I\left(Z_j^l > 0\right).$$

and choose the sequence $\{a_n\}_{n\in\mathbb{N}}$ *so that*

$$n^k \rho^n \left(1 - F_I(a_n)\right) \xrightarrow[n\to\infty]{} 1.$$

Then

$$\mathbb{P}(R_n \le a_n x) \xrightarrow[n\to\infty]{} \mathbb{E}[e^{-\zeta W(I) x^{-q}}]$$

Remark 3.2. As in the irreducible case, the existence of a_n satisfying (2.8) is guaranteed by the result of de Bruijn [16]. Here if $L^\#$ is the de Bruijn conjugate of L, we can take $a_n = L^\#\left(n^{\frac{k}{r}}\rho^{\frac{n}{r}}\right) n^{\frac{k}{r}}\rho^{\frac{n}{r}}$. In particular, this guarantees that for any

$\varepsilon > 0$,

$$\rho^{\frac{n}{r}(1-\varepsilon)} < a_n < \rho^{\frac{n}{r}(1+\varepsilon)} \tag{3.7}$$

for sufficiently large n.

Proof of the Theorem 3.1. Similarly to the irreducible case, we begin with a lemma on the total population. Recall that for $i \in \mathcal{C}$,

$$Y_n^i = \left| \bigcup_{k=1}^{n} \{v \in \mathbb{T}_k \ : \ \sigma(v) = i, (\exists w \in \mathbb{T}_n)(w_k = v)\} \right|$$

is the total number of particles of type i that have offspring in the n-th generation.

Lemma 3.3. *Assume* (3.2) *and* (3.3) *and let*

$$\zeta_i = \sum_{j>0} \rho^{-j} \sum_{l \in \mathcal{C}} \mathbb{P}_i \left(Z_j^l > 0\right).$$

Then for all $i \in \mathcal{C}$,

$$\frac{Y_n^i}{\rho_i^n n^{k_i}} \to \zeta_i W(i).$$

where $\rho_i, k_i, W(i)$ *are as in* (3.5).

Proof. The Lemma is proven with analogous arguments as in Lemma 2.4, but we provide full argument for convenience of the reader. First decompose

$$\frac{Y_n^i}{\rho^n n^{k_i}} = \sum_{j=0}^{n-1} \rho^{-j} \sum_{l \in \mathcal{C}} \left(\frac{n-j}{n}\right)^{k_i} \frac{Z_{n-j}^i}{\rho^{n-j}(n-j)^{k_i}} \frac{1}{Z_{n-j}^i} \sum_{k=1}^{Z_{n-j}^i} \mathbb{1}_{\{Z_j^l(i,k)>0\}}$$

where for any j and l, $\{Z_j^l(i,k)\}_{k>0}$ are i.i.d. distributed as Z_j^l under $\mathbb{P}_i$, and for $i_1 \neq i_2$, $\{Z_j^l(i_1,k)\}_{k>0}$ are independent of $\{Z_j^l(i_2,k)\}_{k>0}$. Now we denote

$$D_{n-j} = \left(\frac{n-j}{n}\right)^{k_i} \frac{Z_{n-j}^i}{\rho^{n-j}(n-j)^{k_i}}$$

and

$$E_{n-j}^l = \frac{1}{Z_{n-j}^i} \sum_{k=1}^{Z_{n-j}^i} \mathbb{1}_{\{Z_j^l(i,k)>0\}}$$

By the strong law of large numbers, for any fixed $j > 0$,

$$E_{n-j}^l \xrightarrow[n\to\infty]{a.s.} \mathbb{P}_i(Z_j^l > 0)$$

We also know from (3.5), that for any fixed $j > 0$,

$$D_{n-j} \xrightarrow[n\to\infty]{a.s.} W(i)$$

Now fix $N > 0$. Then for $n > N$

$$\begin{aligned}\frac{Y_n^i}{\rho^n n^{k_i}} &\le \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l + \sum_{l\in\mathcal{C}} \sup_{i\ge N+1} \{D_i\} \sum_{i=N+1}^{\infty} \rho^{-i} \\ &= \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l + d\frac{\rho^{-N}}{\rho-1} \sup_{i\ge N+1} \{D_i\}\end{aligned}$$

So, $\mathbb{P}$-almost surely,

$$\limsup_n \frac{Y_n^i}{\rho^n n^{k_i}} \le W(i) \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} \mathbb{P}_i(Z_j^l > 0) + \sum_{l\in\mathcal{C}} \sup_{j\ge N+1} \{D_j\} \sum_{j=N+1}^{\infty} \rho^{-j}$$

Letting $N \to \infty$ we get

$$\limsup_n \frac{Y_n^i}{\rho^n n^{k_i}} \le \zeta_i W(i) \quad \mathbb{P} - \text{a.s.}$$

For the bound from below, we note that

$$\frac{Y_n^i}{\rho^n} \ge \sum_{j=0}^{N} \rho^{-j} \sum_{l\in\mathcal{C}} D_{n-j} E_{n-j}^l$$

so taking $\liminf_n$ and then letting $N \to \infty$ we get

$$\liminf_n \frac{Y_n^i}{\rho^n n^{k_i}} \ge \zeta_i W(i) \quad \text{a.s.}$$

concluding the proof of the lemma. □

Now define for $i \in \mathcal{C}$

$$M_n^i = \max\{\xi_{v_k} \;:\; v \in \mathbb{T}_n,\; v_k \sim i,\; k \le n\}$$

We will show that

$$\mathbb{P}\left(M_n^I \le a_n x\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left[\exp\left\{ -\zeta W(I) x^{-r}\right\}\right] \tag{3.8}$$

and if $i \neq I$

$$\mathbb{P}\left(M_n^i \leq a_n x\right) \xrightarrow[n\to\infty]{} 1. \tag{3.9}$$

As a consequence, of course

$$\mathbb{P}\left(M_n \leq a_n x\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left[\exp\left\{-\zeta W(I) x^{-r}\right\}\right] \tag{3.10}$$

where $M_n = \max_{i\in\mathcal{C}}\{M_n^i\}$. We readily calculate

$$\mathbb{P}\left(M_n^i \leq a_n x\right) = \mathbb{E}\left[F_i(a_n x)^{Y_n^i}\right] = \exp\left\{\frac{Y_n^i}{\rho^n n^k}\rho^n n^k \log F_i(a_n x)\right\}.$$

Now, a_n was chosen so that $\rho^n n^k(1 - F_I(a_n)) \xrightarrow[n\to\infty]{} 1$, so we proceed to

$$\rho^n n^k \log F_I(a_n x) \sim -\rho^n n^k(1-F_I(a_n x)) = -n^k\rho^n(1-F_I(a_n))\frac{(1-F_I(a_n x))}{(1-F_I(a_n))} \xrightarrow[n\to\infty]{} -x^{-r}.$$

Hence by Lemma 3.3,

$$F_I(a_n x)^{Y_n^I} \xrightarrow[n\to\infty]{a.s.} \exp\left\{-\zeta W(I) x^{-r}\right\}.$$

Using the dominated convergence theorem, this proves (3.8). Similarly, for $i \neq I$,

$$\frac{Y_n^i}{\rho_i^n n^{k_i}}\frac{\rho_i^n n^{k_i}}{\rho^n n^k}\rho^n n^k\left(1 - F_i(a_n x)\right) = \frac{Y_n^i}{\rho_i^n n^{k_i}}\frac{\rho_i^n n^{k_i}}{\rho^n n^k}\rho^n n^k(1 - F_I(a_n x))\frac{(1-F_i(a_n x))}{(1-F_I(a_n x))} \tag{3.11}$$

Note again that $\rho^n n^k(1-F_I(a_n)) \xrightarrow[n\to\infty]{} 1$, and by Lemma 3.3 $\rho_i^{-n} n^{-k_i} Y_n^i$ has a finite limit. To take care of the remaining terms, we note that

$$\frac{\rho_i^n n^{k_i}}{\rho^n n^k}\frac{(1-F_i(a_n x))}{(1-F_I(a_n x))} \sim \left(\frac{\rho_i}{\rho}\right)^n \left(\rho^{1-\frac{r_i}{r}}\right)^n h(n) \tag{3.12}$$

where (see Remark 3.2)

$$h(n) = \frac{L_i\left(L^{\#}\left(\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right)\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right) L^{\#}\left(\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right)^{r_i} n^{\frac{k}{r}r_i}}{L\left(L^{\#}\left(\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right)\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right) L^{\#}\left(\rho^{\frac{n}{r}} n^{\frac{k}{r}}\right)^{r} n^{k}}.$$

Observe

$$\left(\frac{\rho_i}{\rho}\right)^n \left(\rho^{1-\frac{r_i}{r}}\right)^n = \rho_i^n \rho^{-n\frac{r_i}{r}}$$

and since ρ and r were chosen to satisfy

$$\rho^{\frac{1}{r}} = \max\{\rho_a^{\frac{1}{r_i}} \ : \ a \preceq i\}$$

we have for some $\varepsilon > 0$

$$\rho^{\frac{1-\varepsilon}{r}} > \rho^{\frac{1}{r_i}}.$$

Hence,

$$\rho_i^n \rho^{-n\frac{r_i}{r}} = \left(\rho_i^{\frac{1}{r_i}}\right)^{nr_i} \rho^{-n\frac{r_i}{r}} \leq \left(\rho^{\frac{1-\varepsilon}{r}}\right)^{nr_i} \rho^{-n\frac{r_i}{r}} = \left(\rho^{\frac{r_i\varepsilon}{r}}\right)^{-n}.$$

As $h(n)$ satisfies $\frac{h(n)}{\rho^{\delta n}} \to 0$ for any $\delta > 0$, the right-hand side in (3.12) goes to 0 as $n \to \infty$. Using the dominated convergence theorem again, (3.9) is proven.

Remark 3.4. As we see from the proof, we can in fact make even stronger statement than (3.9). That is, for $i \neq I$ and small enough $\varepsilon > 0$,

$$\mathbb{P}\left(M_n^i \leq \rho^{\varepsilon n} a_n x\right) \xrightarrow[n\to\infty]{} 1.$$

From here we proceed by induction. If $m = 1$, the theorem reduces to Theorem 2.1, so the base case is proven. Assume now that the theorem holds for processes with $m-1$ classes for $m > 1$. Then we can write

$$R_n = \max(R_n^1, R_n^2) \tag{3.13}$$

where

$$R_n^1 = \max\{S_v \ : \ |v| = n, \ \sigma(v) = i, i \in \bigcup_{i \leq m-1} \mathcal{C}_i\}$$
$$R_n^2 = \max\{S_v \ : \ |v| = n, \ \sigma(v) = i, i \in \mathcal{C}_m\}.$$

Let (β, J) be a pair of a class and a type attaining

$$\max\{\rho_a^{\frac{1}{r_i}} \ : \ a \preceq i, a \leq m-1\},$$

and denote $\gamma = \rho_\beta$ and $q = r_J$. First consider the case when $\gamma^{\frac{1}{q}} = \rho^{\frac{1}{r}} > \rho_m^{\frac{1}{r(m)}}$, where $r(m) = \min_{i \in \mathcal{C}_m} r_i$. By induction assumption

$$\limsup_{n\to\infty} \mathbb{P}\left(\frac{R_n}{a_n} \leq x\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\frac{R_n^1}{a_n} \leq x\right) = \mathbb{E}[e^{-\zeta W(I) x^{-q}}]. \tag{3.14}$$

For the lower bound, consider a modified process $\tilde{S}_v = \sum_{k=1}^n \max(\xi_{v_i}, 0)$ where the displacements are nonnegative. Define analogously

$$\begin{aligned}
\tilde{R}_n^1 &= \max\{\tilde{S}_v \ : \ |v| = n, \ \sigma(v) \in \bigcup_{i \le m-1} \mathcal{C}_i\} \\
\tilde{R}_n^2 &= \max\{\tilde{S}_v \ : \ |v| = n, \ \sigma(v) \in \mathcal{C}_m\} \\
\tilde{R}_n &= \max(\tilde{R}_n^1, \tilde{R}_n^2).
\end{aligned}$$

Note that $\tilde{S}$ satisfies the same assumptions we made on S, so all previous results hold. Now for a particle $v \in \mathbb{T}_n$, $\sigma(v) \in \mathcal{C}_m$, we have

$$S_v = S_v - S_{v^*} + S_{v^*} \le S_v - S_{v^*} + R_{k^*}^1 \le n \max_{i \in \mathcal{C}_m} \tilde{M}_n^i + \tilde{R}_n^1 \tag{3.15}$$

where v^* is the last ancestor of v from the first $m-1$ classes, $k^* = |v^*|$ is its generation, and

$$\tilde{M}_n^i = \max\{\xi_{v_k} \ : \ v \in \mathbb{T}_n, \ v_k \sim i, i \in \mathcal{C}_m \ k \le n\}.$$

Taking the supremum over v in (3.15), we have

$$R_n^2 \le n \max_{i \in \mathcal{C}_m} \tilde{M}_n^i + \tilde{R}_n^1,$$

and trivially,

$$R_n^1 \le n \max_{i \in \mathcal{C}_m} \tilde{M}_n^i + \tilde{R}_n^1.$$

Since $I \notin \mathcal{C}_m$, by Remark 3.4,

$$\frac{n \max_{i \in \mathcal{C}_m} \tilde{M}_n^i}{a_n} \xrightarrow[n \to \infty]{d} 0.$$

Hence,

$$\begin{aligned}
\liminf_{n \to \infty} \mathbb{P}\left(\frac{R_n}{a_n} \le x\right) &\ge \liminf_{n \to \infty} \mathbb{P}\left(n \max_{i \in \mathcal{C}_m} \tilde{M}_n^i + \tilde{R}_n^1 \le a_n x\right) \\
&= \liminf_{n \to \infty} \mathbb{P}\left(\frac{\tilde{R}_n^1}{a_n} \le x\right) = \mathbb{E}[e^{-\zeta W(I) x^{-q}}]
\end{aligned}$$

Together with (3.14), we conclude

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{R_n}{a_n} \le x\right) = \mathbb{E}[e^{-\zeta W(I) x^{-q}}].$$

Now consider the case when $\gamma^{\frac{1}{q}} < \rho^{\frac{1}{r}}$. Then by induction assumption we know that $\frac{R_n^1}{a_n^1}$ converges in distribution, where for any $\varepsilon > 0$, $a_n^1 < \gamma^{\frac{n}{q}(1+\varepsilon)}$ for sufficiently large n. Since $a_n > \rho^{\frac{n}{r}(1-\varepsilon)}$ for any $\varepsilon > 0$ and sufficiently large n, we have $\frac{a_n^1}{a_n} \to 0$, therefore $\frac{R_n^1}{a_n}$ converges in distribution to 0. Hence, we only need to examine

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{R_n^2}{a_n} \le x\right). \tag{3.16}$$

We begin by showing that for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(R_n^2 > a_n x\right) \le 1 - \mathbb{E}[e^{-\zeta W(I) x^{-q}}] \tag{3.17}$$

Take $v \in \mathbb{T}_n$, $\sigma(v) \in \mathcal{C}_m$. Then

$$S_v = S_v - S_{v^*} + S_{v^*} \le \sum_{k=k^*(v)+1}^{n} \max\{\xi_{v_k}, 0\} + n \max_{i\ne I} M_n^i.$$

Where v^* is the last ancestor of v from the preceding $m-1$ classes and $k^* = |v^*|$. Note that $X_v := \sum_{k=k^*+1}^{n} \max\{\xi_{v_k}, 0\}$ is a random sum of independent random variables with regularly varying tails, where the heaviest tail is of the order r_I (since now $I \in \mathcal{C}_m$). Taking maximum over v, we have

$$R_n^2 \le R_n^* + n \max_{i\ne I} M_n^i$$

where

$$R_n^* = \max_{\sigma(v)\in\mathcal{C}_m} X_v.$$

Again by Remark 3.4,

$$\frac{n \max_{i\ne I} M_n^i}{a_n} \xrightarrow[n\to\infty]{d} 0.$$

Hence, we have reduced (3.17) to showing

$$\limsup_{n\to\infty} \mathbb{P}\left(R_n^* > a_n x\right) \le 1 - \mathbb{E}[e^{-\zeta W(I) x^{-q}}]. \tag{3.18}$$

Let

$$M_n^* = \max_{i\in\mathcal{C}_M} M_n^i.$$

By (3.8) and (3.9),

$$\lim_{n\to\infty} \mathbb{P}\left(M_n^* \ge a_n x\right) = \mathbb{E}[e^{-\zeta W(I) x^{-q}}]. \tag{3.19}$$

From here we proceed similarly to the irreducible case. Since

$$\limsup_{n\to\infty} \mathbb{P}\left(R_n^* > a_n x\right) \le \lim_{n\to\infty} \mathbb{P}\left(M_n^* \ge a_n x\right) + \mathbb{P}\left(R_n^* > a_n x, M_n^* \le a_n(x-\varepsilon)\right),$$

by (3.19) we only need to show

$$\lim_{n\to\infty} \mathbb{P}\left(R_n^* > a_n x, M_n^* \leq a_n(x-\varepsilon)\right) = 0$$

Let $X_n(a_n x, \infty) = \#\{v \ : \ |v| = n, \ \sigma(v) \in \mathcal{C}_m, \ X_v > a_n x\}$. Then

$$\mathbb{P}\left(R_n^* > a_n x, M_n^* \leq a_n(x-\varepsilon)\right) \leq \mathbb{E}\left[X_n(a_n x, \infty)\mathbb{1}_{\{M_n^* \leq a_n(x-\varepsilon)\}}\right]$$
$$\leq \sum_{l=1}^{n} \sum_{\vec{l}\ :\ |\vec{l}|_1 = l} A_{\vec{l}}\left(F_{\vec{l}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{l}}^{(a_n(x-\varepsilon))}(a_n x)\right)$$

where the inner sum goes over all vectors $\vec{l} = (l_1, \dots l_{|\mathcal{C}_m|})$ satisfying $|\vec{l}|_1 = \sum_{j=1}^{|\mathcal{C}_m|} l_j = l$, $A_{\vec{l}}(n)$ is the expected number of class $\mathcal{C}_m$ particles in the n-th generation, that had l_j ancestors of each respective type in $\mathcal{C}_m$, and

$$F_{\vec{l}}^{y}(x) = F_{d-m+1,l_1}^{y} * F_{d-m,l_2}^{y} * \cdots * F_{d,l_m}^{y}(x)$$

Since the step distributions in $F_{\vec{l}}$ now only involve distributions with tails not heavier than $r_I = r$, we can follow the argument from the proof of (2.22), which yields for $x, \varepsilon, \delta > 0$, $s \in (0, r)$ and $C > 0$,

$$\left(F_{\vec{l}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{l}}^{(a_n(x-\varepsilon))}(a_n x)\right) \leq C\left(\frac{lC_s}{a_n^s(x-\varepsilon)^s}\right)^{\frac{x(1-\delta)}{(x-\varepsilon)}}$$

and as a consequence, for some $C' > 0$ and $p' > 1$,

$$\sum_{l=1}^{n} \sum_{\vec{l}\ :\ |\vec{l}|_1 = l} A_{\vec{l}}\left(F_{\vec{l}}^{(a_n(x-\varepsilon))}(\infty) - F_{\vec{l}}^{(a_n(x-\varepsilon))}(a_n x)\right)$$
$$\leq C'\rho^{-np'}\mathbb{E}[|Z_n(m)|] =\leq C'\rho^{-np'}\sum_{j=1}^{m}(M^n Z_0)(d-m+j)$$

By Jordan decomposition, for some $k > 0$ and all $j = 1, \dots, m$, $\rho^{-n} n^{-k}(M^n Z_0)(d - m + j)$ converges to a finite limit, hence

$$\lim_{n\to\infty} \mathbb{P}\left(R_n^* > a_n x, M_n^* \leq a_n(x-\varepsilon)\right) = 0$$

This concludes the proof of (3.18), and in result (3.17). To finish the proof of the Theorem we are left to show

$$\liminf_{n\to\infty} \mathbb{P}(R_n > a_n x) \geq 1 - \mathbb{E}[e^{-\zeta W(I) x^{-q}}].$$

Since

$$\mathbb{P}(R_n > a_n x) \geq \mathbb{P}(M_n > a_n(x+\varepsilon)) - \mathbb{P}(R_n \leq a_n x, M_n > a_n(x+\varepsilon)).$$

and, by (3.10),

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \geq a_n x\right) = \mathbb{E}[e^{-\zeta W(I) x^{-q}}], \tag{3.20}$$

the proof is reduced to showing

$$\mathbb{P}(R_n \leq a_n x, M_n \geq a_n(x+\varepsilon)) \xrightarrow[n\to\infty]{} 0. \tag{3.21}$$

Here we note that the arguments used to show (2.14) in the proof of Theorem 2.1 were based solely on the logarithmic bound on the lower tails of the distributions and the exponential growth of a_n. Since these conditions are still satisfied, the calculations can be repeated with no significant modifications, concluding the proof of the Theorem. □

3.2 Displacements with semi-exponential tails

Analogously to the irreducible case, throughout this section we assume that the displacements are independent and admit semi-exponential tails:

$$\mathbb{P}\left(\xi^j \geq t\right) = a_j(t)\exp\{-L_j(t)t^{r_j}\}, \tag{3.22}$$

where L_j, a_j are slowly varying functions such that $\frac{L_j(t)}{t^{1-r_j}}$ are eventually nonincreasing, and $r_j \in (0,1)$. We also assume that the displacements have finite moments. Let

$$\begin{aligned} r &= \min\{r_j \mid j \in \mathcal{C}\}, \\ \mathcal{B} &= \{j \in \mathcal{C} \mid r_j = r\}, \\ \mathcal{A} &= \{a \preceq j \mid j \in \mathcal{B}\} \end{aligned}$$

so that $\mathcal{A}$ is the set of classes preceding types that attain r. Our result is as follows.

Theorem 3.5. *Let $L(x) = \min\{L_j(x) \mid j \in \mathcal{B}\}$, $\rho = \max_{a\in\mathcal{A}} \rho(a)$, and choose $\psi(n)$ to be a positive function satisfying*

$$\frac{L(\psi(n))\psi(n)^r}{n} \to 1. \tag{3.23}$$

Then

$$\frac{R_n}{\psi(n)} \xrightarrow{a.s.} (\log\rho)^{\frac{1}{r}}.$$

Remark 3.6. As in the irreducible case, the existence of $\psi(n)$ satisfying (3.23) is guaranteed by the result of de Bruijn [16]. Indeed, if $K(x)$ is the de Bruijn conjugate of $x \mapsto L\left(x^{\frac{1}{r}}\right)$, then we can take$\psi(n) = K(n)^{\frac{1}{r}} n^{\frac{1}{r}}$. In particular, this implies that for any $\varepsilon > 0$,

$$n^{\frac{1}{r}(1-\varepsilon)} \leq \psi(n) \leq n^{\frac{1}{r}(1+\varepsilon)}$$

for large enough n.

The main difference between this Theorem and Theorem 2.5 is that ρ is not necessarily the principal eigenvalue of M. This is because the limit behavior is driven by the heaviest tail and the asymptotic number of particles attaining it. In the irreducible case, all types share the same growth rate, but as seen in [37], the growth rate of particles of any given type is also driven by the types preceding it. To illustrate the issue, consider the following heuristic argument: start with an irreducible process as class 1, and append to it another process as class 2, which follows class 1. Denote by r_1 and r_2 the heaviest tails that appear in classes 1 and 2, respectively. Then there are 3 cases to consider:

1. $r_2 < r_1$. Then the normalization factor $\psi(n)$ should adhere to class 2, as it has a heavier maximum tail. As the number of class 2 particles grows as $\max\{\rho(1), \rho(2)\}^n$, ρ in Theorem 3.5 is, in fact, the principal eigenvalue of M. It is perhaps worth noting that the number of class 2 particles can even grow as $n^k \rho^n$ for some $k > 0$ if $\rho_1 = \rho_2$ (see [37] for details), but this is of no consequence here, since the bounds used in our proof work on exponential scale.

2. $r_2 = r_1$. Since both classes attain the heaviest tail in the process, this is in essence the same as case 1. The only difference lies in $\psi(n)$, but it is limited to a slowly varying function.

3. $r_2 > r_1$. Since the normalizing factor $\psi(n)$ grows as $n^{\frac{1}{r_1}}$, class 2 is essentially irrelevant. In this case $\rho = \rho(1)$ even if $\rho(2)$ is greater.

This argument can be iterated by adding more classes. This should provide some intuition useful for understanding the rationale behind the structure of the proof. The key ingredient will be the following lemma.

Lemma 3.7. *Denote the number of particles of class a in the n-th generation as $Z_n(a)$. If $a \preceq j$, then, for any $\varepsilon > 0$, there exist $k \in \mathbb{N}$, $\delta > 0$ and $\beta, \gamma \in (0,1)$, such that for all n large enough*

$$\mathbb{P}\left(Z_n^j \leq \delta Z_{n-k}(a)\right) \leq \beta^n. \tag{3.24}$$

and

$$\mathbb{P}\left(\frac{Z_n^j}{\rho(a)^n} < (1-\varepsilon)^n\right) \leq \gamma^n \tag{3.25}$$

Proof of the Lemma 3.7. We prove the lemma by induction. First observe that if $m = 1$, the model reduces to the irreducible one considered in previous section and the statement of the lemma is an immediate consequence of Lemmas 2.7 and 2.8. Now assume that it holds for processes with m classes and consider one with $m+1$ classes. Observe that the statement follows immediately from the induction assumption if $j \in \mathcal{C}_l$ for some $l \leq m$, as the last type does not contribute to the previous ones, hence we only consider the case when $j \in \mathcal{C}_{m+1}$. We start by showing (3.24). First note that it is trivially true if $a = m+1$, so assume $a \leq m$. Fix $i \in \mathcal{C}_a$ and $k \in \mathbb{N}$ such that $M^k(i,j) > 0$ (recall that $a \preceq j$ means that such k exists for all $i \in \mathcal{C}_a$). Since (3.24) holds for first m classes, the problem can be reduced to showing existence of $\beta \in (0,1)$ satisfying

$$\mathbb{P}\left(Z_n^j < \delta Z_{n-k}^i\right) \leq \beta^n. \tag{3.26}$$

The proof is again similar to that of Lemma 2.8. We denote by $Z_k^{i \to j}$ a random variable distributed as Z_k^j under $\mathbb{P}_i$. Consequently, $q = \mathbb{P}\left(Z_k^{i\to j} = 0\right) < 1$. Observe

$$\mathbb{P}\left(Z_n^j < \delta Z_{n-k}^i\right) = \mathbb{P}\left(\sum_{r\in\mathcal{C}}\sum_{l=1}^{Z_{n-k}^r} Z_k^{i\to j}(l) < \delta Z_{n-k}^i\right)$$
$$\leq \mathbb{P}\left(\sum_{l=1}^{Z_{n-k}^i} Z_k^{i\to j}(l) < \delta Z_{n-k}^i\right) = \mathbb{E}\left[\Phi\left(Z_{n-k}^i\right)\right]$$

where

$$\Phi(s) = \mathbb{P}\left(\sum_{l=1}^{s} Z_k^{i\to j}(l) < \delta s\right).$$

and $\{Z_k^{i\to j}(l)\}_{l>0}$ are indpendent copies of $Z_k^{i\to j}$. If $q = 0$, the statement of the Lemma is trivially true with $\delta = 1$. Hence, we only consider the case where $q \in (0,1)$. Denote $K_n = \#\{l \leq s : Z_k^{i\to j}(l) > 0\}$. Then

$$\Phi(s) = \mathbb{P}\left(\sum_{l=1}^{s} Z_k^{i\to j}(l) < \delta s\right) = \sum_{t=0}^{\infty}\mathbb{P}\left(\sum_{l=1}^{s} Z_k^{i\to j}(l) < \delta s\ ,\ K_n = t\right)$$
$$\leq \sum_{t=0}^{\infty}\mathbb{P}(t < \delta s, K_n = t) = \sum_{t=0}^{\lfloor \delta s\rfloor}\binom{s}{t}(1-q)^t q^{s-t} \leq q^s \sum_{t=0}^{\lfloor \delta s\rfloor}\frac{(\delta s)^t}{t!}\left(\frac{1-q}{\delta q}\right)^t$$

Choosing $\delta > 0$ small enough so that

$$\frac{1-q}{\delta q} > 1,$$

and

$$\beta_0 = q\left(\frac{1-q}{\delta q}\right)^{\delta} < 1.$$

we get the bound

$$\Phi(s) \leq \beta_0^s.$$

Then

$$\mathbb{E}\left[\Phi\left(Z_{n-k}^i\right)\right] \leq \mathbb{E}\left[\beta_0^{Z_{n-k}^i}\right] \leq \beta_0^n + \mathbb{P}\left(Z_{n-k}^i < n\right).$$

Since $i \in \mathcal{C}_a$ for $a \leq m$, $\mathbb{P}\left(Z_{n-k}^i < n\right)$ decays exponentially fast by induction assumption, which ends the proof of (3.24). Having proved that, (3.25) follows easily:

$$\mathbb{P}\left(\frac{Z_n^j}{\rho(a)} < (1-\varepsilon)^n\right) \leq \beta^n + \mathbb{P}\left(\frac{\delta Z_n^i}{\rho(a)} < (1-\varepsilon)^n\right) \leq \beta^n + \gamma_0^{\frac{n}{\delta}} \leq \gamma^n$$

for appropriate choice of $\gamma \in (0,1)$ and large enough n. □

Proof of the Theorem 3.5. We start with the upper bound. For this part, we can assume without loss of generality (replacing ξ^i with $\max\{\xi^i, 0\}$) that all ξ^i's are nonnegative. Denote

$$\theta = \max\{\alpha \mid \mathcal{C}_\alpha \subset \mathcal{B}\}.$$

Note that we may assume that if α is not comparable (with respect to relation $\preceq$) with any class contained in $\mathcal{B}$, then $\alpha > \theta$. This is achieved by simple renumbering of some classes and is consistent with the ordering we assumed in (3.1). Let

$$R_n^\theta = \max\{S_v \mid |v| = n,\ \sigma(v) \in \mathcal{C}_\alpha,\ \alpha \leq \theta\}$$

Denote by M_θ the minor of M that includes only the subset of classes $\mathcal{C}_\sim^\theta = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\theta\}$, so

$$M_\theta = \begin{pmatrix} M[1] & M[1,2] & \ldots & M[1,\theta] \\ 0 & M[2] & \ldots & M[2,\theta] \\ \vdots & \vdots & \vdots & \ldots \\ 0 & \ldots & 0 & M[\theta] \end{pmatrix}. \tag{3.27}$$

Since the subsequent classes do not contribute to the previous ones, the sub-process consisting only of particles of these types is a multi-type Galton-Watson process with mean matrix M_θ. We now repeat the construction from the first part of the proof of Theorem 2.5. Recall that we bounded the tails of S_v by a tail of an i.i.d. sum of random variables with the heaviest tails. When applied to the considered subprocess, this yields the following inequality. For any $\varepsilon > 0$, there is $\delta > 0$ such that for all large enough n,

$$\mathbb{P}\left(S_v \geq \psi(n)(\log\rho + \varepsilon)^{\frac{1}{r}}\right) \leq \exp\{-n(\log\rho + \delta)\} = \rho^{-n}e^{-n\delta} \tag{3.28}$$

for any $v \in \mathbb{T}_n^\theta = \{v \in \mathbb{T}_n \mid \sigma(v) \in \mathcal{C}_\alpha, \alpha \leq \theta\}$. Calculation analogous to (2.33) yields the inequality

$$\mathbb{P}\left(\exists v \in \mathbb{T}_n : S_v \geq \psi(n)(\log\rho + \varepsilon)^{\frac{1}{r}}\right) \leq \mathbb{E}[|Z_n^\theta|]\rho^{-n}e^{-n\delta}, \tag{3.29}$$

where $|Z_n^\theta|$ is the number of particles in our sub-process. Now observe that

$$\mathbb{E}[|Z_n^\theta|] = \mathbb{E}\left[|M_\theta^n\,(Z_0)|\right] = |M_\theta^n\,(Z_0)|\,.$$

As ρ is the largest eigenvalue of M_θ, by bringing M_θ to its Jordan form we see that for some $k \leq d$, $n^{-k}\rho^{-n}M_\theta^n$ has a finite limit, hence

$$\mathbb{E}[|Z_n^\theta|]\rho^{-n}e^{-n\delta} \leq Ce^{-n\delta'}$$

for some $C > 0, \delta' > 0$, which by the Borel-Cantelli lemma proves

$$\limsup_{n\to\infty} \frac{R_n^\theta}{\psi(n)} \leq (\log\rho)^{\frac{1}{r}} \tag{3.30}$$

Now we consider

$$M_c = \begin{pmatrix} M[\theta+1] & M[\theta+1,\theta+2] & \dots & M[\theta+1,m] \\ 0 & M[\theta+2] & \dots & M[\theta+2,m] \\ \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & M[m] \end{pmatrix} \tag{3.31}$$

Observe that if $v \in \mathbb{T}_n/\mathbb{T}_n^\theta$, then it necessarily has the last ancestor from $\mathbb{T}_{k_v}^\theta$ (as the initial particle comes from the first class) for some k_v. We denote this ancestor v^θ. Hence

$$\frac{S_v}{\psi(n)} = \frac{S_v - S_{v^\theta} + S_{v^\theta}}{\psi(n)} \leq \frac{\sum_{i=k_v+1}^n \xi_{v_i} + R_{k_v}^\theta}{\psi(n)} \leq \frac{X_v}{\psi(n)} + \frac{R_n^\theta}{\psi(n)} \tag{3.32}$$

where $\sum_{i=k_v+1}^n \xi_{v_i}$. Taking maximum over $v \in \mathbb{T}_n/\mathbb{T}_n^\theta$ and letting $n \to \infty$ in (3.32), we see that with (3.30), we only need to show

$$\frac{R_n^*}{\psi(n)} \xrightarrow[n\to\infty]{a.s.} 0, \tag{3.33}$$

where

$$R_n^* = \max_{v\in\mathbb{T}_n/\mathbb{T}_n^\theta} X_v.$$

From here we again deploy the strategy used in the proof of (2.33). Let

$$\begin{aligned} r' &= \min\{r_i \mid i \in \bigcup_{\alpha>\theta} \mathcal{C}_\alpha\} > r \\ L'(x) &= \min\{L_i(x) \mid i \in \bigcup_{\alpha>\theta} \mathcal{C}_\alpha, r_i = r'\} \\ \rho' &= \max\{\rho(\alpha) \mid \alpha > \theta\}. \end{aligned}$$

furthermore, let Ψ be a function satisfying

$$\frac{L'(\Psi(n))\Psi(n)^{r'}}{n} \to 1. \tag{3.34}$$

Analogously to (2.33), we bound

$$\mathbb{P}(X_v > x) \leq \mathbb{P}\left(\tilde{X}_n > x\right)$$

where

$$\tilde{X}_n = \sum_{i=1}^{n} \tilde{\xi}_i$$

is a sum of independent nonnegative random variables with cumulative distribution function F, satisfying

$$a_1(x)\exp\{-L'(x)x^{r'}\} \leq 1 - F(x) \leq a_2(x)\exp\{-L'(x)x^{r'}\}$$

for some slowly varying a_1, a_2. As a consequence, by Theorem 3 from [28], for any $\varepsilon > 0$, there is $\delta > 0$ such that for all large enough n,

$$\mathbb{P}\left(X_v \geq \{\Psi(n)(\log\rho' + \varepsilon)^{\frac{1}{r'}}\right) \leq \exp\{-n(\log\rho' + \delta)\} = (\rho')^{-n}e^{-n\delta}. \qquad (3.35)$$

As a consequence,

$$\mathbb{P}\left(\exists v \in \mathbb{T}_n : X_v \geq \{\Psi(n)(\log\rho' + \varepsilon)^{\frac{1}{r'}}\right) \leq \mathbb{E}\left[\sum_{\alpha>\theta} Z_n(\alpha)\right](\rho')^{-n}e^{-n\delta}.$$

Since

$$\mathbb{E}\left[\sum_{\alpha>\theta} Z_n(\alpha)\right] = \sum_{i\in\bigcup_{\alpha>\theta}\mathcal{C}_\alpha} M_n(i)$$

Again by Jordan decomposition, for all

$$i \in \bigcup_{\alpha>\theta}\mathcal{C}_\alpha,$$

and some $k > 0$, $n^{-k}(\rho')^{-n}M_n(i)$ converges to a finite limit. We conclude

$$\limsup_{n\to\infty} \frac{R*_n}{\Psi(n)} \leq (\log\rho')^{\frac{1}{r'}} \quad \text{a.s.}$$

By Remark 3.6, since $r < r'$, we have

$$\frac{\Psi(n)}{\psi(n)} \xrightarrow[n\to\infty]{} 0$$

therefore

$$\limsup_{n\to\infty} \frac{R^*_n}{\psi(n)} \leq 0 \quad \text{a.s.} \qquad (3.36)$$

Thus we have proved

$$\limsup_{n\to\infty} \frac{R_n}{\psi(n)} \leq (\log\rho)^{\frac{1}{r}} \quad \text{a.s.}$$

For the lower bound we again apply the trimming procedure used in the proof of Theorem 2.5 and reduce the problem to the convergence of the series

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{Z_n^I}{\rho^n} \leq \left(1 - \frac{\varepsilon_1}{2}\right)^n\right)$$

where $I \in \mathcal{B}$. Recall that ρ was the maximum eigenvalue among the classes followed by the types in $\mathcal{B}$, so applying Lemma 3.7 (more specifically (3.25)) ends the proof. $\square$

4 Perturbed branching random walk

In this chapter, which is based on the article [38], we consider a one-dimensional perturbed branching random walk

$$S_v^* = S_v + X_v,$$

where $\{X_v\}_{v\in\mathbb{T}}$ are i.i.d. random variables independent of S. We study the model introduced by Bandyopadhyay and Ghosh in [5], where the perturbations have the form

$$X_v(\theta) = \frac{1}{\theta}\log\frac{Y_v}{E_v}$$

for a given positive real number θ, and $\{Y_v\}_{v\in\mathbb{T}}$ which are independent positive random variables with distribution μ, and given $\mathbb{T}$ are independent of $\{E_v\}_{v\in\mathbb{T}}$, which are independent with distribution $\mathrm{Exp}(1)$. We denote

$$\theta_0 = \inf\{\theta > 0 : \nu(\theta) = \theta\nu'(\theta)\}$$

where

$$\nu(\theta) = \log\mathbb{E}\left[\sum_{i=1}^{N} e^{\theta\xi_i}\right]$$

is the log-Laplace transform of $\mathcal{Z}$, and $\nu'(\theta) = e^{-\nu(\theta)}\mathbb{E}\left[\sum_{i=1}^{N}\xi_i e^{\theta\xi_i}\right]$. Note that ν does not have to be differentiable at θ for this quantity to exist, and that in general θ_0 may be infinite.

In [5] branching random walks with such perturbations where studied in the case when μ has finite mean. In particular, the authors proved that

$$\frac{R_n^*(\theta)}{n} \xrightarrow[n\to\infty]{a.s.} \begin{cases} \frac{\nu(\theta)}{\theta} & \theta < \theta_0 \\ \frac{\nu(\theta_0)}{\theta_0} & \theta \geq \theta_0 \end{cases}$$

and identified weak centered asymptotics for $\theta \leq \theta_0$. However, the result for $\theta > \theta_0$ was only obtained for the degenerated perturbations with $\mu = \delta_1$. In this chapter we present a series of Theorems, that complete the results from [5] by providing the missing weak centered asymptotics for the so called above the boundary case, and extend them to μ with infinite mean, with special focus on distributions with regularly varying tails.

Let $\gamma \in (0,1)$. Our main assumption for μ is that

$$x^\gamma(1 - F(x)) \xrightarrow[x\to+\infty]{} c_+ > 0, \tag{\textbf{H}}$$

where F is the probability distribution function of μ.

This assumption tells us that μ belongs to the domain of attraction of a stable law with characteristic function

$$\tilde{g}(t) = e^{-k|t|^{\gamma}\left(1 - i\tan\left(\frac{\pi\gamma}{2}\right)\mathrm{sign}t\right)} \tag{4.1}$$

where

$$k = \frac{\pi c_+}{2\Gamma(\gamma)\sin(\pi\gamma/2)} > 0$$

and $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt$ is the Gamma function.

Furthermore, it yields that if Y has a distribution μ, then $\mathbb{E}[Y^\gamma] = \infty$, but $\mathbb{E}[Y^r] < \infty$ for any $r \in (0, \gamma)$. (**H**) will be assumed in majority of the Theorems, however the result for the above the boundary case will be stated under a more general assumption.

We assume throughout this chapter that $\mathrm{supp}(\mu) \subset \mathbb{R}_+$, the system survives with probability 1 ($\mathbb{P}(N = 0) = 0$) and $\mathbb{E}N \in (1, \infty]$. The first assumption $\mathbb{P}(N = 0) = 0$ is only made to simplify the notation, it can be easily avoided through conditioning on the survival set, whereas the second one in particular entails that the branching mechanism is not degenerated ($\mathbb{P}(N > 1) > 0$). We also assume that ν is finite on some open interval I containing 0. Since θ_0 is finite, the last assumption guarantees, by convexity of ν, that ν is differentiable on $(-s, \theta_0)$ for some $s > 0$, and has a left derivative at θ_0. One can also characterize θ_0 as the unique argument minimizing $\frac{\nu(\theta)}{\theta}$ over $\theta > 0$. Throughout this chapter, existence of finite θ_0 will only be assumed when necessary.

As proved in [9], if θ_0 is finite, then

$$\frac{R_n}{n} \xrightarrow[n\to\infty]{} \frac{\nu(\theta_0)}{\theta_0}, \quad \text{a.s.} \tag{4.2}$$

For θ such that $\nu(\theta) < \infty$, let

$$W_n(\theta) = e^{-n\nu(\theta)} \sum_{|v|=n} e^{\theta S_v}. \tag{4.3}$$

$W_n(\theta)$ is called the additive martingale associated with S. We denote $W_n = W_n(\theta_0)$. Note that as a positive martingale $W_n(\theta)$ converges almost surely to some finite limit. If $\nu'(\theta) < \infty$, then Biggins martingale convergence theorem [10] states, that the almost sure limit of $W_n(\theta)$ is non-degenerate if and only if $\nu'(\theta) < \nu(\theta)/\theta$ and

$$\mathbb{E}\left[W_1(\theta)\log_+ W_1(\theta)\right] < \infty. \tag{4.4}$$

Furthermore, the limit is then positive almost surely. The first condition is equivalent to $\theta < \theta_0$, thus

$$W_n(\theta) \xrightarrow[n\to\infty]{a.s.} \begin{cases} W_\theta^\infty & \text{if } \theta < \theta_0 \text{ and (4.4) is satisfied,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

where W_θ^∞ is finite and positive almost surely. We also define the **derivative martingale** associated with S as

$$D_n = -\sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0))\, e^{\theta_0 S_v - n\nu(\theta_0)}$$

As seen in Proposition A.3 from [1], under assumptions

(L1) $\theta_0 < \infty$ and

$$\mathbb{E}\left[\sum_{i=1}^{N} e^{\theta_0 \xi_i} \xi_i^2\right] < \infty.$$

(L2) $\theta_0 < \infty$, and for $\tilde{X} = \sum_{i=1}^N e^{\theta_0\xi_i}\xi_i^+$, $X = \sum_{i=1}^N e^{\theta_0\xi_i}$

$$\mathbb{E}\left[\tilde{X}\log_+ \tilde{X}\right] < \infty,$$
$$\mathbb{E}\left[X\log_+^2 X\right] < \infty,$$

where $\log_+ x = \max\{0, \log x\}$, we have

$$D_n \xrightarrow[n\to\infty]{a.s.} D_\infty \tag{4.6}$$

for some random variable D_∞ that is finite and positive almost surely.

These two martingales are connected through Theorem 1.1 from [2], which states that under (**L1**) and (**L2**)

$$n^{\frac{1}{2}} W_n(\theta_0) \xrightarrow[n\to\infty]{\mathbb{P}} c_\infty D_\infty \tag{4.7}$$

where

$$c_\infty = \left(\frac{2}{\pi\sigma^2}\right)^{\frac{1}{2}} \quad \text{and} \quad \sigma^2 = \mathbb{E}\left[\sum_{i=1}^{N} (\theta_0\xi_i - \nu(\theta_0))^2\, e^{\theta_0\xi_i - \nu(\theta_0)}\right].$$

For more results on these martingales and their limits see for example Chapter 3 in [43].

4.1 Main results

In this section we present our main results.

Theorem 4.1 (Almost sure convergence below the boundary)**.** *Assume that* $\theta < \frac{\theta_0}{\gamma}$*, condition* (***H***) *is satisfied and*

$$\mathbb{E}\left[W_1(\gamma\theta)\log_+ W_1(\gamma\theta)\right] < \infty.$$

Then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\gamma\theta)}{\gamma\theta}.$$

Theorem 4.2 (Almost sure convergence above and at the boundary)**.** *If* $\theta_0 \leq \theta$ *and* μ *has finite r-th moments for all* $r < \frac{\theta_0}{\theta}$*, then*

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\theta_0)}{\theta_0}.$$

In particular, the conditions of Theorem 4.2 hold if μ satisfies (**H**) and $\theta \geq \frac{\theta_0}{\gamma}$. The results concerning convergence in distribution we split into three cases.

Theorem 4.3 (Convergence in distribution below the boundary)**.** *Assume that* $\theta < \frac{\theta_0}{\gamma}$*, condition* (***H***) *is satisfied and*

$$\mathbb{E}\left[W_1(\gamma\theta)\log_+ W_1(\gamma\theta)\right] < \infty.$$

Then

$$R_n^* - n\frac{\nu(\gamma\theta)}{\gamma\theta} \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log H_\theta - \log E\right),$$

where H_θ *is finite and positive almost surely, and* E *is exponential with intensity 1, independent of* H_θ*. Furthermore,* H_θ *has the characteristic function* $\mathbb{E}\left[\tilde{g}\left(t\left(W_{\gamma\theta}^\infty\right)^\gamma\right)\right]$ *where* $W_{\gamma\theta}^\infty$ *is the limit from (4.5) and* $\tilde{g}$ *is defined in* (4.1)*.*

Theorem 4.4 (Convergence in distribution at the boundary)**.** *Assume* (***L1***) *and* (***L2***)*. If* μ *satisfies* (***H***) *and* $\theta = \frac{\theta_0}{\gamma}$*, then*

$$R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log H_{\theta_0} - \log E\right)$$

where H_{θ_0} *is finite and positive almost surely and* E *is exponential with intensity 1, independent of* H_{θ_0}*. Furthermore,* H_{θ_0} *has the characteristic function* $\mathbb{E}\left[\tilde{g}\left(t(c_\infty D_\infty)^\gamma\right)\right]$*, where* $\tilde{g}$ *is defined in* (4.1)*.*

It is a natural question to ask whether assumption (**H**) in Theorems 2.1, 2.3

and 2.4 can be weakened by adding a slowly varying function. This is addressed in Remark 4.7.

Theorem 4.5 (Convergence in distribution above the boundary)**.** *Assume* (***L1***)*,* (***L2***) *and that for all* $s \in \mathbb{R}$*,*

$$\mathbb{P}\left(\xi_1, \xi_2, \ldots, \in s\mathbb{Z}\right) < 1.$$

If $\theta > \theta_0$ *and* μ *has a finite* r*-th moment for some* $r > \frac{\theta_0}{\theta}$ *and is not concentrated on a single point, then*

$$R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} \xrightarrow[n\to\infty]{d} \frac{1}{\theta}\left(\log Z_{\frac{\theta_0}{\theta}} - \log E\right)$$

where $Z_{\frac{\theta_0}{\theta}}$ *is finite and positive almost surely and* E *is exponential with intensity 1, independent of* $Z_{\frac{\theta_0}{\theta}}$*. Furthermore,* $Z_{\frac{\theta_0}{\theta}} \overset{d}{=} (D_\infty)^{\frac{\theta}{\theta_0}} U_{\frac{\theta_0}{\theta}}$*, where* D_∞ *is the almost sure limit of the derivative martingale defined in* (4.6) *and* $U_{\frac{\theta_0}{\theta}}$ *is strictly* $\frac{\theta_0}{\theta}$*-stable independent of* D_∞*.*

In particular, if μ has finite mean and $\theta > \theta_0$, or if μ satisfies (**H**) and $\theta > \frac{\theta_0}{\gamma}$, then the assumption in the last theorem is satisfied. This result is also more extensive than Theorem 2.6 in [5], where the asymptotics were only given for the case $\mu = \delta_1$. It is worth noting that the logarithmic correction term in Theorems 4.4 and 4.5 or its absence in Theorem 4.3 correspond to corrections in classical settings, see e.g. [1, 7])

4.2 Proofs of Theorems 4.3, 4.4, and 4.5

We start with a short proof of the following identity.

$$\theta R_n^*(\theta) \stackrel{d}{=} \log Y_n(\theta) - \log E \tag{4.8}$$

where $Y_n(\theta) = \sum_{|v|=n} e^{\theta S_v} Y_v$ and E is exponential with parameter 1, independent of $Y_n(\theta)$. The equation is proven in [5] as Theorem 3.6, but we include it here for completeness.

Proof of (4.8). Take $f \in C_b(\mathbb{R})$ and let $\mathcal{F}_n = \sigma(S_v, Y_v : |v| = n)$ be the σ-algebra generated by $\{S_v\}_{|v|=n}$ and $\{Y_v\}_{|v|=n}$. Then

$$\begin{aligned}\mathbb{E}\left[f\left(\theta R_n^*(\theta)\right)\right] &= \mathbb{E}\left[f\left(\sup_{|v|=n} \theta S_v + \log \frac{Y_v}{E_v}\right)\right] = \mathbb{E}\left[\mathbb{E}\left[f\left(-\log \inf_{|v|=n} \frac{E_v}{e^{\theta S_v} Y_v}\right)\middle| \mathcal{F}_n\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f\left(-\log \frac{E_1}{\sum_{|v|=n} e^{\theta S_v} Y_v}\right)\middle| \mathcal{F}_n\right]\right] = \mathbb{E}\left[f\left(\log Y_n - \log E_1\right)\right]\end{aligned}$$

where the penultimate equality follows from the fact, that the minimum of independent exponential random variables with parameters λ_i, $i = 1 \dots n$, is again an exponential random variable, with parameter $\sum_{i=1}^n \lambda_i$. $\square$

Now we recall Lemma 4.1 from [13] (presented here in a slightly more accessible form for our use), that will be useful to understand behaviour of the asymptotics of Y_n.

Lemma 4.6. *Let $\{Y_v\}_{v\in\mathbb{T}}$ be i.i.d. random variables with distribution μ satisfying ($\boldsymbol{H}$), and $\{A_v\}_{v\in\mathbb{T}}$ be a sequence of positive random variables, independent of $\{Y_v\}_{v\in\mathbb{T}}$, such that*

$$\sum_{|v|=n} A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A \quad \text{and} \quad \sup_{|v|=n} A_v \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

for some positive random variable A. Then

$$\sum_{|v|=n} A_v Y_v \xrightarrow[n\to\infty]{d} H$$

where H has the characteristic function $\varphi_H(t) = \mathbb{E}\left[\tilde{g}(tA^{\frac{1}{\gamma}})\right]$.

Proof of Theorem 4.3. First we will prove that

$$Y_n(\theta) e^{-n\frac{\nu(\gamma\theta)}{\gamma}} \xrightarrow[n\to\infty]{d} H_\theta \tag{4.9}$$

where H_θ has the characteristic function $\mathbb{E}\tilde{g}\big(t(W^\infty_{\gamma\theta})^{\frac{1}{\gamma}}\big)$ and moreover H_θ is positive almost surely. For this purpose we will use Lemma 4.6 for $A_v = e^{\theta S_v - |v|\frac{\nu(\theta\gamma)}{\gamma}}$. To check its hypotheses observe

$$\sup_{|v|=n} A_v = \sup_{|v|=n} e^{n\theta\left(\frac{S_v}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right)} = e^{n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right)}.$$

As $\frac{R_n}{n} \xrightarrow[n\to\infty]{a.s.} \frac{\nu(\theta_0)}{\theta_0}$ and θ_0 is the unique argument minimizing $\frac{\nu(t)}{t}$, then $n\theta\left(\frac{R_n}{n} - \frac{\nu(\theta\gamma)}{\theta\gamma}\right) \xrightarrow[n\to\infty]{a.s.} -\infty$ and so $\sup_{|v|=n} A_v \xrightarrow[n\to\infty]{a.s.} 0$. Furthermore, in view of (4.5),

$$\sum_{|v|=n} A_v^\gamma = W_n\left(\theta\gamma\right)$$

converges to $W^\infty_{\gamma\theta}$, which is positive almost surely, because $\gamma\theta < \theta_0$. Summarizing, Lemma 4.6 entails (4.9).

To see that H_θ is positive almost surely, choose any $\varepsilon > 0$. Then using Exercise 3.3.2 from [21]

$$\begin{aligned}
\mathbb{P}\left(H_\theta = 0\right) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E}\left[\tilde{g}\big(t(W^\infty_{\gamma\theta})^{\frac{1}{\gamma}}\big)\right] dt \\
&\le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{E}\left[e^{-k|t|^\gamma W^\infty_{\gamma\theta}}\right] dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\mathbb{E}\left[e^{-k|t|^\gamma |W^\infty_{\gamma\theta}|}; W^\infty_{\gamma\theta} \le \varepsilon\right] + \mathbb{E}\left[e^{-k|t|^\gamma W^\infty_{\gamma\theta}}; W^\infty_{\gamma\theta} > \varepsilon\right]\right) dt \\
&\le \mathbb{P}\left(W^\infty_{\gamma\theta} \le \varepsilon\right) + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-k|t|^\gamma \varepsilon} dt
\end{aligned}$$

Now the function $t \to e^{-k\varepsilon|t|^\gamma}$ is integrable, hence the limit of the second term is 0 for any ε. The first term can be made arbitrarily small through the choice of ε. Since we know that $W^\infty_{\gamma\theta}$ is positive almost surely for $\gamma\theta < \theta_0$, we conclude positivity of H_θ.

Next, recalling (4.8)

$$R_n^* - n\frac{\nu\left(\gamma\theta\right)}{\gamma\theta} \stackrel{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu\left(\gamma\theta\right)}{\gamma\theta} = \frac{1}{\theta}\left(\log Y_n(\theta) e^{-n\frac{\nu(\gamma\theta)}{\gamma}} - \log E\right)$$

with $E \sim Exp\left(1\right)$ independent of Y_n. Finally, by (4.9) and the continuous mapping theorem

$$\log\left(Y_n(\theta) e^{-n\frac{\nu(\gamma\theta)}{\gamma}}\right) \xrightarrow[n\to\infty]{d} \log H_\theta$$

where the distribution of H_θ is as specified in the statement. □

Proof of Theorem 4.4(the boundary case). Define $A_v = e^{\theta S_v - n\frac{\nu(\theta_0)\theta}{\theta_0} + \frac{\theta}{2\theta_0}\log n}$. Then, by (4.7), we have

$$\sum_{|v|=n} A_v^\gamma = n^{\frac{1}{2}} W_n \xrightarrow[n\to\infty]{\mathbb{P}} \left(\frac{2}{\pi\sigma^2}\right)^{\frac{1}{2}} D_\infty = c_\infty D_\infty$$

The second condition of Lemma 4.6, $\sup_{|v|=n} A_v \xrightarrow[n\to\infty]{\mathbb{P}} 0$, follows by applying Proposition A.3. in [33] to $V_u = \theta_0 S_u - |u|\nu(\theta_0)$. Then once again by Lemma 4.6

$$Y_n(\theta) n^{\frac{\theta}{2\theta_0}} e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} = \sum_{|v|=n} A_v Y_v \xrightarrow[n\to\infty]{d} H_{\theta_0}. \tag{4.10}$$

Next, by (4.8)

$$R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n \overset{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu(\theta_0)}{\theta_0} + \frac{1}{2\theta_0}\log n$$
$$= \frac{1}{\theta}\left(\log\left(n^{\frac{\theta}{2\theta_0}} Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) - \log E\right)$$

and by (4.10) and the continuous mapping theorem

$$\log\left(n^{\frac{\theta}{2\theta_0}} Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) \xrightarrow[n\to\infty]{d} \log H_{\theta_0}$$

where distribution of H_{θ_0} is as specified in the statement. □

Remark 4.7. *It is clear that proofs of Theorems 4.3 and 4.4 rely on Lemma 4.6. If one were to allow a slowly varying function $L(x)$ in the assumption (**H**), then a close examination of the proof available in* [13] *reveals that the assumption $\sum_{|v|=n} A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A$ needs to be replaced with $\sum_{|v|=n} L(A_v^{-1})^\gamma A_v^\gamma \xrightarrow[n\to\infty]{\mathbb{P}} A$ and we have no tools to study convergence of such sequences without the martingale property.*

Proof of Theorem 4.5 (above the boundary case). The proof relies on Proposition 3.2 in [17]. The assumptions for Theorem 4.5 with condition $\theta_0\nu'(\theta_0) = \nu(\theta_0)$ for S are equivalent to assumptions **(A1)** through **(A3)** from [17] for a BRW

$$V_u = -\theta\left(S_u - |u|\frac{\nu(\theta_0)}{\theta_0}\right)$$

with critical parameter $\vartheta = \frac{\theta_0}{\theta}$. Proposition 3.2 in [17] entails

$$n^{\frac{3}{2}\vartheta} \sum_{|u|=n} e^{-V_u} Y_u \xrightarrow[n\to\infty]{d} Z_{\frac{\theta_0}{\theta}}$$

where $Z_{\frac{\theta_0}{\theta}}$ is positive almost surely. Furthermore, by equation (1.13) in [17], we have $Z_{\frac{\theta_0}{\theta}} \overset{d}{=} D^{\frac{1}{\vartheta}} U_\vartheta$, where D is the limit of the derivative martingale associated with $-\vartheta V$, and U_ϑ is strictly ϑ-stable independent of D. If we let ψ be the log-Laplace transform of $-\vartheta V$, then it satisfies the equation $\psi(1) = 0 = \psi'(1)$, so the derivative martingale associated with $-\vartheta V$ is

$$\sum_{|u|=n} \vartheta V_u e^{-\vartheta V_u} = -\sum_{|v|=n} (\theta_0 S_v - n\nu(\theta_0))\, e^{\theta_0 S_v - n\nu(\theta_0)} = D_n$$

so $D^{\frac{1}{\vartheta}} = (D_\infty)^{\frac{\theta}{\theta_0}}$. Therefore

$$n^{\frac{3\theta}{2\theta_0}} Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} = n^{\frac{3}{2}\vartheta} \sum_{|u|=n} e^{-V_u} Y_u \xrightarrow[n\to\infty]{d} Z_{\frac{\theta_0}{\theta}}$$

where the distribution of $Z_{\frac{\theta_0}{\theta}}$ is as in the statement. By (4.8) we have

$$\begin{aligned} R_n^* - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} &\overset{d}{=} \frac{1}{\theta}\left(\log Y_n(\theta) - \log E\right) - n\frac{\nu(\theta_0)}{\theta_0} + \frac{3\log n}{2\theta_0} \\ &= \frac{1}{\theta}\left(\log n^{\frac{3\theta}{2\theta_0}} Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} - \log E\right) \end{aligned}$$

and by the continuous mapping theorem

$$\log n^{\frac{3\theta}{2\theta_0}} Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}} \xrightarrow[n\to\infty]{d} \log Z_{\frac{\theta_0}{\theta}}$$

which completes the proof. □

4.3 Proof of Theorems 4.1 and 4.2

We start with the following Lemma, which gives the convergence in probability. It is an essential step in the proof of Theorems 4.1 and 4.2, as it provides the bound in (4.15).

Lemma 4.8.

(a) If conditions of Theorem 4.1 hold, then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{\nu(\gamma\theta)}{\gamma\theta}$$

(b) If conditions of Theorem 4.2 hold, then

$$\frac{R_n^*}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{\nu(\theta_0)}{\theta_0}$$

Proof. Let $\beta = \gamma\theta$ in case (a) and $\beta = \theta_0$ in case (b). We will prove first that

$$\frac{1}{n\theta} \log\left(Y_n(\theta) e^{-n\frac{\theta\nu(\beta)}{\beta}}\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{4.11}$$

We consider first the case (b). Fix an arbitrary $\varepsilon > 0$ and choose $\delta < \frac{\theta_0}{\theta}$ satisfying

$$\frac{\nu(\theta_0)}{\theta_0} - \frac{\nu(\delta\theta)}{\delta\theta} + \varepsilon > 0.$$

Such δ always exists, since ν is continuous and θ_0 is the unique argument minimizing $\frac{\nu(t)}{t}$ over $t > 0$. The Markov inequality yields

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{n\theta} \log\left(Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > \varepsilon\right) &= \mathbb{P}\left(\delta \log\left(Y_n(\theta) e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > n\theta\delta\varepsilon\right) \\
&= \mathbb{P}\left(Y_n(\theta)^\delta e^{-n\delta\frac{\theta\nu(\theta_0)}{\theta_0}} > e^{n\theta\delta\varepsilon}\right) \\
&\le \mathbb{E}[Y_n(\theta)^\delta] e^{-\delta n\theta\left(\frac{\nu(\theta_0)}{\theta_0}+\varepsilon\right)}.
\end{aligned}$$

Applying the well-known inequality $(a+b)^\delta \le a^\delta + b^\delta$, valid for any positive a, b and $\delta < 1$ and the fact that for any v the random variable Y_v is independent of S_v, we obtain

$$\mathbb{E}[Y_n(\theta)^\delta] = \mathbb{E}\left[\left(\sum_{|v|=n} e^{\theta S_v} Y_v\right)^\delta\right] \le \mathbb{E}\left[\sum_{|v|=n} e^{\theta\delta S_v} Y_v^\delta\right] = e^{n\nu(\theta\delta)} \mathbb{E}[Y^\delta], \tag{4.12}$$

where the last expectation is finite. Summarizing

$$\mathbb{P}\left(\frac{1}{n\theta}\log\left(Y_n(\theta)e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) > \varepsilon\right) \le \mathbb{E}[Y^\delta]e^{-\delta n\theta\left(\frac{\nu(\theta_0)}{\theta_0}-\frac{\nu(\delta\theta)}{\delta\theta}+\varepsilon\right)}$$

and thanks to our choice of δ the above expression converges to 0 as n tends to $+\infty$. To prove the remaining bound, denote $v_n = \arg\max_{|v|=n} S_v$. Since

$$\frac{1}{n\theta}\log\left(Y_n(\theta)e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) \ge \frac{1}{n\theta}\log\left(e^{n\theta\left(\frac{S_{v_n}}{n}-\frac{\nu(\theta_0)}{\theta_0}\right)}Y_{v_n}\right) = \frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \frac{1}{n\theta}\log Y_{v_n},$$

for any parameters $0<\delta<\varepsilon$ we obtain

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n\theta}\log\left(Y_n(\theta)e^{-n\frac{\theta\nu(\theta_0)}{\theta_0}}\right) < -\varepsilon\right) &\le \mathbb{P}\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \frac{1}{n\theta}\log Y_{v_n} < -\varepsilon\right)\\ &= \mathbb{P}\left(e^{n\theta\left(\frac{R_n}{n}-\frac{\nu(\theta_0)}{\theta_0}\right)}Y_{v_n} < e^{-\varepsilon n\theta}\right)\\ &= \mathbb{P}\left(e^{n\theta\left(\frac{R_n}{n}-\frac{\nu(\theta_0)}{\theta_0}+\varepsilon\right)}Y_{v_n} < 1\right)\\ &\le \mathbb{P}\left(e^{n\theta\delta}Y_{v_n}<1\right) + \mathbb{P}\left(\frac{R_n}{n} - \frac{\nu(\theta_0)}{\theta_0} + \varepsilon < \delta\right).\end{aligned}$$

Now, since $\frac{R_n}{n}$ converges almost surely to $\frac{\nu(\theta_0)}{\theta_0}$ and $\delta<\varepsilon$, the second term converges to 0. For the first term we have

$$\mathbb{P}\left(e^{n\theta\delta}Y_{v_n}<1\right) = \mathbb{P}\left(Y < e^{-n\theta\delta}\right) \to 0.$$

Thus, we conclude the proof of (4.11) for case (b).

For case (a), by Theorem 4.3, $\log Y_n(\theta)e^{-n\frac{\nu(\gamma\theta)}{\gamma}}$ converges in distribution to $\log H_\theta$ and this limit is finite almost surely. Therefore $\frac{1}{n\theta}\log\left(Y_n(\theta)e^{-n\frac{\nu(\gamma\theta)}{\gamma}}\right)$ converges in distribution to 0, and hence the convergence holds in probability as well. Thus, the proof of (4.11) is completed.

To prove Lemma 4.8 notice that using (4.8) we can write

$$\frac{R_n^*(\theta)}{n} \overset{d}{=} \frac{\log Y_n(\theta)}{n\theta} - \frac{\log E}{n\theta} = \frac{1}{n\theta}\log\left(Y_n(\theta)e^{-n\frac{\theta\nu(\beta)}{\beta}}\right) + \frac{\nu(\beta)}{\beta} - \frac{\log E}{n\theta}.$$

Now $\frac{\log E}{n\theta}$ converges to 0 almost surely and by (4.11), $\frac{1}{n\theta}\log Y_n(\theta)e^{-n\frac{\theta\nu(\beta)}{\beta}}$ converges to 0 in probability. That completes the proof of the Lemma.

□

Proof of Theorems 4.1 and 4.2 (almost sure convergence). To prove the almost sure convergence we utilize here the arguments given in the proof of Theorem 2.1 in [5]. For the sake of completeness, we present a complete proof. We note that the main difference is that, due to Lemma 4.8, there is no need to treat separately the cases below and above the boundary. Again, let $\beta = \gamma\theta$ if conditions of Theorem 4.1 are satisfied and $\beta = \theta_0$ if conditions of Theorem 4.2 are satisfied. We start with the upper bound

$$\limsup_{n\to\infty} \frac{R_n^*(\theta)}{n} \leq \frac{\nu(\beta)}{\beta} \qquad \text{a.s.} \tag{4.13}$$

Fix any $\varepsilon > 0$. By (4.8) and the Markov inequality we get that for any $\delta < \min(\frac{\theta_0}{\theta}, 1)$

$$\begin{aligned}
\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} > \varepsilon\right) &= \mathbb{P}\left(\theta\delta R_n^*(\theta) - \frac{\theta\delta n\nu(\beta)}{\beta} > n\delta\theta\varepsilon\right) \\
&= \mathbb{P}\left(\log\frac{Y_n(\theta)^\delta}{E^\delta} - \frac{\theta\delta n\nu(\beta)}{\beta} > n\delta\theta\varepsilon\right) \\
&\leq e^{-\delta n\theta\left(\frac{\nu(\beta)}{\beta}+\varepsilon\right)}\mathbb{E}\left[E^{-\delta}\right]\mathbb{E}\left[Y_n(\theta)^\delta\right] \\
&\leq e^{-\delta n\theta\left(\frac{\nu(\beta)}{\beta}-\frac{\nu(\theta\delta)}{\theta\delta}+\varepsilon\right)}\Gamma(1-\delta)\,\mathbb{E}\left[Y^\delta\right],
\end{aligned}$$

where the last inequality follows from (4.12).

Since ν is continuous, we can choose δ so that

$$\frac{\nu(\beta)}{\beta} - \frac{\nu(\delta\theta)}{\delta\theta} + \varepsilon > 0.$$

Therefore the series

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} > \varepsilon\right)$$

converges. The Borel-Cantelli lemma and arbitrariness of ε entails (4.13).

Finally our goal is to prove the lower bound

$$\liminf_{n\to\infty} \frac{R_n^*(\theta)}{n} \geq \frac{\nu(\beta)}{\beta} \qquad \text{a.s.} \tag{4.14}$$

For u such that $|u| = m \leq n$, we define

$$R_{n-m}^{*(u)}(\theta) := \max_{v>u,|v|=n}\left(S(v) + \frac{1}{\theta}\log(Y_v/E_v)\right) - S(u),$$

where $v > u$ means that v is a descendant of u. Note that, due to the branching property of S, $\{R_{n-m}^{*(u)}(\theta)\}_{|u|=m}$ are i.i.d. and have the same distribution as $R_{n-m}^*(\theta)$.

Now,

$$\begin{aligned}R_n^*(\theta) &= \max_{|u|=m} \max_{v>u,|v|=n} \left(S(v) + \frac{1}{\theta}\log(Y_v/E_v) \right) \\ &= \max_{|u|=m} \left(S(u) + R_{n-m}^{*(u)}(\theta) \right) \\ &\geq S(\tilde{u}_m) + \max_{|u|=m} \left(R_{n-m}^{*(u)}(\theta) \right),\end{aligned}$$

where

$$\tilde{u}_m := \arg\max_{|u|=m} \left(R_{n-m}^{*(u)}(\theta) \right).$$

Now, for any $\varepsilon \in (0,1)$ and small s such that $\nu(-s/2)$ is finite,

$$\begin{aligned}&\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon \right) \\ \leq\ & \mathbb{P}\left(S\left(\tilde{u}_{[\sqrt{n}]}\right) + \max_{|u|=[\sqrt{n}]} \left(R_{n-[\sqrt{n}]}^{*(u)}(\theta) \right) < n\left(\frac{\nu(\beta)}{\beta} - \varepsilon \right) \right) \\ \leq\ & \mathbb{P}\left(\max_{|u|=[\sqrt{n}]} \left(R_{n-[\sqrt{n}]}^{*(u)}(\theta) \right) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2} \right) \right) + \mathbb{P}\left(S\left(\tilde{u}_{[\sqrt{n}]}\right) < -\frac{n\varepsilon}{2} \right) \\ \leq\ & \mathbb{E}\left[\mathbb{P}\left(R_{n-[\sqrt{n}]}^{*}(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2} \right) \right)^{N_{[\sqrt{n}]}} \right] + e^{-n\varepsilon s/4} \cdot \mathbb{E}\left[e^{-sS\left(\tilde{u}_{[\sqrt{n}]}\right)/2} \right],\end{aligned}$$

where N_k is the number of offspring in k-th generation. Recalling Lemma 4.8, for all large enough n,

$$\mathbb{P}\left(R_{n-[\sqrt{n}]}^{*}(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2} \right) \right) < \varepsilon. \tag{4.15}$$

We have

$$\begin{aligned}\mathbb{E}\left[\mathbb{P}\left(R_{n-[\sqrt{n}]}^{*}(\theta) < n\left(\frac{\nu(\beta)}{\beta} - \frac{\varepsilon}{2} \right) \right)^{N_{[\sqrt{n}]}} \right] &\leq \mathbb{E}[\varepsilon^{N_{[\sqrt{n}]}}] \leq \mathbb{E}\left[\varepsilon^n; N_{[\sqrt{n}]} \geq n \right] + \mathbb{P}\left(N_{[\sqrt{n}]} < n \right) \\ &\leq \varepsilon^n + \mathbb{P}\left(N_{[\sqrt{n}]} < n \right)\end{aligned}$$

If $\mathbb{P}(N=1) = 0$, then $N_{[\sqrt{n}]} \geq 2^{[\sqrt{n}]}$, so $\mathbb{P}\left(N_{[\sqrt{n}]} < n\right)$ obviously disappears. Otherwise, if $\mathbb{P}[N=1] > 0$, then as seen in [27] (Corollary 5 with equations (29) and (4b)) , there are positive constants $C > 0$ and $\alpha > 0$, such that for all large enough $k \in \mathbb{N}$,

$$\mathbb{P}\left(N_k < k^2 \right) \leq C m^{-\alpha k},$$

where $m = \mathbb{E}[N]$. Therefore

$$\varepsilon^n + \mathbb{P}\left(N_{[\sqrt{n}]} < n \right) \leq \varepsilon_1^{\lceil \sqrt{n} \rceil}$$

for some $\varepsilon_1 < 1$. To estimate the second term, we bound supremum by the sum and we have

$$\mathbb{E}\left[e^{-sS\left(\tilde{u}_{[\sqrt{n}]}\right)/2}\right] \leq \mathbb{E}\left[\sum_{|v|=\lceil\sqrt{n}\rceil} e^{-\frac{s}{2}S_v}\right] = e^{[\sqrt{n}]\nu(-s/2)}.$$

Therefore we have for all large enough n,

$$\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon\right) \leq \varepsilon_1^{\lceil\sqrt{n}\rceil} + e^{-n\varepsilon s/4+[\sqrt{n}]\nu(-s/2)}.$$

Since for every $\varepsilon \in (0,1)$,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\frac{R_n^*(\theta)}{n} - \frac{\nu(\beta)}{\beta} < -\varepsilon\right) < \infty,$$

using the Borel-Cantelli Lemma once again we deduce (4.14), completing the proof. □

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