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Asymptotics of functionals of Gaussian and Lévy processes with a view towards risk and queueing models

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Asymptotyki funkcjonałów procesów gaussowskich i Lévy'ego z uwzględnieniem modeli ryzyka i kolejek

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Chapter 1

Introduction

Research on the distributional properties of extremes of stochastic processes has attracted growing attention in recent literature. These studies are motivated by both the theoretical aspects of extreme value theory [7, 33, 37, 45, 51–55, 57, 61, 64, 71, 75, 79–81, 90, 94, 97, 101–103, 113–117, 133] and applied probability problems, including finance, physics, risk theory, and queueing theory [3–6, 9, 24, 25, 38, 40, 42–44, 48, 49, 64, 67, 70, 89, 93, 101, 102, 104, 110, 123]. In recent years, the asymptotic behavior of various functionals of stochastic processes was intensively studied for both Gaussian and Lévy models [5, 7, 9, 17, 18, 25, 30, 33–35, 37–39, 42–44, 47, 50–54, 57–59, 71, 75, 77, 79–84, 89–91, 93–95, 97–100, 104, 107–111, 116, 134], which commonly describe a broad class of phenomena and require different proof techniques based on the specific distributional properties of each of class of stochastic processes. Early studies mainly focused on one-dimensional problems for supremum, infimum and sojourn-time functionals; see e.g. [10–14, 113–115, 117]. Contemporary works, such as [2, 6, 7, 17, 31–46, 50, 51, 86, 87, 89–91, 97–100], emphasize developing methods for general functionals of stochastic processes, often multidimensional, and on extremes over random time horizons. In the most general form, the problems analyzed in the aforementioned literature can be formulated as investigation of

$$\mathbb{P}\{\Theta(\{X(\boldsymbol{t}) : \boldsymbol{t} \in E\}) \in K_u\},$$

where $\{X(\boldsymbol{t}) : \boldsymbol{t} \in \mathbb{R}^d\}$ is a stochastic process or a field, E is a subset of $\mathbb{R}^d$, $\Theta(\cdot)$ is a $\mathbb{R}^q$-valued functional, and where $K_u \subset \mathbb{R}^q$ is a family of sets depending on u (usually $K_u = [u, \infty)$) satifying the *rare event regime*, i.e. where the above probability goes to zero as u tends to infinity.

Gaussian processes play a central role in applied probability due to their versatile covariance structures and their significant occurrence in central limit theorems. The celebrated Donsker's Theorem given in [62] demonstrates the convergence of certain random walks to Brownian motion, which provides a fundamental motivation to study various settings leading to functional limit theorems. A seminal work of Iglehart [83] shows that a certain sequence of Cramér-Lundberg processes under the so-called *diffusion approximation regime* converges to the *Brownian risk model*. Further research (e.g. [108, 127]) demonstrates that some sequences of processes converge to a self-similar Gaussian process, and, in particular, to fractional Brownian motion $\{B_H(t) : t \geq 0\}$ with Hurst parameter $H \in (0, 1]$, defined as a Gaussian process that satisfies

1. $B_H(0) = 0$ a.s.,
2. $B_H(t) \sim \mathcal{N}(0, t^{2H})$,
3. $\mathbb{C}ov(B_H(t), B_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$,
4. $\{B_H(t) : t \geq 0\}$ has almost surely continuous paths.

Naturally, Brownian motion is a specific case of fractional Brownian motion with $H = \frac{1}{2}$.

The foundational papers of James Pickands III [113–115] initiated intensive research on the extremes of Gaussian processes and provided many proof techniques for their analysis. In particular, Pickands [114] derived the asymptotic behavior of the probability of high excursions, as $u \to \infty$,

$$\mathbb{P}\{\sup_{t\in[0,T]} (X(t)) > u\},$$

for a broad class of centered stationary Gaussian processes $\{X(t) : t \geq 0\}$. This analysis was based on techniques now known as the Pickands lemma and the double-sum method. These techniques, along with the Borell-TIS inequality and comparison inequalities (e.g., the Slepian, Gordon, and Sudakov-Fernique inequalities), are essential for studying the extremes of Gaussian stochastic processes. They were later extended in the fundamental monograph of Piterbarg [117] and synthesized in the books of Lifshits [105] and Adler & Taylor [1]. Recent research on Gaussian extremes is focused on the multidimensional problems and more general functionals of Gaussian processes and fields, extending these classical techniques; see e.g. [7, 10–14, 17, 33–47, 50–54, 56, 59, 71, 77, 82, 89–91, 94, 104, 111, 116].

In contrast to Gaussian processes, Lévy processes are well-suited for modeling phenomena with discontinuities. As a result, Lévy processes are widely studied in applied probability, including such fields such as physics, communication networks, finance, and insurance [3–6, 9, 49, 64, 69, 95, 101, 102, 109]. Particular attention is given to light-tailed Lévy processes $\{X(t) - ct : t \geq 0\}$ satisfying a *Cramér condition*, namely the existence of $\omega > 0$ such that

$$\mathbb{E}\{e^{\omega(X(1)-c)}\} = 1 \ \text{ and } \ \mathbb{E}\{e^{\omega(X(1)-c)}|X(1) - c|\} < \infty.$$

Several methods for investigating Lévy processes based on their distributional properties were widely developed in the literature [3, 4, 101, 102]. These methods include, Doob's techniques, stopping-time arguments, the Markov property, the exponential change of measure technique, and the Wiener-Hopf factorization, among others.

Building on this foundation, this dissertation investigates specific problems related to risk and queueing models. It introduces new techniques, such as an extension of Breiman's lemma, and develops known methods to prove results in various areas, as described below.

In Chapter 2, we analyze the sojourn-time functional of a two-dimensional correlated Brownian motion with drift on the finite time horizon $[0, T]$. Specifically, let $\{\boldsymbol{W}(t) \equiv (W_1(t), W_2(t)) : t \geq 0\}$, be a two-dimensional Brownian motion with constant correlation $\rho \in (-1, 1)$, and let $\boldsymbol{a}, \boldsymbol{c} > (0, 0)$. We consider the sojourn time functional

$$\int_{[0,T]} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}\, t > \boldsymbol{a}\} dt,$$

that in the risk theory appears in the context of the *cumulative Parisian ruin* probability, which describes the probability of the event that ruin persists for a significant period of time. We focus on the so-called *many-source regime*, in which the risk process $\boldsymbol{R}(t)$ is composed of a large number of i.i.d. sub-risk processes $\boldsymbol{R}^{(k)}(t)$ representing independent companies. More precisely, let $\{\boldsymbol{W}^{(k)}(t) : t \geq 0\}$ for $k = 1, \ldots, N$ be i.i.d. copies of $\{\boldsymbol{W}(t) : t \geq 0\}$, and define the aggregate process

$$\boldsymbol{R}(t) = \sum_{k=1}^{N} \left(\boldsymbol{a} + \boldsymbol{c}t - \boldsymbol{W}^{(k)}(t) \right) \stackrel{d}{=} \boldsymbol{a}N + \boldsymbol{c}Nt - \sqrt{N}\boldsymbol{W}(t),$$

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where $\{\boldsymbol{W}(t) : t \geq 0\}$ is a two-dimensional correlated Brownian motion with a constant correlation $\rho \in (-1, 1)$. Then, the probability of interest can be written as

$$\mathbb{P}\{\int_{[0,T]} \mathbb{I}\{\boldsymbol{R}(t) < \boldsymbol{0}\}dt > H(u)\} = \mathbb{P}\{\int_{[0,T]} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\}dt > H(u)\},$$

with $u = \sqrt{N}$ and some function $H(u) \geq 0$. Note that the above expression describes the probability that the bivariate risk process $\boldsymbol{R}(t)$ spends below zero at least $H(u)$ time units within the time interval $[0, T]$. In particular, the case $H(u) \equiv 0$ recovers the *simultaneous ruin* probability.

In this chapter, we derive the exact asymptotic behavior of the cumulative Parisian ruin probability, as $u \to \infty$, for $H(u) = xu^{-2}$. The analysis reveals two regimes in which the relationship between the two components leads to either a dimension-reduction or full-dimensional scenario.

In Chapter 3, we study a stationary Gaussian queue with a fractional Brownian input. Following Reich's construction [120], the stationary buffer content process is represented as follows

$$Q(t) = \sup_{-\infty < s \leq t} \left(B_H(t) - B_H(s) - c(t-s)\right),$$

where $\{B_H(t) : t \in \mathbb{R}\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ and $c > 0$ is a constant service rate. The asymptotic behavior of the buffer overflow probability

$$\mathbb{P}\{\sup_{t \in [0, \mathcal{T}_u]} (Q(t)) > u\}$$

over deterministic time horizons $[0, \mathcal{T}_u]$ (which may depend on u) was extensively analyzed in the literature; see e.g. [46, 79, 118]. More precisely, it was shown that for broad classes of deterministic functions $\mathcal{T}_u$ satysfying either $\mathcal{T}_u \to c \in [0, \infty)$ or $\mathcal{T}_u \to \infty$, as $u \to \infty$, one can obtain the exact asymptotics of

$$\mathbb{P}\{\sup_{t \in [0, \mathcal{T}_u]} (Q(t)) > u\}, \quad u \to \infty.$$

Motivated by the study of the models with random resettings [67, 112, 123], we focus on the analysis of the overflow probability over a random time interval. Specifically, let $\{T_k : k \in \mathbb{N}\}$ be i.i.d. copies of a non-negative random variable T, which are independent of $\{Q(t) : t \geq 0\}$, and consider the asymptotic behavior of the probability

$$\mathbb{P}\{\sup_{t \in [0, u^\gamma \sum_{k=1}^{u^\beta} T_k]} (Q(t)) > u\}, \quad u \to \infty,$$

for $\beta \geq 0, \gamma \in \mathbb{R}$. We analyze the above quantity in three regimes distinguishing the heaviness of a random variable T:

D1: integrable, i.e. $\mathbb{E}\{T\} < \infty$,

D2: regularly varying tail with index $\alpha \in (0,1)$,

D3: slowly varying tail.

We demonstrate that each regime leads to a qualitatively different asymptotic behavior and needs distinct proof techniques. In scenario **D1**, the asymptotic results over deterministic intervals and Karamata's theorem play a crucial role. The heavy-tailed scenarios (**D2** and **D3**) require an extension of Lemma 3 in [79] to the uniform convergence (with respect to x) in each compact subset of $(0,\infty)$ for "very long" intervals $[0,\mathcal{T}_u]$, where

$$\mathcal{T}_u = x/\mathbb{P}\{\sup_{t\in[0,1]}(Q(t)) > u\} \sim x\frac{1}{c_2}u^{-h}e^{\frac{1}{2}A^2u^{2(1-H)}},$$

with suitable constants $c_2, h, A > 0$.

Chapter 4 is devoted to the extremes of the functional of a stationary queue fed by a Lévy process $\{X(t) : t \in \mathbb{R}\}$ satisfying the Cramér condition. Similarly to Chapter 3, we consider the stationary queue

$$\{Q(t) \equiv \sup_{-\infty<s\leq t}(X(t) - X(s) - c(t-s)) : t \geq 0\},$$

with the service rate $c > 0$. We study the asymptotic behavior of general functionals of the sample paths of $\{Q(t) : t \geq 0\}$ restricted to a compact domain $E \subset [0,\infty)$. More precisely, we analyze a functional $\Theta : D(E) \to \mathbb{R}$ that satisfies

F1: $\Theta(f) \leq \sup_{t\in E}(f(t))$ for any $f \in D(E)$,

F2: $\Theta(af + b) = a\Theta(f) + b$ for any $f \in D(E)$ and $a > 0$, $b \in \mathbb{R}$,

and derive the exact asymptotics of

$$\mathbb{P}\{\Theta(\{Q(t) : t \in E\}) > u\}, \quad u \to \infty.$$

Our results generalize the work of Baurdoux et al. [9] on the light-tailed Lévy queues, in which the supremum functional on a constant time interval was analyzed. In the proof presented in this thesis, we use the celebrated Breiman lemma to reduce the tail analysis to the product of independent random components - the expected value and the tail distribution function of two independent random variables.

In Chapter 5, we extend Fougeres & Mercadier's version of Breiman's lemma [68] on products of the random matrices and the random vectors indexed by u (and possibly by an additional parameter τ_u), and apply this result to derive exact asymptotics for several Gaussian models. Specifically, we consider two random families defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$: $\mathbb{R}^d$-valued vectors $\{\boldsymbol{X}_{u,\tau_u} : u \geq 0\}$ and $\mathbb{R}^{q\times d}$-valued matrices $\{\boldsymbol{M}_{u,\tau_u} : u \geq 0\}$. For suitable normalizing functions $a_{u,\tau_u}, b_{u,\tau_u} > 0$, we analyze the vague convergence of

$$b_{u,\tau_u}\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}\cdot\}, \quad u \to \infty.$$

Under certain assumptions, we obtain a limit measure that describes the extremal behavior of such products.

The extension of Breiman's lemma allows to analyze a wide range of problems in extreme value theory. In particular, it provides an alternative, direct proof of asymptotic behavior for various Gaussian models. As a corollary to the main finding of this chapter, we present an independent proof of the celebrated *Pickands lemma*, that is, for a centered stationary Gaussian process $\{X(t) : t \geq 0\}$ with a covariance function that satisfies

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0,$$

we show that for any relatively compact Borel set $K \subset (0, \infty]$ with $\int_{\partial K} dx = 0$,

$$\mathbb{P}\{\sup_{t\in[0,Tu^{-2/\alpha}]}(X(t)) \in u + \tfrac{1}{u}\log(K)\} \sim \mathcal{H}^{\sup}_{B_{\alpha/2},0}([0,T]) \int_K \frac{1}{x^2} dx\, \Psi(u), \text{ as } u \to \infty,$$

where $\mathcal{H}^{\sup}_{B_{\alpha/2},0}([0,T])$ is a Pickands constant on $[0,T]$ ans $\Psi(\cdot)$ is the tail distribution function of a standard normal random variable. Note that the above result generalizes the Pickands lemma to a broad class of sets of the form $u + \frac{1}{u}\log(K)$, whereas the Pickands lemma only covers the special case $K = [1, \infty)$; see e.g. [114] and [117][Lemma D.1].

We also prove uniform Pickands lemma for homogeneous functionals of Gaussian fields, and analyze extremes of the supremum of self-standardized Gaussian processes, which are related to Gamma bridge [20, 65, 66, 74, 122, 132] and Gaussian processes with random variance [75, 80, 116].

1.1 Notation

We introduce some basic mathematical notation that will be used consistently throughout the thesis. Let $f(\cdot)$ and $g(\cdot)$ be positive functions. As $x \to \infty$, we write

- $f(x) \sim g(x)$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
- $f(x) = o(g(x))$ if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$,
- $f(x) = O(g(x))$ if there exists $M > 0$ such that $\left|\frac{f(x)}{g(x)}\right| \le M$ for all x.

For vectors $\boldsymbol{x} = (x_1, \cdots, x_n)$, $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^d$, we define the component-wise product as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}\boldsymbol{y} = (x_1 y_1, \cdots, x_n y_n)$$

and write

$$\boldsymbol{x} \ge \boldsymbol{y} \text{ if and only if } x_i \ge y_i \text{ for all } i \in \{1, \cdots, n\}.$$

For an invertable matrix $\boldsymbol{M} \in \mathbb{R}^{q\times d}$ and a $K \subset \mathbb{R}^q$, define the preimage under M as

$$\boldsymbol{M}^{-1}K = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{M}\boldsymbol{x} \in K\}.$$

Let $\Gamma(\cdot)$ denote the *Gamma function*, defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

The notation $\mathcal{X} \stackrel{d}{=} \mathcal{Y}$ denotes equality in distribution both between two random variables, vectors, or stochastic processes.

Let (X, Y) be a centered bivariate normal random vector with $\mathbb{V}ar\{X\} = \mathbb{V}ar\{Y\} = t$ and $\mathbb{C}ov(X, Y) = \rho t$. Denote by $\varphi_t(\cdot, \cdot)$ the joint probability density function of (X, Y), and by $\varphi_t(\cdot)$ the marginal density function of X. Additionally let $\Psi(\cdot)$ be the tail distribution function of a standard normal random variable. Recall that, as $x \to \infty$,

$$\Psi(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}}.$$

We denote the extended real line by $\overline{\mathbb{R}} = [-\infty, \infty]$.

For a metric space X (e.g., $\mathbb{R}^d$, $\overline{\mathbb{R}^d}$ or $[0,\infty)^d$), we define the following function spaces

- $C(X)$ – the space of continuous functions $f(\cdot)$ on X, equipped with the sup-norm,
- $C_0(X)$ – the space of continuous functions $f(\cdot)$ on X such that $f(0) = 0$, equipped with the sup-norm,
- $D(X)$ – the Skorokhod space of càdlàg functions $f(\cdot)$ on X, equipped with the Skorokhod norm.

Let $E \subset X$ be a measurable subset and $\mathcal{B}_E$ the Borel σ–algebra on E. A sequence of measures $\{\nu_n : n \in \mathbb{N}\}$ is said to converge vaguely to a Radon measure ν on $\mathcal{B}_E$, denoted $\nu_n(\cdot) \xrightarrow{v} \nu(\cdot)$, if

$$\int_E f(\boldsymbol{x}) d\nu_n(\boldsymbol{x}) \to \int_E f(\boldsymbol{x}) d\nu(\boldsymbol{x}), \text{ as } n \to \infty,$$

for all continuous function $f(\cdot)$ with compact support E. Equivalently, $\nu_n(K) \to \nu(K)$, as $n \to \infty$, for all relatively compact subsets $K \subset E$ such that $\nu(\partial K) = 0$. Recall that a Radon measure on $\mathcal{B}_E$ is finite on all compact subsets of E.

Let $\{\boldsymbol{X}_u : u \geq 0\}$ be a family of random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $\{\boldsymbol{X}_u : u \geq 0\}$ converges vaguely to a random vector $\boldsymbol{X}$, if the associated family of distributions converges vaguely to the distribution of $\boldsymbol{X}$, as $u \to \infty$.

A Radon measure ν on $\mathcal{B}_E$ is said to be homogeneous with index $\alpha > 0$, if

$$\nu(tK) = t^{-\alpha}\nu(K), \quad \text{for all } t > 0 \text{ and all relatively compact } K \subset E.$$

A positive function $f(\cdot)$ is said to be regularly varying at $x_0 \in \overline{\mathbb{R}}$ with index $\alpha \in \mathbb{R}$, which we denote $f \in \mathcal{RV}_{x_0}(\alpha)$, if

$$\lim_{x \to x_0} \frac{f(tx)}{f(x)} = t^{\alpha}, \text{ for all } t > 0.$$

If $\alpha = 0$, then $f(\cdot)$ is said to be slowly varying at x_0 and is denoted $f \in \mathcal{SV}_{x_0}$.

A non-negative random variable X is called regularly (or slowly) varying with index α, if its tail distribution function is regularly (or slowly) varying at ∞ with index $-\alpha$.

Chapter 2

Sojourn time for correlated Brownian motion with drift

2.1 Introduction

Let $\{\boldsymbol{W}(t) \equiv (W_1(t), W_2(t)) : t \geq 0\}$ be a two-dimensional correlated Brownian motion with a constant correlation $\rho \in (-1, 1)$, i.e. $\mathbb{C}ov(W_1(t), W_2(t)) = \rho t$. For a deterministic drift vector $\boldsymbol{c} = (c_1, c_2) > (0, 0)$ and a threshold vector $\boldsymbol{a} = (a_1, a_2) > (0, 0)$, we consider the sojourn-time functional

$$S_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) = \int_{[0,T]} \mathbb{I}\{W_1(s) - c_1us > a_1u,\ W_2(s) - c_2us > a_2u\}ds.$$

In particular, we are interested in the tail distribution function

$$s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; H(u)) := \mathbb{P}\{S_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u) > H(u)\}, \tag{2.1}$$

for some $H(u) \geq 0$, which describes the probability that the bivariate correlated Brownian motion with drift $-\boldsymbol{c}u$ spends above the threshold $\boldsymbol{a}u$ at least $H(u)$ time units within the time interval $[0, T]$.

Functionals of the type $S_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u)$ appear in research related to the risk theory [25, 47, 70, 82, 89, 98, 100, 111], financial mathematics [21, 24, 76, 106, 129], and queueing theory [38, 39] (see also [10, 50, 59]).

The scaling c_iu and a_iu occurs naturally in many applications involving the aggregation of a large number of sources, companies or customers, and has been investigated in the various application contexts, for example, in [48, 49, 82, 93, 94, 104].

In the risk theory, the surplus processes of two lines of business can be modeled as:

$$R_i(t) = a_i + c_i t - W_i(t),\ i = 1, 2,$$

where $W_i(t)$ represents the amount of cumulative claims in time interval $[0, t)$, c_i is the premium rate, and a_i is the initial reserve; see [83]. The ruin event occurs in the interval $[0, T]$ when $R_i(t) < 0$, or equivalently, $W_i(t) - c_i t > a_i$, for some $t \in [0, T]$. Recently, this model was extended to a many-source regime, in which the risk process $\boldsymbol{R}(t)$ is composed of a large number of i.i.d. sub-risk processes $\boldsymbol{R}^{(k)}(t)$ representing independent companies; see e.g. [82, 93]. More precisely, $\boldsymbol{R}(t)$ is defined as follows:

$$\boldsymbol{R}(t) = \sum_{k=1}^{N} \boldsymbol{R}^{(k)}(t) = \sum_{k=1}^{N} \left(\boldsymbol{a} + \boldsymbol{c}t - \boldsymbol{W}^{(k)}(t) \right) \stackrel{d}{=} \boldsymbol{a}N + \boldsymbol{c}Nt - \sqrt{N}\boldsymbol{W}(t),$$

where $\{\boldsymbol{W}^{(k)}(t) : t \geq 0\}$, for $k = 1, \cdots, N$ are mutually independent copies of $\{\boldsymbol{W}(t) : t \geq 0\}$. In this setting, $s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; 0)$ with $u = \sqrt{N}$ denotes the *simultaneous ruin* probability, while $s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; H(u))$ can be interpreted as the *cumulative Parisian ruin* probability, which describes the probability of the event that ruin persists for a significant period of time, the so-called *occupation time in red*; see e.g. [33, 40, 43, 47, 82, 89, 91, 93, 94, 98–100].

In the financial mathematics, similar Brownian models arise in the pricing of exotic derivatives such as Parisian options, where payoffs depend on the time an asset price spends above or below a barrier. In the context of multi-asset portfolios, the common time spent above barriers can model systemic risks or trigger cumulative loss mechanisms [21, 24, 76, 106, 129].

In this chapter, we analyze the exact asymptotic behavior, as $u \to \infty$, of $s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; H(u))$ with $H(u) = zu^{-2}$, $z > 0$. This regime was extensively studied in the literature; see e.g. [39, 47, 50, 59, 98–100, 111].

2.2 Main results

Before proceeding to the main part of this chapter, we introduce notation that goes in line with the one used in the related literature (see e.g. [33, 89]).

For a matrix $M \in \mathbb{R}^{2\times 2}$ and index sets $A, B \subset \{1,2\}$, we write

$$M_{AB} = (m_{ab})_{a\in A, b\in B}$$

and for a vector $v \in \mathbb{R}^2$, we write

$$v_A = (v_a)_{a\in A}.$$

Let $I(t)$ be the essential index set, defined in [33] [Lemma 2.1] and let $I^{\complement}(t)$ be its complement with respect to the set $\{1,2\}$, as follows.

Lemma 2.2.1 *([33] [Lemma 2.1]) For the quadratic optimisation problem*

$$q(t) = \min_{\boldsymbol{x} > (\boldsymbol{a}+\boldsymbol{c}t)} \boldsymbol{x}\Sigma_t^{-1}\boldsymbol{x}^\top \tag{2.2}$$

there exist unique $\tilde{\mathbf{a}}$, $\tilde{\mathbf{c}}$ *and non-empty index set* $I(t) \subset \{1,2\}$ *such that*

$$(\tilde{\boldsymbol{a}} + \tilde{\boldsymbol{c}}t)_{I(t)} = (\boldsymbol{a}+\boldsymbol{c}t)_{I(t)} \neq \mathbf{0}_{I(t)}, (\tilde{\boldsymbol{a}} + \tilde{\boldsymbol{c}})_{I^{\complement}(t)} = (\Sigma_t)_{I^{\complement}(t)I(t)}(\Sigma_t^{-1})_{I(t)I(t)}(\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}$$

so

$$(\Sigma_t^{-1})_{I(t)I(t)}(\boldsymbol{a}+\boldsymbol{c}t)_{I(t)} > \mathbf{0}_{I(t)} \tag{2.3}$$

and

$$\min_{\boldsymbol{x} > (\boldsymbol{a}+\boldsymbol{c}t)} \boldsymbol{x}\Sigma_t^{-1}\boldsymbol{x}^\top = (\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}(\Sigma_t^{-1})_{I(t)I(t)}(\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}^\top.$$

We further define

$$\begin{aligned} K(t) &:= \{k \in I^{\complement}(t) : (\Sigma_t)_{kI(t)}(\Sigma_t^{-1})_{I(t)I(t)}(\boldsymbol{a}+\boldsymbol{c}t)_{I(t)} = a_k + c_k t\}, \\ J(t) &:= \{j \in I^{\complement}(t) : (\Sigma_t)_{jI(t)}(t)(\Sigma_t^{-1})_{I(t)I(t)}(\boldsymbol{a}+\boldsymbol{c}t)_{I(t)} > a_j + c_j t\}, \end{aligned}$$

where $(\Sigma_t)_{jI(t)} = (\Sigma_t)_{\{j\}I(t)} = \big((\sigma_t)_{ji}\big)_{i\in I(t)}$.

Let

$$t_0 := \sqrt{\frac{\boldsymbol{a}_I\Sigma_{II}^{-1}\boldsymbol{a}_I^T}{\boldsymbol{c}_I\Sigma_{II}^{-1}\boldsymbol{c}_I^T}} > 0$$

denotes the unique minimizer of (2.2) in the interval $[0,\infty)$, and define

$$t_0^* := \min(t_0, T). \tag{2.4}$$

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It turns out that t_0^* is the minimizer of (2.2) over $[0, T]$; see Lemma 2.3.1.

Finally, set

$$I := I(t_0^*),\ K := K(t_0^*) \text{ and } J := J(t_0^*).$$

Following e.g. [33], we refer to I as the *essential index set*, K as the *weakly essential index set* (which only contributes to constant terms in the asymptotics), and J as the *unessential index set* (which does not affect the leading asymptotics). Since (2.3) holds for the non-empty index set I, we observe that

$$a_1 - a_2\rho > (\rho c_2 - c_1)\min(t_0, T) \text{ or } a_2 - a_1\rho > (\rho c_1 - c_2)\min(t_0, T). \tag{2.5}$$

We therefore distinguish between two regimes:

(a) *Dimension-reduction case*, where exactly one of the inequalities in (2.5) holds and one coordinate asymptotically dominates the other ($|I| = 1$),

(b) *Full-dimensional case*, where both inequalities in (2.5) hold and both coordinates influence the asymptotics ($|I| = 2$).

For the sake of notational simplicity, and in view of the symmetry of the results, in the following theorems we write $\neg 1 := 2$, $\neg 2 := 1$ and (a_1, a_2) instead of $(1, a_2)$. Let us recall that $\varphi_t(\cdot, \cdot)$ denotes the joint probability density function of $(W_1(t), W_2(t))$, while $\varphi_t(\cdot)$ refers to the marginal density function of $W_i(t)$.

Theorem 2.2.2 (Dimension-reduction case)

(i) If $I = \{i\}$, $J = \{\neg i\}$ and $K = \emptyset$, then $t_0 = \frac{a_i}{c_i}$ and, as $u \to \infty$,

$$s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; zu^{-2}) \sim \begin{cases} \widehat{\mathcal{B}}_I(z) u^{-1} \varphi_T((a_i + c_i T)u), & \text{if } t_0 > T \\ \sqrt{\frac{\pi a_i}{2c_i^3}} \mathcal{B}_I(z) \varphi_{t_0}(2a_i u), & \text{if } t_0 = T \\ \sqrt{\frac{2\pi a_i}{c_i^3}} \mathcal{B}_I(z) \varphi_{t_0}(2a_i u), & \text{if } t_0 < T \end{cases}.$$

(ii) If $I = \{i\}$, $J = \emptyset$ and $K = \{\neg i\}$, then $t_0 = \frac{a_i}{c_i}$ and, as $u \to \infty$,

$$s_{\rho,T}(\boldsymbol{c}, \boldsymbol{a}, u; zu^{-2}) \sim \begin{cases} \frac{1}{2} \widehat{\mathcal{B}}_I(z) u^{-1} \varphi_T((a_i + c_i T)u), & \text{if } t_0 > T \\ \int_0^\infty \Psi\left(-\frac{c_{\neg i} - \rho c_i}{\sqrt{(1-\rho^2)t_0}} x\right) e^{-\frac{c_i^3}{2a_i} x^2} dx \mathcal{B}_I(z) \varphi_{t_0}(2a_i u), & \text{if } t_0 = T \\ \int_{\mathbb{R}} \Psi\left(-\frac{c_{\neg i} - \rho c_i}{\sqrt{(1-\rho^2)t_0}} x\right) e^{-\frac{c_i^3}{2a_i} x^2} dx \mathcal{B}_I(z) \varphi_{t_0}(2a_i u), & \text{if } t_0 < T \end{cases},$$

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where

$$\widehat{\mathcal{B}}_I(z)=\int_{\mathbb{R}}\mathbb{P}\Big\{\int_{[0,\infty)}\mathbb{I}\{W_i(t)-\frac{a_i}{T}t>x\}>z\Big\}e^{\frac{a_i+c_iT}{T}x}dx\in(0,\infty)$$

and

$$\mathcal{B}_I(z)=\lim_{\Delta\to\infty}\frac{1}{\Delta}\int_{\mathbb{R}}\mathbb{P}\Big\{\int_{[0,\Delta]}\mathbb{I}\{W_i(t)-c_it>x\}>z\Big\}e^{2c_ix}dx\in(0,\infty).$$

Theorem 2.2.3 (Full-dimensional case)

If $I=\{1,2\}$, $J=\emptyset$ and $K=\emptyset$, then $t_0=\sqrt{\frac{1-2\rho a_2+a_2^2}{c_1^2-2\rho c_1c_2+c_2^2}}$ and, as $u\to\infty$,

$$s_{\rho,T}(\boldsymbol{c},\boldsymbol{a},u;zu^{-2})\sim\begin{cases}\widehat{\mathcal{B}}_I(z)u^{-2}\varphi_T((1+c_1T)u,(a_2+c_2T)u), & \text{if } t_0>T\\ \frac{1}{2}\sqrt{\frac{2\pi(t_0)^3(1-\rho^2)}{1-2\rho a_2+a_2^2}}\mathcal{B}_I(z)u^{-1}\varphi_{t_0}((1+c_1t_0)u,(a_2+c_2t_0)u), & \text{if } t_0=T\\ \sqrt{\frac{2\pi(t_0)^3(1-\rho^2)}{1-2\rho a_2+a_2^2}}\mathcal{B}_I(z)u^{-1}\varphi_{t_0}((1+c_1t_0)u,(a_2+c_2t_0)u), & \text{if } t_0<T\end{cases},$$

where

$$\widehat{\mathcal{B}}_I(z)=\int_{\mathbb{R}^2}\mathbb{P}\left\{\int_{[0,\infty)}\mathbb{I}\left\{\begin{array}{c}W_1(t)-\frac{t}{T}>x\\ W_2(t)-\frac{a_2t}{T}>y\end{array}\right\}>z\right\}e^{\lambda_1x+\lambda_2y}dxdy\in(0,\infty)$$

and

$$\mathcal{B}_I(z)=\lim_{\Delta\to\infty}\frac{1}{\Delta}\int_{\mathbb{R}^2}\mathbb{P}\left\{\int_{[0,\Delta]}\mathbb{I}\left\{\begin{array}{c}W_1(t)-c_1t>x\\ W_2(t)-c_2t>y\end{array}\right\}>z\right\}e^{\lambda_1x+\lambda_2y}dxdy\in(0,\infty),$$

with $(\lambda_1,\lambda_2)=(\frac{1-a_2\rho+(c_1-\rho c_2)\min(t_0,T)}{(1-\rho^2)\min(t_0,T)},\frac{a_2-\rho+(c_2-\rho c_1)\min(t_0,T)}{(1-\rho^2)\min(t_0,T)})$.

2.3 Proofs

2.3.1 Proofs of Theorems 2.2.2 and 2.2.3

We begin the proof by analyzing properties of the function $q(t)$, defined in (2.2), in the neighborhood of its minimizer.

Lemma 2.3.1 *Let $t_0^*:=\min(t_0,T)$ with $t_0:=\sqrt{\frac{\boldsymbol{a}_I\Sigma_{II}^{-1}\boldsymbol{a}_I^T}{\boldsymbol{c}_I\Sigma_{II}^{-1}\boldsymbol{c}_I^T}}>0$. Then t_0^* is the unique point minimizing the function $q(t)$ in the interval $[0,T]$. Furthermore,*

(i) If $t_0>T$, then

$$q(T-t)=q(T)+q'(T)t(1+o(1)),\ \ \text{as } t\to0+, \tag{2.6}$$

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with $q'(T) = \frac{-\boldsymbol{a}_I \Sigma_{II}^{-1} \boldsymbol{a}_I + \boldsymbol{c}_I \Sigma_{II}^{-1} \boldsymbol{c}_I T^2}{T^2}$.

(ii) If $t_0 \le T$, *then*

$$q(t_0 \pm t) = q(t_0) + \frac{q''(t_0\pm)}{2} t^2(1+o(1)), \ \ \textit{as } t \to 0, \tag{2.7}$$

with $q''(t_0\pm) = 2\frac{\boldsymbol{a}_{I\pm}\Sigma_{I\pm I\pm}^{-1}\boldsymbol{a}_{I\pm}}{(t_0)^3}$.

Proof of Lemma 2.3.1 The proof follows straightforwardly from Taylor expansion and basic analitical properties of the function $q(t)$; see also [33] [Lemma 2.2]. □

Lemma 2.3.2 *Let* t_0 *be the unique point that minimizes the value of* $q(t)$ *on interval* $[0,\infty)$.

(i) If $t_0 > T$, *then*

$$\mathbb{P}\left\{\exists_{t\in[0,T]\setminus[T-\frac{\log(u)}{u},T]} \boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\right\} \le Ce^{-\frac{\tau}{2}\log(u)u} e^{-\frac{q(T)}{2}u^2},$$

for some $\tau > 0$, $C > 0$.

(ii) If $t_0 = T$, *then*

$$\mathbb{P}\left\{\exists_{t\in[0,T]\setminus[T-\frac{\log(u)}{u},T]} \boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\right\} \le Ce^{-\frac{\tau}{2}\log^2(u^2)} e^{-\frac{q(t_0)}{2}u^2},$$

for some $\tau > 0$, $C > 0$.

(iii) If $t_0 < T$, *then*

$$\mathbb{P}\left\{\exists_{t\in[0,\infty)\setminus[t_0-\frac{\log(u)}{u},t_0+\frac{\log(u)}{u}]} \boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\right\} \le Ce^{-\frac{\tau}{2}\log^2(u^2)} e^{-\frac{q(t_0)}{2}u^2},$$

for some $\tau > 0$, $C > 0$.

Proof of Lemma 2.3.2 We present detailed proof only for case (i). The other cases follow in a similar way.

For $\boldsymbol{b}(t) = (\boldsymbol{a}+\boldsymbol{c}t)_{I(t)}(\Sigma_t)^{-1}_{I(t)I(t)}$ with $Z_{I(t)}(t) := \frac{\boldsymbol{b}(t)(W_1(t),W_2(t))^\top_{I(t)}}{\boldsymbol{b}(t)(\boldsymbol{a}+\boldsymbol{c}t)^\top_{I(t)}}$, we obtain that

$$\mathbb{P}\left\{\exists_{t\in[0,T-\frac{\log(u)}{u}]} \boldsymbol{W}(t) - \boldsymbol{c}u^\alpha t > \boldsymbol{a}u^\alpha\right\} \le \mathbb{P}\left\{\exists_{t\in[0,T-\frac{\log(u)}{u}]} Z_{I(t)}(t) > u^\alpha\right\},$$

where we use the fact that $\boldsymbol{b}(t) > \boldsymbol{0}_{I(t)}$. Straightforward calculations give

$$Var(Z_{I(t)}(t)) = \frac{1}{q(t)}.$$

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Further, since the process $\{Z_{I(t)}(t) : t \geq 0\}$ has bounded sample paths a.s., by the Borell-TIS inequality (see, e.g., [1]) it holds that for sufficiently large u

$$\mathbb{P}\Big\{\exists_{t\in[0,T-\frac{\log(u)}{u}]} Z_{I(t)}(t) > u\Big\} \;\leq\; e^{-\inf\limits_{t\in[0,T-\frac{\log(u)}{u}]} q(t)\frac{(u-\mu)^2}{2}},$$

where

$$\begin{aligned}\mu \;&:=\; \mathbb{E}\{\sup_{t\in F_u} Z_{I(t)}(t)\} \leq \mathbb{E}\{\sup_{t\in F_u} \max\{Z_{\{1\}}(t); Z_{\{2\}}(t); Z_{\{1,2\}}(t)\}\} \\ &\leq\; \mathbb{E}\{\sup_{t\in F_u} Z_{\{1\}}(t)\} + \mathbb{E}\{\sup_{t\in F_u} Z_{\{2\}}(t)\} + \mathbb{E}\{\sup_{t\in F_u} Z_{\{1,2\}}(t)\} < \infty,\end{aligned}$$

with $F_u := [0, T - \frac{\log(u)}{u}]$. Using Lemma 2.3.1 we obtain that, for some $\tau > 0$,

$$\inf_{t\in[0,T-\frac{\log(u)}{u}]} q(t) = q(T) + \tau\frac{\log(u)}{u}.$$

Thus, as $u \to \infty$,

$$\mathbb{P}\Big\{\exists_{t\in[0,T-\frac{\log(u)}{u}]} Z_{I(t)}(t) > u\Big\} \;\leq\; e^{-\frac{q(T)}{2}u^2-\frac{\tau}{2}\log(u)u+O(u)} \leq Ce^{-\frac{\tau}{2}\log(u)u}e^{-\frac{q(T)}{2}u^2}.$$

This completes the proof. $\square$

For $\Delta > 0$, let $k_u := t_0^* - \frac{(k-1)\Delta}{u^2}$ and $E_{u,k} := [(k+1)_u, k_u]$. Furthermore, let

$$\boldsymbol{M}_{\{1,2\}} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \boldsymbol{M}_{\{1\}} := \begin{pmatrix} 1 & 0 \\ \rho & 0 \end{pmatrix}, \; \boldsymbol{M}_{\{2\}} := \begin{pmatrix} 0 & \rho \\ 0 & 1 \end{pmatrix}.$$

In Lemmas 2.3.3 and 2.3.4, we analyze the asymptotic behavior of

$$\mathcal{P}_k(u) := \mathbb{P}\Big\{\int_{E_{u,k}} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\} > zu^{-2}\Big\},$$

as $u \to \infty$.

Lemma 2.3.3 *Let $t_0 \leq T$.*

(i) If $I = \{i\}$, $J = \{\neg i\}$ and $K = \emptyset$, then, as $u \to \infty$,

$$\mathcal{P}_k(u) \sim \mathcal{B}_I(z;\Delta)u^{-1}\varphi_{t_0}(2a_iu)e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2},$$

where

$$\mathcal{B}_I(z;\Delta) = \int_{\mathbb{R}} \mathbb{P}\Big\{\int_{[0,\Delta]} \mathbb{I}\{W_i(t) - c_it > x\} > z\Big\}e^{2c_ix}dx \; and \; q_I''(t_0) = 2\frac{c_i^3}{a_i}.$$

(ii) If $I=\{i\}$, $J=\emptyset$ and $K=\{\neg i\}$, then, as $u\to\infty$,

$$\mathcal{P}_k(u)\sim\mathcal{B}_I(z;\Delta)u^{-1}\varphi_{t_0}(2a_iu)\Psi(-\frac{c_{\neg i}-\rho c_i}{\sqrt{(1-\rho^2)t_0}}\frac{k\Delta}{u})e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2},$$

where

$$\mathcal{B}_I(z;\Delta)=\int_{\mathbb{R}}\mathbb{P}\left\{\int_{[0,\Delta]}\mathbb{I}\{W_i(t)-c_it>x\}>z\right\}e^{2c_ix}dx \text{ and } q_I''(t_0)=2\frac{c_i^3}{a_i}.$$

(iii) If $I=\{1,2\}$, $J=\emptyset$ and $K=\emptyset$, then, as $u\to\infty$,

$$\mathcal{P}_k(u)\sim\mathcal{B}_I(z;\Delta)u^{-2}\varphi_{t_0}((a_1+c_1t_0)u,(a_2+c_2t_0)u)e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2},$$

where

$$\mathcal{B}_I(z;\Delta)=\int_{\mathbb{R}^2}\mathbb{P}\left\{\int_{[0,\Delta]}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}t>\boldsymbol{x}\}>z\right\}e^{(\boldsymbol{a}+\boldsymbol{c}T)\Sigma_T^{-1}\boldsymbol{x}^T}d\boldsymbol{x} \text{ and } q_I''(t_0)=2\frac{a_1^2-2\rho a_1a_2+a_2^2}{(1-\rho^2)(t_0)^3}.$$

Proof of Lemma 2.3.3 Let $l_u=(k+1)_u$, $n=|I|$ and $A_{I,u}=\{\boldsymbol{W}_I(l_u)=((\boldsymbol{a}+\boldsymbol{c}l_u)u-\boldsymbol{x}u^{-1})_I\}$, $E=[0,\Delta]$. Then, by the total probability formula, we have

$$\begin{aligned}
\mathcal{P}_k(u) &= \int_{\mathbb{R}^n}\mathbb{P}\left\{\int_{E_{u,k}}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}ut>\boldsymbol{a}u\}>zu^{-2}\middle|A_{I,u}\right\}u^{-n}\varphi_{l_u}((\boldsymbol{a}+\boldsymbol{c}l_u)_Iu-\frac{\boldsymbol{x}_I}{u})d\boldsymbol{x}_I\\
&= \int_{\mathbb{R}^n}\mathbb{P}\left\{\int_{E}\mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+l_u)-\boldsymbol{c}u(\frac{t}{u^2}+l_u)>\boldsymbol{a}u\}>zu^{-2}\middle|A_{I,u}\right\}u^{-n}\varphi_{l_u}((\boldsymbol{a}+\boldsymbol{c}l_u)_Iu-\frac{\boldsymbol{x}_I}{u})d\boldsymbol{x}_I\\
&= \int_{\mathbb{R}^n}\mathbb{P}\left\{\int_{E}\mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+l_u)-\boldsymbol{M}_I\boldsymbol{W}(l_u)-\frac{\boldsymbol{c}t}{u}>(\boldsymbol{Id}-\boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}l_u)u+\frac{\boldsymbol{M}_I\boldsymbol{x}}{u}\}>z\middle|A_{I,u}\right\}\\
&\quad\times u^{-n}\varphi_{l_u}((\boldsymbol{a}+\boldsymbol{c}l_u)_Iu-\frac{\boldsymbol{x}_I}{u})d\boldsymbol{x}_I\\
&= \int_{\mathbb{R}^n}\mathbb{P}\left\{\int_{E}\mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+l_u)-\boldsymbol{M}_I\boldsymbol{W}(l_u)-\frac{\boldsymbol{c}t}{u}>(\boldsymbol{Id}-\boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}l_u)u+\frac{\boldsymbol{M}_I\boldsymbol{x}}{u}\}>z\right\}\\
&\quad\times u^{-n}\varphi_{l_u}((\boldsymbol{a}+\boldsymbol{c}l_u)_Iu-\frac{\boldsymbol{x}_I}{u})d\boldsymbol{x}_I.
\end{aligned}$$

Further, it holds that, as $u\to\infty$,

$$\varphi_{l_u}((\boldsymbol{a}+\boldsymbol{c}l_u)_Iu-\frac{\boldsymbol{x}_I}{u})=\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_Iu)e^{\boldsymbol{\lambda}_I\boldsymbol{x}_I^T}e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2}e^{O(\boldsymbol{x}_I\boldsymbol{x}_I^Tu^{-2})},$$

with $\boldsymbol{\lambda}_I=(\boldsymbol{a}+\boldsymbol{c}t_0)_I(\Sigma_{t_0}^{-1})_{II}$. In order to prove the thesis, it remains to show the finiteness of

$$\begin{aligned}
&\widetilde{\mathcal{B}}_{I,u}(z;\Delta)\\
&:=\int_{\mathbb{R}^n}\mathbb{P}\left\{\int_{E}\mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+l_u)-\boldsymbol{M}_I\boldsymbol{W}(l_u)-\frac{\boldsymbol{c}t}{u}>(\boldsymbol{Id}-\boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}l_u)u+\frac{\boldsymbol{M}_I\boldsymbol{x}}{u}\}>z\right\}
\end{aligned}$$

$$\times e^{\boldsymbol{\lambda}_I \boldsymbol{x}_I^T} d\boldsymbol{x}_I,$$

where, for sufficiently large u,

$$((\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a} + \boldsymbol{c} l_u))_j = \begin{cases} 0, & \text{if } j \in I \\ (a_j + c_j t_0) - \rho(a_{\neg j} + c_{\neg j} t_0) - (c_j - \rho c_{\neg j})\frac{k\Delta}{u^2}, & \text{if } j \in K \\ ((a_j + c_j t_0) - \rho(a_{\neg j} + c_{\neg j} t_0)) - (c_j - \rho c_{\neg j})\frac{k\Delta}{u^2}, & \text{if } j \in J \end{cases}, \quad (2.8)$$

with

$$((a_j + c_j t_0) - \rho(a_{\neg j} + c_{\neg j} t_0)) = 0 \text{ if } j \in K \text{ and } ((a_j + c_j t_0) - \rho(a_{\neg j} + c_{\neg j} t_0)) < 0 \text{ if } j \in J.$$

We define $\boldsymbol{\chi}_u(t) = (\chi_{u;1}(t), \chi_{u;2}(t)) := \boldsymbol{W}(\frac{t}{u^2} + l_u) - \boldsymbol{M}_I \boldsymbol{W}(l_u) - \frac{\boldsymbol{c}t}{u}$, and denote $\bar{\boldsymbol{\mu}}_u := \mathbb{E}\{\boldsymbol{\chi}_u(t)\}$, $\bar{\boldsymbol{\Sigma}}_u := \mathbb{V}ar\{\boldsymbol{\chi}_u(t)\}$.

In the following, we divide the analysis of $\widetilde{\mathcal{B}}_{I,u}(z;\Delta)$ into two cases: (a) $I = \{i\}$ and (b) $I = \{1,2\}$ which require slightly different approaches.

(a) Suppose that $I = \{i\}$. Then

$$\bar{\boldsymbol{\mu}}_{u;i} = -\frac{\boldsymbol{c}t}{u} \text{ and } \bar{\boldsymbol{\Sigma}}_{u;i,i} = \frac{t}{u^2},\ \bar{\boldsymbol{\Sigma}}_{u;\neg i,\neg i} = \frac{t}{u^2} + (1-\rho^2) l_u,\ \bar{\boldsymbol{\Sigma}}_{u;\neg i,i} = \rho \frac{t}{u^2}.$$

Hence

$$\boldsymbol{\chi}_u(t) \stackrel{d}{=} \frac{1}{u}(\boldsymbol{W}(t) - \boldsymbol{c}t) + \sqrt{(1-\rho^2) l_u}\boldsymbol{Z},$$

where $\boldsymbol{Z} = (Z_1, Z_2)$, with $Z_i = 0, Z_{\neg i} = Z$, $Z \sim \mathcal{N}(0,1)$ is a random variable independent of $\{\boldsymbol{W}(t) : t \geq 0\}$.

Let $\boldsymbol{\chi}_u^*(t) = (\chi_{u;1}^*(t), \chi_{u;2}^*(t)) := \boldsymbol{v}_{I,u} \boldsymbol{\chi}_u(t)$ where $\boldsymbol{v}_{I,u} := (v_{I,u;1}, v_{I,u;2})^T$ with $v_{I,u;i} = \begin{cases} u, & \text{if } i \in I \\ 1, & \text{if } i \notin I \end{cases}$.

Then, as $u \to \infty$, $\{\boldsymbol{\chi}_u^*(t) : t \geq 0\} \xrightarrow{d} \{\boldsymbol{\xi}_I(t) \equiv (\xi_1(t), \xi_2(t)) : t \geq 0\}$, where $\xi_i(t) = W_i(t) - c_i t$ and $\xi_{\neg i}(t) = \sqrt{(1-\rho^2) t_0} Z$ with $\mathbb{C}ov(W_i(t), Z) = 0$.

Observe that, for sufficiently large u, we have that $\boldsymbol{\lambda}_I = \lambda_i = 2c_i$ and

$$\begin{aligned} \widetilde{\mathcal{B}}_{I,u}(z;\Delta) &\leq \widetilde{\mathcal{B}}_{I,u}(0;\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t \in E} : \boldsymbol{\chi}_u^*(t) > \boldsymbol{v}_{I,u}((\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a} + \boldsymbol{c} l_u)u + \frac{\boldsymbol{M}_I \boldsymbol{x}}{u})\right\} e^{2c_i x} dx \\ &\leq \int_{\mathbb{R}_-} e^{\lambda_i x_i} dx_i + \int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t \in [0,\Delta]} : W_i(t) - c_i t > x\} e^{2c_i x} dx \end{aligned}$$

$$\leq \frac{1}{2c_i} + \int_{\mathbb{R}} \mathbb{P}\left\{ \sup_{t\in[0,\Delta]} (W_i(t) - c_i t) > x_i \right\} e^{2c_i x} dx < \infty.$$

By the dominated convergence theorem we have that, as $u \to \infty$,

$$\begin{aligned}
\widetilde{\mathcal{B}}_{I,u}(z;\Delta) &\sim \int_{\mathbb{R}} \mathbb{P}\left\{ \int_E \mathbb{I}\left\{ \begin{array}{c} W_i(t) - c_i t > x \\ \sqrt{(1-\rho^2)t_0} Z > ((a_{\neg i} + c_{\neg i} l_u) - \rho(a_i + c_i l_u))u + \frac{\rho x}{u} \end{array} \right\} dt > z \right\} e^{2c_i x} dx \\
&= \int_{\mathbb{R}} \mathbb{P}\left\{ \int_E \mathbb{I}\left\{ W_i(t) - c_i t > x \right\} dt > z \right\} \\
&\quad \times \mathbb{P}\left\{ \sqrt{(1-\rho^2)t_0} Z > ((a_{\neg i} + c_{\neg i} t_0) - \rho(a_i + c_i t_0))u - (c_{\neg i} - \rho c_i)\frac{k\Delta}{u} + \frac{\rho x}{u} \right\} e^{2c_i x} dx \\
&\sim \int_{\mathbb{R}} \mathbb{P}\left\{ \int_E \mathbb{I}\left\{ W_i(t) - c_i t > x \right\} dt > z \right\} e^{2c_i x} dx \\
&\quad \times \mathbb{P}\left\{ \sqrt{(1-\rho^2)t_0} Z > ((a_{\neg i} + c_{\neg i} t_0) - \rho(a_i + c_i t_0))u - (c_{\neg i} - \rho c_i)\frac{k\Delta}{u} \right\} \\
&\sim \mathcal{B}_I(z;\Delta) C_{J,K} =: \widetilde{\mathcal{B}}_I(z;\Delta),
\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$ is independent of $\{W_i(t) : t \geq 0\}$ and

$$\begin{aligned}
\mathcal{B}_I(z;\Delta) &= \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{[0,\Delta]} \mathbb{I}\left\{ W_i(t) - c_i t > x \right\} dt > z \right\} e^{2c_i x} dx, \\
C_{J,K} &= \begin{cases} \Psi\left(-\frac{c_{\neg i} - \rho c_i}{\sqrt{(1-\rho^2)t_0}} \frac{k\Delta}{u}\right), & \text{if } \neg i \in K \\ 1, & \text{if } \neg i \in J \end{cases}.
\end{aligned}$$

Note that the above asymptotic equivalence holds at all continuity points of $\widetilde{\mathcal{B}}_I(z;\Delta)$ in the interval $[0,\Delta)$.

(b) Suppose that $I = \{1,2\}$. Then $\bar{\boldsymbol{\mu}}_u = -\frac{\boldsymbol{c}t}{u}$ and $\bar{\boldsymbol{\Sigma}}_u = \frac{t}{u^2}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Let $\boldsymbol{\chi}_u^*(t) = u\chi_u(t)$. Hence

$$\{\boldsymbol{\chi}_u^*(t) : t \geq 0\} \stackrel{d}{=} \{\boldsymbol{\xi}_I(t) \equiv \boldsymbol{W}(t) - \boldsymbol{c}t : t \geq 0\}$$

and

$$\widetilde{\mathcal{B}}_{I,u}(z;\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}t > \boldsymbol{x}\} dt > z \right\} e^{\boldsymbol{\lambda}\boldsymbol{x}^T} d\boldsymbol{x} =: \mathcal{B}_I(z;\Delta).$$

Continuity of $\mathcal{B}_I(z;\Delta)$. The analysis of the continuity of $\mathcal{B}_I(z;\Delta)$ proceeds in a manner analogous to the proof of Lemma 4.1 stated in [47], we refer also to the proof of Lemma 4.1 in [39]. A comprehensive proof is provided for the sake of completeness.

We show that $\mathcal{B}_I(z;\Delta)$ is continuous at any $z \in [0,\Delta)$. Note that $\mathcal{B}_I(z;\Delta)$ is right-continuous at 0. Next we show the continuity at $z \in (0,\Delta)$. The claimed continuity at z follows if we show

$$\int_{\mathbb{R}^{|I|}} \mathbb{P}\{A_{\boldsymbol{y}_I}\} e^{\boldsymbol{\lambda}_I \boldsymbol{y}_I^T} d\boldsymbol{y}_I = 0 \text{ with } A_{\boldsymbol{y}_I} = \left\{ \int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{\xi}_I(t) > -\boldsymbol{y}_I\} dt = z \right\},\ \boldsymbol{y}_I \in \mathbb{R}^{|I|}.$$

Since $\boldsymbol{\xi}_I(t)$ is continuous over $[0,\Delta]$, then for any $\boldsymbol{y}_I' > \boldsymbol{y}_I$, we have

$$\int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{\xi}_I(t) > -\boldsymbol{y}_I'\} dt > \int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{\xi}_I(t) > -\boldsymbol{y}_I\} dt.$$

Hence

$$A_{\boldsymbol{y}_I} \cap A_{\boldsymbol{y}_I'} = \emptyset, \boldsymbol{y}_I \neq \boldsymbol{y}_I',\ \boldsymbol{y}_I, \boldsymbol{y}_I' \in \mathbb{R}^{|I|}.$$

Note that the continuity of $\{\boldsymbol{\xi}_I(t) : t \geq 0\}$ guarantees the measurability of $A_{\boldsymbol{y}_I}$, and consequently, $\sum_{\boldsymbol{y}_I \in \mathbb{R}^{|I|}} \mathbb{P}\{A_{\boldsymbol{y}_I}\} \leq 1$. Hence $\{\boldsymbol{y}_I : \boldsymbol{y}_I \in \mathbb{R}^{|I|}$ such that $\mathbb{P}\{A_{\boldsymbol{y}_I}\} > 0\}$ is a countable set. Thus, we obtain $\int_{\mathbb{R}^{|I|}} \mathbb{P}\{A_{\boldsymbol{y}_I}\} e^{\boldsymbol{\lambda}_I \boldsymbol{y}_I^T} d\boldsymbol{y}_I = 0$, and therefore $\mathcal{B}_I(z;\Delta)$ is continuous at $z \in (0,\Delta)$.

This completes the proof. □

Lemma 2.3.4 *Let $t_0 > T$.*

(i) If $I = \{i\}$, $J = \{\neg i\}$ and $K = \emptyset$, then, as $u \to \infty$,

$$\mathcal{P}_k(u) \sim \mathcal{B}_I(z;\Delta) u^{-1} \varphi_T((a_i + c_i T)u) e^{\frac{q_I'(T)}{2}(k-1)\Delta},$$

where

$$\mathcal{B}_I(z;\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{[0,\Delta]} \mathbb{I}\{W_i(t) - \frac{a_i}{T} t > x\} > z \right\} e^{\frac{a_i + c_i T}{T} x} dx \text{ and } q_I'(T) = \frac{c_i^2 T^2 - a_i^2}{T^2}.$$

(ii) If $I = \{i\}$, $J = \emptyset$ and $K = \{\neg i\}$, then, as $u \to \infty$,

$$\mathcal{P}_k(u) \sim \mathcal{B}_I(z;\Delta) u^{-1} \varphi_T((a_i + c_i T)u) \Psi\left(-\frac{c_{\neg i} - \rho c_i}{\sqrt{(1-\rho^2)T}} \frac{(k-1)\Delta}{u}\right) e^{\frac{q_I'(T)}{2}(k-1)\Delta},$$

where

$$\mathcal{B}_I(z;\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{[0,\Delta]} \mathbb{I}\{W_i(t) - \frac{a_i}{T} t > x\} > z \right\} e^{\frac{a_i + c_i T}{T} x} dx \text{ and } q_I'(T) = \frac{c_i^2 T^2 - a_i^2}{T^2}.$$

(iii) If $I = \{1,2\}$, $J = \emptyset$ and $K = \emptyset$, then, as $u \to \infty$,

$$\mathcal{P}_k(u) \sim \mathcal{B}_I(z;\Delta) u^{-2} \varphi_T((a_1 + c_1 T)u, (a_2 + c_2 T)u) e^{\frac{q_I'(T)}{2}(k-1)\Delta},$$

where

$$\mathcal{B}_I(z;\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\Big\{ \int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\} > z \Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x}$$

and

$$q_I'(T) = \frac{(c_1^2 - 2\rho c_1 c_2 + c_2^2)T^2 - (a_1^2 - 2\rho a_1 a_2 + a_2^2)}{(1-\rho^2)T^2}.$$

PROOF OF LEMMA 2.3.4 Let $n = |I|$ and $A_{I,u} = \{\boldsymbol{W}_I(k_u) = ((\boldsymbol{a}+\boldsymbol{c}k_u)u - \boldsymbol{x}u^{-1})_I\}$, $E = [-\Delta, 0]$. Then, by the total probability formula, we have

$$\begin{aligned}
\mathcal{P}_k(u) &= \int_{\mathbb{R}^n} \mathbb{P}\Big\{ \int_{E_{u,k}} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\} > zu^{-2} \Big| A_{I,u} \Big\} u^{-n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_I u - \frac{\boldsymbol{x}_I}{u}) d\boldsymbol{x}_I \\
&= \int_{\mathbb{R}^n} \mathbb{P}\Big\{ \int_{E} \mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+k_u) - \boldsymbol{c}u(\frac{t}{u^2}+l_u) > \boldsymbol{a}u\} > zu^{-2} \Big| A_{I,u} \Big\} u^{-n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_I u - \frac{\boldsymbol{x}_I}{u}) d\boldsymbol{x}_I \\
&= \int_{\mathbb{R}^n} \mathbb{P}\Big\{ \int_{E} \mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+k_u) - \boldsymbol{M}_I \boldsymbol{W}(k_u) - \frac{\boldsymbol{c}t}{u} > (\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}k_u)u + \frac{\boldsymbol{M}_I \boldsymbol{x}}{u}\} > z \Big| A_{I,u} \Big\} \\
&\quad \times u^{-n} \varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_I u - \frac{\boldsymbol{x}_I}{u}) d\boldsymbol{x}_I.
\end{aligned}$$

Furthermore, as $u \to \infty$,

$$\varphi_{k_u}((\boldsymbol{a}+\boldsymbol{c}k_u)_I u - \frac{\boldsymbol{x}_I}{u}) = \varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_I u) e^{\boldsymbol{\lambda}_I \boldsymbol{x}_I^T} e^{\frac{q_I'(T)}{2}(k-1)\Delta} e^{O(\boldsymbol{x}_I \boldsymbol{x}_I^T u^{-2})},$$

with $\boldsymbol{\lambda}_I = (\boldsymbol{a}+\boldsymbol{c}T)_I (\Sigma_T^{-1})_{II}$. In order to prove the thesis, it remains to show the finiteness of

$$\begin{aligned}
&\widetilde{\mathcal{B}}_{I,u}(z;\Delta) \\
&:= \int_{\mathbb{R}^n} \mathbb{P}\Big\{ \int_{E} \mathbb{I}\{\boldsymbol{W}(\frac{t}{u^2}+k_u) - \boldsymbol{M}_I \boldsymbol{W}(k_u) - \frac{\boldsymbol{c}t}{u} > (\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}k_u)u + \frac{\boldsymbol{M}_I \boldsymbol{x}}{u}\} > z \Big| A_{I,u} \Big\} \\
&\quad \times e^{\boldsymbol{\lambda}_I \boldsymbol{x}_I^T} d\boldsymbol{x}_I,
\end{aligned}$$

where

$$((\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a}+\boldsymbol{c}k_u))_j = \begin{cases} 0, & \text{if } j \in I \\ (a_j + c_jT) - \rho(a_{\neg j} + c_{\neg j}T) - (c_j - \rho c_{\neg j})\frac{(k-1)\Delta}{u^2}, & \text{if } j \in K \\ ((a_j + c_jT) - \rho(a_{\neg j} + c_{\neg j}T)) - (c_j - \rho c_{\neg j})\frac{(k-1)\Delta}{u^2}, & \text{if } j \in J \end{cases}, \quad (2.9)$$

with

$$((a_j + c_jT) - \rho(a_{\neg j} + c_{\neg j}T)) = 0 \text{ if } j \in K \text{ and } ((a_j + c_jT) - \rho(a_{\neg j} + c_{\neg j}T)) < 0 \text{ if } j \in J.$$

We define $\boldsymbol{\chi}_u(t) = (\chi_{u;1}(t), \chi_{u;2}(t)) := \boldsymbol{W}(\frac{t}{u^2}+k_u) - \boldsymbol{M}_I\boldsymbol{W}(k_u) - \frac{\boldsymbol{c}t}{u}$ and denote $\bar{\boldsymbol{\mu}}_u := \mathbb{E}\{\boldsymbol{\chi}_u(t)|A_{I,u}\}$, $\bar{\boldsymbol{\Sigma}}_u := \mathbb{V}ar\{\boldsymbol{\chi}_u(t)|A_{I,u}\}$. In the following, we divide the analysis of $\widetilde{\mathcal{B}}_{I,u}(z;\Delta)$ into two cases: (a) $I = \{i\}$ and (b) $I = \{1,2\}$ which require slightly different approaches.

(a) Suppose that $I = \{i\}$. Then, as $u \to \infty$,

$$\begin{aligned}
\bar{\boldsymbol{\mu}}_{u;i} &= \frac{1}{u}\frac{a_i}{k_u}t - \frac{1}{u^3}\frac{x_i}{k_u}t = \frac{1}{u}\frac{a_i}{T}t + x_i O\left(u^{-3}\right), \\
\bar{\boldsymbol{\mu}}_{u;\neg i} &= \frac{1}{u}\frac{-c_{\neg i}k_u + \rho(a_i + c_i k_u)}{k_u}t - \frac{1}{u^3}\frac{\rho x_i}{k_u}t = \frac{1}{u}\frac{-c_{\neg i}T + \rho(a_i + c_i T)}{T}t + \rho x_i O\left(u^{-3}\right),
\end{aligned}$$

$$\bar{\boldsymbol{\Sigma}}_{u;i,i} = -\frac{1}{u^2}t + O\left(u^{-4}\right),\ \bar{\boldsymbol{\Sigma}}_{u;\neg i,\neg i} = (1-\rho^2)T - \frac{1}{u^2}(1-2\rho^2)t + O\left(u^{-4}\right),\ \bar{\boldsymbol{\Sigma}}_{u;\neg i,i} = -\frac{1}{u^2}\rho t + O\left(u^{-4}\right).$$

Let $\boldsymbol{\chi}^*_u(t) = (\chi^*_{u;1}(t), \chi^*_{u;2}(t)) := \boldsymbol{v}_{I,u}\boldsymbol{\chi}_u(t)$ where $\boldsymbol{v}_{I,u} := (v_{I,u;1}, v_{I,u;2})^T$ with $v_{I,u;i} = \begin{cases} u, & \text{if } i \in I \\ 1, & \text{if } i \notin I \end{cases}$.

Then, as $u \to \infty$, $\{\boldsymbol{\chi}^*_u(t) : t \le 0\} \xrightarrow{d} \{\boldsymbol{\xi}_I(t) \equiv (\xi_1(t), \xi_2(t)) : t \le 0\}$, where $\xi_i(t) = W_i(-t) + \frac{a_i}{T}t$ and $\xi_{\neg i}(t) = \sqrt{(1-\rho^2)T}Z$ with $Z \sim \mathcal{N}(0,1)$ and $\mathbb{C}ov(W_i(t), Z) = 0$.

Note that $\boldsymbol{\lambda}_I = \lambda_i = \frac{a_i + c_i T}{T}$ and observe that, for sufficiently large u, we obtain

$$\begin{aligned}
\widetilde{\mathcal{B}}_{I,u}(z;\Delta) &\le \widetilde{\mathcal{B}}_{I,u}(0;\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in E} : \boldsymbol{\chi}^*_u(t) > \boldsymbol{v}_{I,u}((\boldsymbol{Id} - \boldsymbol{M}_I)(\boldsymbol{a} + \boldsymbol{c}T)u + \frac{\boldsymbol{M}_I\boldsymbol{x}}{u})\middle| A_{I,u}\right\} e^{2c_i x}dx \\
&\le \int_{\mathbb{R}_-} e^{\lambda_i x}dx + \int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t\in E} : \chi^*_{u;i}(t) > x | A_{I,u}\} e^{\lambda_i x}dx < \infty, \qquad (2.10)
\end{aligned}$$

where (2.10) follows from (2.9) and [117][Thm 8.1] with some constants $C, \widetilde{C} > 0$. Combining the weak convergence of $\boldsymbol{\chi}^*_u(t)$ with the dominated convergence theorem we have that, as $u \to \infty$,

$$\begin{aligned}
\widetilde{\mathcal{B}}_{I,u}(z;\Delta) &\sim \int_{\mathbb{R}} \mathbb{P}\left\{\int_E \mathbb{I}\left\{\begin{array}{c} W_i(-t) + \frac{a_i}{T}t > x \\ \sqrt{(1-\rho^2)T}Z > Cu - (c_{\neg i} - \rho c_i)\frac{(k-1)\Delta}{u} + \frac{\rho x}{u} \end{array}\right\} dt > z\right\} e^{\frac{a_i + c_i T}{T}x}dx \\
&= \int_{\mathbb{R}} \mathbb{P}\left\{\int_E \mathbb{I}\left\{W_i(-t) + \frac{a_i}{T}t > x\right\} dt > z\right\} \\
&\quad \times \mathbb{P}\left\{\sqrt{(1-\rho^2)T}Z > ((a_{\neg i} + c_{\neg i}T) - \rho(a_i + c_i T))u - (c_{\neg i} - \rho c_i)\frac{(k-1)\Delta}{u} + \frac{\rho x}{u}\right\} e^{2c_i x}dx \\
&\sim \int_{\mathbb{R}} \mathbb{P}\left\{\int_E \mathbb{I}\left\{W_i(-t) + \frac{a_i}{T}t > x\right\} dt > z\right\} \\
&\quad \times \mathbb{P}\left\{\sqrt{(1-\rho^2)T}Z > ((a_{\neg i} + c_{\neg i}T) - \rho(a_i + c_i T))u - (c_{\neg i} - \rho c_i)\frac{(k-1)\Delta}{u}\right\} e^{2c_i x}dx \\
&\sim \mathcal{B}_I(z;\Delta) C_{J,K} =: \widetilde{\mathcal{B}}_I(z;\Delta),
\end{aligned}$$

where $C := ((a_{\neg i} + c_{\neg i}T) - \rho(a_i + c_iT))$, $Z \sim \mathcal{N}(0,1)$ is independent of $\{W_i(t) : t \geq 0\}$ and

$$\mathcal{B}_I(z;\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\int_{[0,\Delta]} \mathbb{I}\left\{W_i(t) - \frac{a_i}{T}t > x\right\} dt > z\right\} e^{2c_ix}dx,$$

$$C_{J,K} = \begin{cases} \Psi(-\frac{c_{\neg i}-\rho c_i}{\sqrt{(1-\rho^2)T}}\frac{(k-1)\Delta}{u}), & \text{if } \neg i \in K \\ 1, & \text{if } \neg i \in J \end{cases}.$$

Note that the above asymptotic equivalence holds at all continuity points of $\widetilde{\mathcal{B}}_I(z;\Delta)$ in the interval $[0,\Delta)$.

(b) Suppose that $I = \{1,2\}$. Then, as $u \to \infty$,

$$\bar{\boldsymbol{\mu}}_{u;1} = \frac{1}{u}\frac{a_1}{k_u}t - \frac{1}{u^3}\frac{x_1}{k_u}t = \frac{1}{u}\frac{a_1}{T}t + x_1 O\left(u^{-3}\right), \bar{\boldsymbol{\mu}}_{u;2} = \frac{1}{u}\frac{a_2}{k_u}t - \frac{1}{u^3}\frac{x_2}{k_u}t = \frac{1}{u}\frac{a_2}{T}t + x_2 O\left(u^{-3}\right),$$

$$\bar{\boldsymbol{\Sigma}}_{u;1,1} = -\frac{1}{u^2}t + O\left(u^{-4}\right), \bar{\boldsymbol{\Sigma}}_{u;2,2} = -\frac{1}{u^2}t + O\left(u^{-4}\right), \bar{\boldsymbol{\Sigma}}_{u;1,2} = -\frac{1}{u^2}\rho t + O\left(u^{-4}\right).$$

Let $\boldsymbol{\chi}_u^*(t) := u\boldsymbol{\chi}_u(t)$. Then, as $u \to \infty$, $\{\boldsymbol{\chi}_u^*(t) : t \leq 0\} \xrightarrow{d} \{\boldsymbol{\xi}_I(t) \equiv \boldsymbol{W}(-t) + \frac{\boldsymbol{a}}{T}t : t \leq 0\}$.

Observe that, for sufficiently large u, we obtain

$$\begin{aligned}
&\widetilde{\mathcal{B}}_{I,u}(z;\Delta) \leq \widetilde{\mathcal{B}}_{I,u}(0;\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in E} : \boldsymbol{\chi}_u^*(t) > \boldsymbol{x} \middle| A_{I,u}\right\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\leq \int_{\mathbb{R}_-}\int_{\mathbb{R}_-} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} + \int_{\mathbb{R}_-}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi_{u;1}^*(t) > x_1 | A_{I,u}\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi_{u;2}^*(t) > x_2 | A_{I,u}\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi_{u;1}^*(t) + \chi_{u;2}^*(t) > x_1 + x_2 | A_{I,u}\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\leq \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2}\int_{\mathbb{R}_+} Ce^{-\widetilde{C}x_1^2 + \lambda_1 x_1}dx_1 + \frac{1}{\lambda_1}\int_{\mathbb{R}_+} Ce^{-\widetilde{C}x_2^2+\lambda_2x_2}dx_2 + \int_{\mathbb{R}_+^2} Ce^{-\widetilde{C}(x_1+x_2)^2 + \boldsymbol{\lambda x}^T} d\boldsymbol{x} < \infty,
\end{aligned}$$

where the last inequality follows from (2.9) and [117][Thm 8.1] with some constants $C, \widetilde{C} > 0$. Hence, by the weak convergence of $\boldsymbol{\chi}_u^*(t)$ and the dominated convergence theorem, we have that, as $u \to \infty$,

$$\widetilde{\mathcal{B}}_{I,u}(z;\Delta) \sim \int_{\mathbb{R}^2} \mathbb{P}\left\{\int_{[0,\Delta]} \mathbb{I}\{\boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\}dt > z\right\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} =: \mathcal{B}_{I,u}(z;\Delta).$$

Continuity of $\mathcal{B}_I(z;\Delta)$. The proof is analogous to the proof of Lemma 2.3.3. □

Lemma 2.3.5 *Let $I = \{1,2\}$, $t_0 > T$. Then*

$$\lim_{\Delta\to\infty} \mathcal{B}_I(0;\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\right\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \in (0,\infty).$$

Proof of Lemma 2.3.5 Note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[k,(k+1)]} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&\leq \sum_{k=0}^{\infty} \int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[k,(k+1)]} W_1(t) - \frac{t}{T} > x_1, \exists_{t\in[k,(k+1)]} W_2(t) - \frac{a_2 t}{T} > x_2\Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \\
&= \frac{1}{\lambda_1\lambda_2} \sum_{k=0}^{\infty} \mathbb{E}\{e^{\lambda_1 M_k + \lambda_2 M_k^*}\},
\end{aligned}$$

where

$$(M_k, M_k^*) = \left(\sup_{t\in[k,(k+1)]} (W_1(t) - \frac{t}{T}), \sup_{t\in[k,(k+1)]} (W_2(t) - \frac{a_2 t}{T}) \right).$$

Due to the independence of increments of Brownian motion, we have that

$$\begin{aligned}
(M_k, M_k^*) &\overset{d}{=} \left(\sup_{t\in[0,1]} (W_1(t) - \frac{t}{T}), \sup_{t\in[0,1]} (W_2(t) - \frac{a_2 t}{T}) \right) + (V_1(k) - \frac{k}{T}, V_2(k) - \frac{a_2 k}{T}) \\
&=: (Q_1, Q_2) + (V_1(k) - \frac{k}{T}, V_2(k) - \frac{a_2 k}{T}),
\end{aligned}$$

where $\{(V_1(t), V_2(t)) : t \geq 0\}$ is an independent copy of $\{(W_1(t), W_2(t)) : t \geq 0\}$. Hence

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{E}\{e^{\lambda_1 M_k + \lambda_2 M_k^*}\} &= \sum_{k=0}^{\infty} \mathbb{E}\{e^{\lambda_1 Q_1 + \lambda_2 Q_2}\} \mathbb{E}\{e^{\lambda_1 (V_1(k) - \frac{k}{T}) + \lambda_2 (V_2(k) - \frac{a_2 k}{T})}\} \\
&= \mathbb{E}\{e^{\lambda_1 Q_1 + \lambda_2 Q_2}\} \sum_{k=0}^{\infty} e^{k(-\frac{\lambda_1 - a_2\lambda_2}{T} + \frac{(\lambda_1 + \rho\lambda_2)^2}{2} + (1-\rho^2)\frac{\lambda_2^2}{2})} \\
&= \mathbb{E}\{e^{\lambda_1 Q_1 + \lambda_2 Q_2}\} \sum_{k=0}^{\infty} e^{-\frac{k}{2}\kappa},
\end{aligned}$$

where $\kappa = \frac{2\lambda_1 + 2a_2\lambda_2}{T} - (\lambda_1 + \rho\lambda_2)^2 - (1-\rho^2)\lambda_2^2$. Straightforward calculations give $\lambda_1 + \rho\lambda_2 = \frac{1 + c_1 T}{T}$ and

$$\kappa = \frac{(1 - 2a_2\rho + a_2^2) - (c_1^2 - 2\rho c_1 c_2 + c_2^2)T^2}{T^2(1-\rho^2)} > 0 \text{ iff } t_0 > T.$$

Thus

$$\int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \leq \frac{1}{\lambda_1\lambda_2} \mathbb{E}\{e^{\lambda_1 Q_1 + \lambda_2 Q_2}\} \frac{e^{\frac{\kappa}{2}}}{e^{\frac{\kappa}{2}} - 1} < \infty.$$

Applying the Lebesgue's monotone convergence theorem, we obtain

$$\lim_{\Delta\to\infty} \mathcal{B}_I(0;\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\Big\{\exists_{t\in[0,\infty)} \boldsymbol{W}(t) - \frac{\boldsymbol{a}}{T}t > \boldsymbol{x}\Big\} e^{\boldsymbol{\lambda x}^T} d\boldsymbol{x} \in (0,\infty).$$

This completes the proof. $\square$

PROOFS OF THEOREMS 2.2.2 AND 2.2.3 We shall prove both theorems simultaneously, and divide the proof into three scenarios: $t_0 < T$, $t_0 = T$ and $t_0 > T$.

Recall that for $\Delta > 0$, we write $k_{u;\Delta} = t_0^* - \frac{(k-1)\Delta}{u^2}$, $E_{k,u;\Delta} = [(k+1)_{u;\Delta}, k_{u;\Delta}]$, and let $N_u := \lfloor \frac{u\log(u)}{\Delta} \rfloor$.

Case (i): $t_0 < T$. Note that $t_0^* = t_0$, and for any $u > 0, z \geq 0$ and $\widetilde{\Delta}, \widehat{\Delta} > 0$

$$\Sigma_1(u;z;\widetilde{\Delta}) - \Sigma_2(u;\widetilde{\Delta}) \leq s_{\rho,T}(\boldsymbol{c},\boldsymbol{a},u;z) \leq \Sigma_1(u;z;\widehat{\Delta}) + \Sigma_2(u;\widehat{\Delta}) + \Sigma_3(u),$$

where

$$\begin{aligned}
\Sigma_1(u;z;\Delta) &= \sum_{k=-N_u}^{N_u} \mathbb{P}\{\int_{E_{k,u;\Delta}} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\}dt > zu^{-2}\},\\
\Sigma_2(u;\Delta) &= 2\sum_{k=-N_u}^{N_u}\sum_{l=k+1}^{N_u} \mathbb{P}\{\sup_{t\in E_{k,u;\Delta}} (\boldsymbol{W}(t) - \boldsymbol{c}ut) > \boldsymbol{a}u, \sup_{s\in E_{l,u;\Delta}} (\boldsymbol{W}(s) - \boldsymbol{c}us) > \boldsymbol{a}u\},\\
\Sigma_3(u) &= \mathbb{P}\{\sup_{[0,T]\setminus F_u} (\boldsymbol{W}(t) - \boldsymbol{c}ut) > \boldsymbol{a}u\}.
\end{aligned}$$

with $F_u = [t_0 - \frac{\log(u)}{u}, \min(t_0 + \frac{\log(u)}{u}, T)]$.

In the following, we calculate the asymptotic behavior of $\Sigma_1(u;z;\Delta)$, $\Sigma_2(u;\Delta)$ and $\Sigma_3(u)$, as $u \to \infty$.

Asymptotics of $\Sigma_1(u;z;\Delta)$. Lemma 2.3.3 gives, as $u \to \infty$,

$$\begin{aligned}
\Sigma_1(u;z;\Delta) &= \sum_{k=-N_u}^{N_u} \mathbb{P}\{\int_{E_{k,u}} \mathbb{I}\{\boldsymbol{W}(t) - \boldsymbol{c}ut > \boldsymbol{a}u\}dt > zu^{-2}\}\\
&\sim \sum_{k=-N_u}^{N_u} \mathcal{B}_I(z;\Delta)\mathcal{C}(\frac{k\Delta}{u})u^{-|I|}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2}\\
&= \mathcal{B}_I(z;\Delta)u^{-|I|}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\sum_{k=-N_u}^{N_u} \mathcal{C}(\frac{k\Delta}{u})e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2}\\
&= \frac{1}{\Delta}\mathcal{B}_I(z;\Delta)u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\sum_{k=-N_u}^{N_u} \frac{\Delta}{u}\mathcal{C}(\frac{k\Delta}{u})e^{-\frac{q_I''(t_0)}{4}\left(\frac{k\Delta}{u}\right)^2}\\
&\sim \frac{1}{\Delta}\mathcal{B}_I(z;\Delta)u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\int_{-\log(u)}^{\log(u)} \mathcal{C}(x)e^{-\frac{q_I''(t_0)}{4}x^2}dx\\
&\sim \frac{1}{\Delta}\mathcal{B}_I(z;\Delta)u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\int_{\mathbb{R}} \mathcal{C}(x)e^{-\frac{q_I''(t_0)}{4}x^2}dx,
\end{aligned}$$

where

$$g_I''(t_0) = 2\frac{\boldsymbol{a}_I^T\Sigma_{II}^{-1}\boldsymbol{a}_I}{(t_0)^3}, \text{ with } \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \text{ and } \mathcal{C}(x) = \begin{cases} \Psi(-\frac{c_{\neg i}-\rho c_i}{\sqrt{(1-\rho^2)t_0}}x), & \text{if } K \neq \emptyset \\ 1, & \text{if } K = \emptyset \end{cases}.$$

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Thus,

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\left|\frac{\Sigma_1(u;z;\Delta)}{u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\int_{\mathbb{R}}\mathcal{C}(x)e^{-\frac{g_I''(t_0)}{4}x^2}dx}-\frac{1}{\Delta}\mathcal{B}_I(z;\Delta)\right|=0, \tag{2.11}$$

where $g_I''(t_0)=2\frac{\boldsymbol{a}_I^T\Sigma_{II}^{-1}\boldsymbol{a}_I}{(t_0)^3}$.

Asymptotics of $\Sigma_2(u;\Delta)$. The analysis of the sum $\Sigma_2(u;\Delta)$ follows in a similar way to the proof of Theorem 3.1 in [33] and leads to

$$\Sigma_2(u;\Delta)=o\left(u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\right),$$

as $u\to\infty$.

Asymptotics of $\Sigma_3(u)$. Lemma 2.3.2 (iii) gives, as $u\to\infty$,

$$\Sigma_3(u) \le Ce^{-\frac{\tau}{2}\log^2(u^2)}e^{-\frac{q(t_0)}{2}u^2}=o(u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u))=o(\Sigma_1(u;z)),$$

for some $\tau, C>0$.

By (2.11), we obtain that, for any $\widetilde{\Delta},\widehat{\Delta}\ge z$, as $u\to\infty$

$$\begin{aligned}\frac{1}{\widetilde{\Delta}}\mathcal{B}_I(z;\widetilde{\Delta}) &= \frac{\Sigma_1(u;z;\widetilde{\Delta})-\Sigma_2(u;\widetilde{\Delta})}{u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)}\\ &\le \frac{s_{\rho,T}(\boldsymbol{c},\boldsymbol{a},u;z)}{u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)}\\ &\le \frac{\Sigma_1(u;z;\widehat{\Delta})+\Sigma_2(u;\widehat{\Delta})+\Sigma_3(u)}{u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)}=\frac{1}{\widehat{\Delta}}\mathcal{B}_I(z;\widehat{\Delta}).\end{aligned}$$

The above implies that

$$0<\frac{1}{\widetilde{\Delta}}\int_{\mathbb{R}^{|I|}}\mathbb{P}\{\inf_{t\in[0,\widetilde{\Delta}]}(\boldsymbol{W}_I(t)-\boldsymbol{c}_I t)>\boldsymbol{x}_I\}e^{\boldsymbol{\lambda}_I\boldsymbol{x}_I^T}d\boldsymbol{x}_I\le\limsup_{\widetilde{\Delta}\to\infty}\frac{1}{\widetilde{\Delta}}\mathcal{B}_I(z;\widetilde{\Delta})=\liminf_{\widehat{\Delta}\to\infty}\frac{1}{\widehat{\Delta}}\mathcal{B}_I(z;\widehat{\Delta})<\infty.$$

Thus,

$$\mathcal{B}(z)=\lim_{\Delta\to\infty}\frac{1}{\Delta}\mathcal{B}(z;\Delta)\in(0,\infty).$$

Case (ii): $t_0=T$. Note that $t_0^*=T$, and for any $u>0, z\ge 0$ and $\widetilde{\Delta},\widehat{\Delta}>0$

$$\Sigma_1(u;z;\widetilde{\Delta})-\Sigma_2(u;\widetilde{\Delta})\le s_{\rho,T}(\boldsymbol{c},\boldsymbol{a},u;z)\le\Sigma_1(u;z;\widehat{\Delta})+\Sigma_2(u;\widehat{\Delta})+\Sigma_3(u),$$

where

$$\Sigma_1(u;z;\Delta) = \sum_{k=1}^{N_u}\mathbb{P}\{\int_{E_{k,u;\Delta}}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}ut>\boldsymbol{a}u\}dt>zu^{-2}\},$$

$$\begin{aligned}\Sigma_2(u;\Delta) &= 2\sum_{k=-N_u}^{N_u}\sum_{l=k+1}^{N_u}\mathbb{P}\{\sup_{t\in E_{k,u;\Delta}}(\boldsymbol{W}(t)-\boldsymbol{c}ut)>\boldsymbol{a}u, \sup_{s\in E_{l,u;\Delta}}(\boldsymbol{W}(s)-\boldsymbol{c}us)>\boldsymbol{a}u\},\\ \Sigma_3(u) &= \mathbb{P}\{\sup_{[0,T]\setminus F_u}(\boldsymbol{W}(t)-\boldsymbol{c}ut)>\boldsymbol{a}u\}.\end{aligned}$$

with $F_u=[t_0-\frac{\log(u)}{u}, \min(t_0+\frac{\log(u)}{u},T)]$.

Following the analysis similar to the case (i), we conclude that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\Sigma_1(u;z;\Delta)}{\mathcal{B}_I(z)u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)\int_{[0,\infty)}\mathcal{C}(x)e^{-\frac{g_I''(t_0)}{4}x^2}dx}=1$$

and, as $u\to\infty$,

$$\Sigma_2(u)=o(u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)) \text{ and } \Sigma_3(u)=o(u^{-|I|+1}\varphi_{t_0}((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)),$$

where

$$g_I''(t_0)=2\frac{\boldsymbol{a}_I^T\Sigma_{II}^{-1}\boldsymbol{a}_I}{(t_0)^3}, \text{ with } \Sigma=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix} \text{ and } \mathcal{C}(x)=\begin{cases}\Psi(-\frac{c_{\neg i}-\rho c_i}{\sqrt{(1-\rho^2)t_0}}x), & \text{if } K\neq\emptyset\\ 1, & \text{if } K=\emptyset\end{cases}.$$

Case (iii): $t_0>T$. Note that $t_0^*=T$, and for any $u>0, z\geq 0$

$$\Sigma_1(u;z)\leq s_{\rho,T}(\boldsymbol{c},\boldsymbol{a},u;z)\leq\Sigma_1(u;z)+\Sigma_2(u)+\Sigma_3(u),$$

where

$$\begin{aligned}\Sigma_1(u;z) &= \mathbb{P}\{\int_{E_{1,u}}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}ut>\boldsymbol{a}u\}dt>zu^{-2}\},\\ \Sigma_2(u) &= \mathbb{P}\{\sup_{t\in F_u}(\boldsymbol{W}(t)-\boldsymbol{c}ut)>\boldsymbol{a}u\},\\ \Sigma_3(u;z) &= \mathbb{P}\{\int_{G_u}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}ut>\boldsymbol{a}u\}>zu^{-2}\},\end{aligned}$$

with $F_u=[0,T-\frac{\log(u)}{u}]$ and $G_u=[T-\frac{\log(u)}{u},T-\frac{\Delta}{u^2}]$.

Next, we calculate the asymptotic behavior of $\Sigma_1(u;z)$, $\Sigma_2(u)$ and $\Sigma_3(u;z)$, as $u\to\infty$.

Asymptotics of $\Sigma_1(u;z)$. Lemma 2.3.4 gives, as $u\to\infty$,

$$\Sigma_1(u;z)\sim\mathcal{B}_I(z;\Delta)\mathcal{C}_K u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}t_0)_I u),$$

where

$$\mathcal{C}_K=\begin{cases}\frac{1}{2}, & \text{if } K\neq\emptyset\\ 1, & \text{if } K=\emptyset\end{cases}.$$

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Hence, we obtain that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\left|\frac{\Sigma_1(u;z)}{\mathcal{C}_K u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}t_0)_I u)}-\mathcal{B}_I(z;\Delta)\right|=0.$$

Futher, we have that, for $\Delta>0$,

$$0<\mathcal{B}_I(z;\Delta)\le\mathcal{B}_I(0;\Delta)\le\int_{\mathbb{R}^{|I|}}\mathbb{P}\left\{\sup_{t\in[0,\infty)}(\boldsymbol{W}_I(t)-\frac{\boldsymbol{a}_I}{T}t)>\boldsymbol{x}_I\right\}e^{\boldsymbol{\lambda}_I\boldsymbol{x}_I^T}d\boldsymbol{x}_I<\infty,$$

where the last inequality follows from Lemma 3.5 in [35] for $|I|=1$ and Lemma 2.3.5 for $|I|=2$. Thus, due to the monotonicity of $\mathcal{B}_I(z;\Delta)$ in Δ, it holds

$$\widehat{\mathcal{B}}_I(z)=\lim_{\Delta\to\infty}\mathcal{B}_I(z;\Delta)\in(0,\infty).$$

Asymptotics of $\Sigma_2(u)$. Lemma 2.3.2 (i) gives, as $u\to\infty$,

$$\Sigma_2(u)\quad\le\quad Ce^{-\frac{\tau}{2}\log^2(u^2)}e^{-\frac{q(T)}{2}u^2}=o(u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}t_0)_I u))=o(\Sigma_1(u;z)),$$

for some $\tau,C>0$.

Asymptotics of $\Sigma_3(u;z)$. Lemma 2.3.4 implies, as $u\to\infty$,

$$\begin{aligned}\Sigma_3(u;z)\quad&\le\quad\sum_{k=2}^{N_u+1}\mathbb{P}\{\int_{E_{k,u}}\mathbb{I}\{\boldsymbol{W}(t)-\boldsymbol{c}ut>\boldsymbol{a}u\}>zu^{-2}\}\\&\le\quad\sum_{k=2}^{N_u+1}\mathcal{C}(\frac{(k-1)\Delta}{u})\mathcal{B}_I(z;\Delta)u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_I u)e^{\frac{q_I'(T)}{2}(k-1)\Delta}\\&\le\quad\sum_{k=2}^{N_u+1}\mathcal{B}_I(z;\Delta)u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_I u)e^{\frac{q_I'(T)}{2}(k-1)\Delta}\\&\le\quad\mathcal{B}_I(z;\Delta)u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_I u)\sum_{k=2}^{\infty}e^{\frac{q_I'(T)}{2}(k-1)\Delta}\\&=\quad\mathcal{B}_I(z;\Delta)u^{-|I|}\varphi_T((\boldsymbol{a}+\boldsymbol{c}T)_I u)e^{\frac{q_I'(T)}{2}\Delta}\frac{1}{1-e^{\frac{q_I'(T)}{2}\Delta}},\end{aligned}$$

with

$$q_I'(T)<0,\ \mathcal{C}(x)=\begin{cases}\Psi(-\frac{c_{\neg i}-\rho c_i}{\sqrt{(1-\rho^2)T}}x), & \text{if } K\neq\emptyset\\ 1, & \text{if } K=\emptyset\end{cases}\quad\text{and } \mathcal{C}(x)\le 1.$$

Hence, for some $\widetilde{C}>0$, we have that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\Sigma_3(u;z)}{\Sigma_1(u;z)}=\lim_{\Delta\to\infty}\widetilde{C}e^{\frac{q'(T)}{2}\Delta}=0.$$

This completes the proof. ∎

Chapter 3

Stationary Gaussian queues over a random time interval

3.1 Introduction

Consider a single-node fluid queue with infinite buffer capacity which is fed by a fractional Brownian motion $\{B_H(t) : t \in \mathbb{R}\}$ with Hurst index $H \in (0,1)$ and emptied with rate $c > 0$. We study the stationary buffer content process $\{Q(t) : t \geq 0\}$ defined as the stationary solution to the *Skorokhod problem* given in [124, 125] (see also [49][Chapter 2]). We recall that a pair (Q, L) with $Q \equiv \{Q(t) : t \geq 0\}$, $L \equiv \{L(t) : t \geq 0\}$ is a solution to the Skorokhod problem for the process $\{B_H(t) : t \in \mathbb{R}\}$ with drift c if the following conditions hold:

S0: The process $\{L(t) : t \geq 0\}$ is non-decreasing, right-continuous and $L(0) = 0$,

S1: The workload process $\{Q(t) : t \geq 0\}$, defined through

$$Q(0) := x \text{ and } Q(t) := Q(0) + (B_H(t) - ct) + L(t)$$

is non-negative for all $t \geq 0$,

S2: $L(t)$ can only increase when $Q(t) = 0$, that is,

$$\int_0^T Q(t)dL(t) = 0, \text{ for all } T > 0.$$

Note that since

$$\lim_{t\to\infty}(B_H(t)-ct)=-\infty \text{ a.s.},$$

there exists a stationary solution to the *Skorokhod problem* (**S0**–**S2**).

Following [120], the stationary buffer content process $\{Q(t):t\geq 0\}$ has the following representation:

$$Q(t)=\sup_{s\leq t}\left(B_H(t)-B_H(s)-c(t-s)\right).$$

The modelling of the input to the fluid queue by fractional Brownian motion stems both from the observation that the traffic in modern communication networks has a self-similar nature [128], and by theory based results on functional limit theorems, as shown in [127].

Distributional properties of the process $\{Q(t):t\geq 0\}$ were intensively analyzed. In particular, Theorem 5 in [118] establishes the exact asymptotics of $\mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}$, with the celebrated *Piterbarg property* for $H>1/2$, which says that, as $u\to\infty$,

$$\mathbb{P}\{Q(0)>u\}\sim\mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}.$$

Further, in [118](see also equation (9) in [79]) it was proved that, as $u\to\infty$,

$$\mathbb{P}\{\sup_{t\in[0,1]}(Q(t))>u\}\sim c_2u^he^{-\frac{1}{2}A^2u^{2(1-H)}}, \tag{3.1}$$

where

$$c_2=\left(\frac{H}{c(1-H)}\right)^{-4}2^{-\frac{2}{H}}\sqrt{\frac{A}{B}}\mathcal{H}_{2H}^2A^{\frac{2}{H}-2},\ h=\frac{2(1-H)^2}{H}-1 \tag{3.2}$$

with

$$A=\left(\frac{H}{c(1-H)}\right)^{-H}\frac{1}{1-H},\ B=\left(\frac{H}{c(1-H)}\right)^{-(H+2)}H \tag{3.3}$$

and $\mathcal{H}_{2H}$ being the *Pickands constant*, i.e.

$$\mathcal{H}_{2H}=\lim_{t\to\infty}\frac{1}{T}\mathcal{H}_{X_H}[0,T]\in(0,\infty) \text{ with } \mathcal{H}_{2H}[T]=\mathbb{E}\{e^{\sup_{t\in[0,T]}\left(\sqrt{2}B_H(t)-t^{2H}\right)}\},\ T\in(0,\infty).$$

In this chapter, we abuse slightly the notation used in the rest of the thesis. Instead, we adopt the standard notation found in the literature on Gaussian processes (see, e.g. [48, 79, 117, 118]). More specifically, we write $\mathcal{H}_{2H}$ and $\mathcal{H}_{2H}[T]$ instead of $\mathcal{H}_{X_H}$ and $\mathcal{H}_{X_H}[0,T]$ with $X_H(t) = \sqrt{2}B_H(t) - t^{2H}$, as in the other chapters.

We note that $\mathcal{H}_{2H} \in (0, \infty)$ for $H \in (0,1)$ and $\mathcal{H}_1 = 1$; see, e.g., [48, 117].

In the following, we shall investigate the asymptotics of

$$p(\mathcal{T}_u; u) := \mathbb{P}\{\sup_{t\in[0,\mathcal{T}_u]} (Q(t)) > u\}, \tag{3.4}$$

as $u \to \infty$, where $\mathcal{T}_u$ is a non-negative random variable, possibly depending on u, independent of $\{Q(t) : t \geq 0\}$. The motivation to analyze the behavior of $\{Q(t) : t \geq 0\}$ over a random time interval stems from recently investigated models with the so called "resetting" [67, 112, 123], where the system is terminated at a random time.

We note that for $\mathcal{T}_u$ a deterministic function of u, the probability $p(\mathcal{T}_u; u)$ was investigated, among others, in [46, 79, 118]. The following proposition shows the results obtained in [46][Corollaries 4.1–4.3]; see also [118].

Here, we write $\Psi(x) = 1 - \Phi(x) := \mathbb{P}\{\mathcal{N} > x\}$, where $\mathcal{N}$ is a standard normal random variable.

Proposition 3.1.1 *Let* $\tau_0 = \frac{H}{c(1-H)}$ *and* $m^*(u) = \left(\frac{c}{H}\right)^H \frac{1}{(1-H)^{1-H}} u^{1-H}$.
(i) If $\mathcal{T}_u u^{\frac{1-2H}{H}} \sim \rho \left(\frac{\sqrt{2}(\tau_0)^{2H}}{1+c\tau_0}\right)^{\frac{1}{H}}$ *as* $u \to \infty$, *with* $\rho \in [0,\infty)$, *then*

$$p(\mathcal{T}_u; u) \sim \mathcal{H}_{2H}[\rho]\mathcal{H}_{2H} 2^{\frac{H-1}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{1-H}{H}}}{(\tau_0)^{2-H}} u^{\frac{(1-H)^2}{H}} \Psi\left(Au^{1-H}\right).$$

(ii) If $\mathcal{T}_u u^{\frac{1-2H}{H}} \to \infty$ *and* $\mathcal{T}_u = o(e^{\beta_1 (m^*(u))^2})$ *with* $\beta_1 \in (0, 1/2)$, *then*

$$p(\mathcal{T}_u; u) \sim (\mathcal{H}_{2H})^2 2^{\frac{H-2}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{2-H}{H}}}{(\tau_0)^{4-H}} \mathcal{T}_u u^{\frac{(2-H)^2-2}{H}} \Psi\left(Au^{1-H}\right).$$

In this chapter we shall consider the following three classes of random variables T, that differ by the heaviness of the tail distribution and that give qualitatively different types of the asymptotics of (3.4):

D1: T is integrable, i.e. $\mathbb{E}\{T\} < \infty$;

D2: T is regularly varying with parameter $\alpha \in (0, 1)$;

D3: T is slowly varying.

We analyze asymptotic behavior of $p(\mathcal{T}_u; u)$, as $u \to \infty$, where

$$\mathcal{T}_u = u^\gamma \sum_{k=1}^{u^\beta} T_k,$$

with $\beta \geq 0$, $\gamma \in \mathbb{R}$, $\{T_k : k = 1, 2, \dots\}$ being i.i.d. copies of T, which are independent of the process $\{Q(t) : t \geq 0\}$ non-negative random variable with distribution function $F_T(\cdot)$.

3.2 Main results

In this section, we derive the exact asymptotics of (3.4), as $u \to \infty$. The analysis is divided into three cases, depending on the probabilistic properties of the random variable T, which determines the distribution of $\mathcal{T}_u = u^\gamma \sum_{k=1}^{u^\beta} T_k$.

As we shall prove below, if T satisfies **D1**, then its distribution contributes to the asymptotics through a constant and, due to the dependence of $\mathcal{T}_u$ on u, through a polynomial factor. Under **D2**, the regularly varying index of T affects the logarithmic asymptotics, resulting in heavier - but still exponentially decaying - asymptotics. Finally, when T satisfies **D3**, the asymptotic behavior becomes significantly heavier; in particular, $p(\mathcal{T}_u; u)$ may be regularly varying itself.

The results below formalize these distinctions and emphasize the crucial role of the heaviness of T in the exact asymptotics of $p(\mathcal{T}_u; u)$, as $u \to \infty$.

3.2.1 Integrable random time horizon.

In scenario **D1**, the distribution of the random variable T contributes to the asymptotic behavior only via a constant. Meanwhile, the relationship between γ, β and $\frac{2H-1}{H}$ determines which scenario in Proposition 3.1.1 applies, which affects the polynomial factor as well.

Theorem 3.2.1 *Suppose that T satisfies **D1** and $\gamma \in \mathbb{R}$.*

(i) If $\beta = 0$, then, as $u \to \infty$,

$$p(u^\gamma T; u) \sim \begin{cases} \mathcal{H}_{2H} 2^{\frac{H-1}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{1-H}{H}}}{(\tau_0)^{2-H}} u^{\frac{(1-H)^2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma < \frac{2H-1}{H} \\ \mathbb{E}\{\mathcal{H}_{2H}\left[\mathcal{C}_H T\right]\} \mathcal{H}_{2H} 2^{\frac{H-1}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{1-H}{H}}}{(\tau_0)^{2-H}} u^{\frac{(1-H)^2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma = \frac{2H-1}{H} \\ \mathbb{E}\{T\} (\mathcal{H}_{2H})^2 2^{\frac{H-2}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{2-H}{H}}}{(\tau_0)^{4-H}} u^{\gamma + \frac{(2-H)^2-2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma > \frac{2H-1}{H} \end{cases},$$

with $\mathcal{C}_H = 2^{-\frac{1}{2H}} \left(\frac{c}{H}\right)^2 (1-H)^{\frac{2H-1}{H}}$.

(ii) If $\beta > 0$*, then, as* $u \to \infty$*,*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim \begin{cases} \mathcal{H}_{2H} 2^{\frac{H-1}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{1-H}{H}}}{(\tau_0)^{2-H}} u^{\frac{(1-H)^2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma+\beta < \frac{2H-1}{H} \\ \mathcal{H}_{2H}\left[\mathcal{C}_H \mathbb{E}\{T\}\right] \mathcal{H}_{2H} 2^{\frac{H-1}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{1-H}{H}}}{(\tau_0)^{2-H}} u^{\frac{(1-H)^2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma+\beta = \frac{2H-1}{H} \\ \mathbb{E}\{T\}(\mathcal{H}_{2H})^2 2^{\frac{H-2}{2H}} \sqrt{\frac{A\pi}{B}} \frac{(1+c\tau_0)^{\frac{2-H}{H}}}{(\tau_0)^{4-H}} u^{\gamma+\beta+\frac{(2-H)^2-2}{H}} \Psi\left(Au^{1-H}\right), & \text{if } \gamma+\beta > \frac{2H-1}{H} \end{cases},$$

with $\mathcal{C}_H = 2^{-\frac{1}{2H}} \left(\frac{c}{H}\right)^2 (1-H)^{\frac{2H-1}{H}}$.

Remark 3.2.2 *If* T *satisfies* ***D1*** *and* $\beta > 0$*, the asymptotics derived in Theorem 3.2.1 takes the following form, as* $u \to \infty$*,*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim p(u^{\gamma+\beta}\mathbb{E}\{T\}; u).$$

where the right hand side of the above asymptotics follows from Proposition 3.1.1.

In some special cases, the exact asymptotics given in Theorem 3.2.1 can be expressed explicitly. The results for $H = \frac{1}{2}$ and an exponentially distributed T are shown below.

Example 3.2.3 *Suppose that* $T \sim Exp(\lambda)$*,* $\lambda > 0$ *and let* $\gamma \in \mathbb{R}$*,* $H = \frac{1}{2}$*.*

(i) If $\beta = 0$*, then, as* $u \to \infty$*,*

$$p(u^\gamma T; u) \sim \begin{cases} \sqrt{\frac{A\pi}{B}}(2c)^{\frac{3}{2}}\sqrt{u}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma < 0 \\ \left[\frac{c^2+\lambda}{\lambda} + \frac{c(c^2+3\lambda)}{\lambda\sqrt{2\lambda+c^2}}\right]\sqrt{\frac{A\pi}{B}}(2c)^{\frac{3}{2}}\sqrt{u}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma = 0 \\ \frac{1}{\lambda}\sqrt{\frac{A\pi}{B}}(2c)^{3/2}u^{\gamma+\frac{1}{2}}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma > 0 \end{cases}.$$

(ii) If $\beta > 0$*, then, as* $u \to \infty$*,*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim \begin{cases} \sqrt{\frac{A\pi}{B}}(2c)^{\frac{3}{2}}\sqrt{u}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma+\beta < 0 \\ \left[2\frac{c^2+\lambda}{\lambda}\Phi\left(\frac{c}{\sqrt{\lambda}}\right) + c\sqrt{\frac{2}{\lambda\pi}}\exp\left(-\frac{c^2}{2\lambda}\right)\right]\sqrt{\frac{A\pi}{B}}(2c)^{\frac{3}{2}}\sqrt{u}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma+\beta = 0 \\ \frac{1}{\lambda}\sqrt{\frac{A\pi}{B}}(2c)^{3/2}u^{\gamma+\beta+\frac{1}{2}}\Psi\left(A\sqrt{u}\right), & \text{if } \gamma+\beta > 0 \end{cases}.$$

Remark 3.2.4 *If* T *satisfies* ***D1****, the random variable* $\mathcal{T}_u$ *does not affect the logarithmic asymptotics of the probability* $p(\mathcal{T}_u; u)$ *stated in Theorem 3.2.1. More precisely, for* $\beta \geq 0$*,* $\gamma \in \mathbb{R}$*, it holds, as* $u \to \infty$*,*

$$\log(p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u)) \sim -\frac{1}{2}A^2 u^{2(1-H)} \sim \log(p(1; u)).$$

3.2.2 Regularly varying random time horizon.

In scenario **D2**, the contribution of the event that the random variable T attains large values is crucial. More precisely, the asymptotics is dominated by the event that T exceeds the threshold

$$N(u) := 1/p(1;u).$$

Recall that, by (3.1), we have, as $u \to \infty$,

$$N(u) \sim \frac{1}{c_2} u^{-h} e^{\frac{1}{2}A^2 u^{2(1-H)}},$$

where the constants c_2, h and A are defined in (3.2) and (3.3).

Before stating the result, recall that $\Gamma(\cdot)$ denotes *Gamma function*, i.e. $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.

Theorem 3.2.5 *Suppose that T satisfies* ***D2*** *and $\beta \geq 0$, $\gamma \in \mathbb{R}$. Then, as $u \to \infty$,*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim \Gamma(1-\alpha) u^\beta \mathbb{P}\{T > u^{-\gamma} N(u)\}.$$

Remark 3.2.6 *If T satisfies* ***D2****, the index of regular variation α of T, affects the logarithmic asymptotics of $p(\mathcal{T}_u; u)$, while the dependence of $\mathcal{T}_u$ on u does not. Concretely, for $\beta \geq 0$, $\gamma \in \mathbb{R}$, we have, as $u \to \infty$,*

$$\log(p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u)) \sim -\frac{1}{2} A^2 \alpha u^{2(1-H)} \sim \alpha \log(p(1;u)).$$

3.2.3 Slowly varying random time horizon.

In scenario **D3**, a crucial role plays the following asymptotic equivalence, as $u \to \infty$,

$$\mathbb{P}\{u^\gamma \sum_{k=1}^{u^\beta} T_k > u\} \sim u^\beta \mathbb{P}\{u^\gamma T > u\}.$$

To guarantee the above, we need an additional condition on the tail distribution function of T. Following Theorem 3.3 in [26], we assume that

D4: There exists a function $t(u)$ such that

$$\lim_{u\to\infty} u^\beta \mathbb{P}\{T > t(u)\} = 0 \text{ and } t(u) = o(N(u)),$$

as $u \to \infty$.

In order to ilustrate condition **D4**, note that if $\beta < 2(1-H)$ and the random variable T satisfies $\mathbb{P}\{T > u\} \sim \frac{1}{\log(u)}$, as $u \to \infty$, then the function $t(u) = N(u)u^{-1} = \frac{1}{c_2}u^{-h-1}e^{\frac{1}{2}A^2u^{2(1-H)}}$ satisfies **D4**.

Theorem 3.2.7 *Suppose that T satisfies **D3**–**D4** and $\beta \geq 0$, $\gamma \in \mathbb{R}$. Then, as $u \to \infty$,*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim u^\beta \mathbb{P}\{T > u^{-\gamma}N(u)\}.$$

Remark 3.2.8 *In the slowly varying regime (**D3**–**D4**), the asymptotics of $p(\mathcal{T}_u; u)$ is much heavier than in scenarios **D1** and **D2**. For example, if $\beta < 2(1-H)$, $\gamma \in \mathbb{R}$ and T satisfies $\mathbb{P}\{T > u\} \sim \frac{1}{\log(u)}$, as $u \to \infty$, then*

$$p(u^\gamma \sum_{k=1}^{u^\beta} T_k; u) \sim \frac{2}{A^2}u^{\beta-2(1-H)},$$

as $u \to \infty$, where A was defined in (3.3).

3.3 Proofs

3.3.1 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1 We split the proof into two cases: $\beta = 0$ and $\beta > 0$.

(i) Case $\beta = 0$: Let $0 < A_0 < A_\infty < \infty$ and $I_1 = [0, A_0]$, $I_2 = [A_0, A_\infty]$, $I_3 = [A_\infty, \infty)$. Since for $\beta = 0$ it holds $\sum_{k=1}^{u^\beta} T_k = T_1 \stackrel{d}{=} T$, we have

$$\mathbb{P}\{\sup_{t\in[0,u^\gamma T]}(Q(t)) > u\} = \sum_{m=1}^{3} \mathbb{P}\{\sup_{t\in[0,u^\gamma T]}(Q(t)) > u, T \in I_m\} =: \sum_{m=1}^{3} \mathcal{P}_m(u).$$

Asymptotics of $\mathcal{P}_1(u)$. Note that, by the stationarity of $\{Q(t) : t \geq 0\}$, as $u \to \infty$,

$$\frac{\mathcal{P}_1(u)}{p(u^\gamma; u)} \leq \frac{p(u^\gamma A_0; u)}{p(u^\gamma; u)} \leq A_0(1 + o(1)) = O(A_0).$$

Asymptotics of $\mathcal{P}_2(u)$. By Proposition 3.1.1, there exists a constant $C > 0$ such that for each $u \geq 0$,

$$\sup_{x\in[A_0,A_\infty]} \frac{p(u^\gamma x; u)}{p(u^\gamma; u)} \leq A_\infty(1 + o(1)) \leq C.$$

Lebesgue's dominated convergence theorem gives that

$$\lim_{u\to\infty} \frac{\mathcal{P}_2(u)}{p(u^\gamma; u)} = \lim_{u\to\infty} \int_{A_0}^{A_\infty} \frac{p(u^\gamma x; u)}{p(u^\gamma; u)} dF_T(x) = \int_{A_0}^{A_\infty} \lim_{u\to\infty} \frac{p(u^\gamma x; u)}{p(u^\gamma; u)} dF_T(x) =: \int_{A_0}^{A_\infty} f(x)dF_T(x).$$

Asymptotics of $\mathcal{P}_3(u)$. By the stationarity of $\{Q(t) : t \geq 0\}$ we have, as $A_\infty \to \infty$,

$$\frac{\mathcal{P}_3(u)}{p(u^\gamma; u)} = \int_{A_\infty}^{\infty} \frac{p(u^\gamma x; u)}{p(u^\gamma; u)} dF_T(x) \leq \int_{A_\infty}^{\infty} (x+1) dF_T(x) = o(1).$$

Since $\int_0^\infty f(x) dF_T(x) = \mathbb{E}\{f(T)\} \leq \mathbb{E}\{T\} < \infty$, by taking $A_0 \to 0$ and $A_\infty \to \infty$, we obtain the thesis.

(ii) Case $\beta > 0$: Let $S_{u^\beta} := \sum_{k=1}^{u^\beta} T_k$ and $\mu_T := \mathbb{E}\{T\}$. For $\epsilon > 0$ define

$$B_{1,\epsilon} := \left\{ \left| \frac{1}{u^\beta} S_{u^\beta} - \mu_T \right| > \epsilon \right\},\ B_{2,\epsilon} := \left\{ \left| \frac{1}{u^\beta} S_{u^\beta} - \mu_T \right| \leq \epsilon \right\}.$$

We have

$$\mathbb{P}\{ \sup_{t \in [0, u^\gamma \sum_{k=1}^{u^\beta} T_k]} (Q(t)) > u\} = \sum_{m=1}^{2} \mathbb{P}\{ \sup_{t \in [0, u^{\gamma+\beta} \frac{1}{u^\beta} S_{u^\beta}]} (Q(t)) > u, \frac{1}{u^\beta} S_{u^\beta} \in B_{m,\epsilon}\} =: \sum_{m=1}^{2} \mathcal{P}_m(u).$$

Asymptotics of $\mathcal{P}_1(u)$. By the stationarity of $\{Q(t) : t \geq 0\}$ we obtain, as $u \to \infty$,

$$\begin{aligned}
\frac{\mathcal{P}_1(u)}{p(u^{\gamma+\beta}; u)} &= \int_{B_{1,\epsilon}} \frac{p(u^{\gamma+\beta} x; u)}{p(u^{\gamma+\beta}; u)} dF_{\frac{1}{u^\beta} S_{u^\beta}}(x) \leq \int_{B_{1,\epsilon}} (x+1) dF_{\frac{1}{u^\beta} S_{u^\beta}}(x) \\
&\leq \mathbb{E}\{\left(\frac{1}{u^\beta} S_{u^\beta} + 1 \right) \mathbb{I}_{B_{1,\epsilon}}\} = \mathbb{E}\{\frac{1}{u^\beta} \sum_{k=1}^{u^\beta} T_k \mathbb{I}_{B_{1,\epsilon}}\} + \mathbb{P}\{B_{1,\epsilon}\} \\
&= \frac{1}{u^\beta} \sum_{k=1}^{u^\beta} \mathbb{E}\{T_k \mathbb{I}_{B_{1,\epsilon}}\} + \mathbb{P}\{B_{1,\epsilon}\} = \mathbb{E}\{T \mathbb{I}_{B_{1,\epsilon}}\} + \mathbb{P}\{B_{1,\epsilon}\} = o(\epsilon),
\end{aligned}$$

where the last equation follows from the fact that $\mathbb{E}\{T\} < \infty$ and $\frac{1}{u^\beta} S_{u^\beta} \xrightarrow{\mathbb{P}} \mu_T$, as $u \to \infty$.

Asymptotics of $\mathcal{P}_2(u)$. For each $u > 0$, we obtain

$$p(u^{\gamma+\beta}(\mu_T - \epsilon); u) \mathbb{P}\{ \left| \frac{1}{u^\beta} S_{u^\beta} - \mu_T \right| \leq \epsilon\} \leq \mathcal{P}_2(u) \leq p(u^{\gamma+\beta}(\mu_T + \epsilon); u) \mathbb{P}\{ \left| \frac{1}{u^\beta} S_{u^\beta} - \mu_T \right| \leq \epsilon\},$$

where for each $\epsilon > 0$, it holds $\mathbb{P}\{ \left| \frac{1}{u^\beta} S_{u^\beta} - \mu_T \right| \leq \epsilon\} \to 1$, as $u \to \infty$.

Using Proposition 3.1.1 we have

$$\frac{\mu_T - \epsilon}{\mu_T} \leq \lim_{u \to \infty} \frac{p(u^{\gamma+\beta}(\mu_T - \epsilon); u)}{p(u^{\gamma+\beta} \mu_T; u)} \leq \lim_{u \to \infty} \frac{p(u^{\gamma+\beta}(\mu_T + \epsilon); u)}{p(u^{\gamma+\beta} \mu_T; u)} \leq \frac{\mu_T + \epsilon}{\mu_T}.$$

Hence, by taking $\epsilon \to 0$ we obtain the thesis. □

3.3.2 Proof of Example 3.2.3

Proof of Example 3.2.3 We shall calculate $\mathcal{H}_1 [2c^2 T]$. Applying Lemma 2.2 in [55] for $x > 0$, we have

$$\mathcal{H}_1 [x] = (x+2) \Phi \left(\sqrt{\frac{x}{2}} \right) + \sqrt{\frac{x}{\pi}} \exp \left(-\frac{x}{4} \right).$$

Thus, for $T \sim Exp(\lambda)$ we obtain

$$
\begin{aligned}
\mathbb{E}\{\mathcal{H}_1\left[2c^2T\right]\} &= \int_0^\infty \lambda \exp\left(-\lambda x\right) \mathcal{H}_1\left[0, 2c^2x\right] dx \\
&= \int_0^\infty \lambda \exp\left(-\lambda x\right) \left(\left(2c^2x+2\right)\Phi\left(\sqrt{\frac{2c^2x}{2}}\right) + \sqrt{\frac{2c^2x}{\pi}} \exp\left(-\frac{2c^2x}{4}\right)\right) dx \\
&= \int_0^\infty \frac{\lambda}{2c^2} \exp\left(-\frac{\lambda}{2c^2}x\right) \left(\left(x+2\right)\Phi\left(\sqrt{\frac{x}{2}}\right) + \sqrt{\frac{x}{\pi}} \exp\left(-\frac{x}{4}\right)\right) dx \\
&= \int_0^\infty \frac{\lambda}{2c^2} \exp\left(-\frac{\lambda}{2c^2}x\right) \left(x+2\right)\Phi\left(\sqrt{\frac{x}{2}}\right) dx \\
&+ \int_0^\infty \frac{\lambda}{2c^2} \exp\left(-\frac{\lambda}{2c^2}x\right) \sqrt{\frac{x}{\pi}} \exp\left(-\frac{x}{4}\right) dx =: I_1 + I_2.
\end{aligned}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1 &= \left[\left(x + 2\frac{c^2+\lambda}{\lambda}\right) \exp\left(-\frac{\lambda}{2c^2}x\right) \Phi\left(\sqrt{\frac{x}{2}}\right)\right]\Bigg|_{x=0}^{x=\infty} \\
&- \int_0^\infty \left(-\left(x + 2\frac{c^2+\lambda}{\lambda}\right)\right) \exp\left(-\frac{\lambda}{2c^2}x\right) \frac{1}{4\sqrt{\pi}} x^{-1/2} \exp\left(-\frac{x}{4}\right) dx \\
&= \frac{c^2+\lambda}{\lambda} + \frac{1}{4\sqrt{\pi}} \int_0^\infty x^{1/2} \exp\left(-\frac{2\lambda+c^2}{4c^2}x\right) dx + 2\frac{c^2+\lambda}{4\lambda\sqrt{\pi}} \int_0^\infty x^{-1/2} \exp\left(-\frac{2\lambda+c^2}{4c^2}x\right) dx
\end{aligned}
$$

and

$$
I_2 = \frac{\lambda}{2c^2\sqrt{\pi}} \int_0^\infty x^{1/2} \exp\left(-\frac{2\lambda+c^2}{4c^2}x\right) dx.
$$

Since for $A > 0$

$$
\int_0^\infty x^{-1/2} \exp\left(-Ax\right) dx = \sqrt{\pi} A^{-1/2} \text{ and } \int_0^\infty x^{1/2} \exp\left(-Ax\right) dx = \frac{\sqrt{\pi}}{2} A^{-3/2},
$$

we obtain

$$
\begin{aligned}
\mathbb{E}\{\mathcal{H}_1\left[2c^2T\right]\} &= \frac{c^2+\lambda}{\lambda} + \frac{1}{\sqrt{\pi}}\left(\frac{1}{4} + \frac{\lambda}{2c^2}\right) \frac{\sqrt{\pi}}{2} \left(\frac{2\lambda+c^2}{4c^2}\right)^{-3/2} + \frac{c^2+\lambda}{2\lambda\sqrt{\pi}} \sqrt{\pi} \left(\frac{2\lambda+c^2}{4c^2}\right)^{-1/2} \\
&= \frac{c^2+\lambda}{\lambda} + \left(\frac{2\lambda+c^2}{4c^2}\right)^{-1/2} \left(\frac{2\lambda+c^2}{4c^2}\left(\frac{2\lambda+c^2}{4c^2}\right)^{-1} + \frac{c^2+\lambda}{2\lambda}\right) \\
&= \frac{c^2+\lambda}{\lambda} + \frac{c(c^2+3\lambda)}{\lambda\sqrt{2\lambda+c^2}}.
\end{aligned}
$$

This completes the proof. □

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3.3.3 Proofs of Theorems 3.2.5 and 3.2.7

Proof of Theorem 3.2.5 Let $S_{u^\beta} := \sum_{k=1}^{u^\beta} T_k$ and $0 < A_0 < A_\infty < \infty$, $I_1(u) = [0, A_0 N(u)]$, $I_2(u) = [A_0 N(u), A_\infty N(u)]$, $I_3(u) = [A_\infty N(u), \infty)$. We have

$$\mathbb{P}\{\sup_{t\in[0,u^\gamma S_{u^\beta}]} (Q(t)) > u\} = \sum_{m=1}^{3} \mathbb{P}\{\sup_{t\in[0,u^\gamma S_{u^\beta}]} (Q(t)) > u, u^\gamma S_{u^\beta} \in I_k(u)\} =: \sum_{m=1}^{3} \mathcal{P}_m(u).$$

Below we analyze the asymptotics of $\mathcal{P}_1(u)$, $\mathcal{P}_2(u)$ and $\mathcal{P}_3(u)$, as $u \to \infty$.

Asymptotics of $\mathcal{P}_1(u)$. Using stationarity of the process $\{Q(t) : t \geq 0\}$, we obtain

$$\begin{aligned} \mathcal{P}_1(u) &\leq p(1;u)\left[\int_0^{A_0 N(u)} x dF_{u^\gamma S_{u^\beta}}(x) + 1\right] \\ &= p(1;u)\left[\int_0^{A_0 N(u)} \mathbb{P}\{u^\gamma S_{u^\beta} > x\}dx - A_0 N(u)\mathbb{P}\{u^\gamma S_{u^\beta} > A_0 N(u)\} + 1\right]. \end{aligned}$$

Note that $t(u) = u^{1+\gamma}$ satisfies the thesis of Lemma 3.3.1, therefore for some $C > 0$, we have

$$\begin{aligned} \int_0^{A_0 N(u)} \mathbb{P}\{u^\gamma S_{u^\beta} > x\}dx &= \int_0^{t(u)} \mathbb{P}\{u^\gamma S_{u^\beta} > x\}dx + \int_{t(u)}^{A_0 N(u)} \mathbb{P}\{u^\gamma S_{u^\beta} > x\}dx \\ &\leq t(u) + C\int_{t(u)}^{A_0 N(u)} u^\beta \mathbb{P}\{T > xu^{-\gamma}\}dx \leq t(u) + C\int_0^{A_0 N(u)} u^\beta \mathbb{P}\{T > xu^{-\gamma}\}dx \\ &= t(u) + Cu^{\beta+\gamma}\int_0^{A_0 u^{-\gamma} N(u)} \mathbb{P}\{T > x\}dx. \end{aligned}$$

By Karamata's theorem (see, e.g., Proposition 1.5.8 in [16]) we obtain, as $u \to \infty$,

$$\int_0^{A_0 u^{-\gamma} N(u)} \mathbb{P}\{T > x\}dx \sim \frac{1}{1-\alpha} A_0 u^{-\gamma} N(u)\mathbb{P}\{T > A_0 u^{-\gamma} N(u)\}.$$

Proposition 3.1.1 gives the following upper bound for $\mathcal{P}_1(u)$

$$\mathcal{P}_1(u) \leq \frac{1}{1-\alpha} A_0 u^\beta \mathbb{P}\{T > A_0 u^{-\gamma} N(u)\}(1 + o(1)) \sim \frac{1}{1-\alpha} A_0^{1-\alpha} u^\beta \mathbb{P}\{T > u^{-\gamma} N(u)\},$$

as $u \to \infty$.

Asymptotics of $\mathcal{P}_2(u)$. Let $\epsilon > 0$. Applying Lemma 3.3.2, we obtain for sufficiently large $u \geq 0$

$$\begin{aligned} &(1-\epsilon)\left(\int_{A_0}^{A_\infty} e^{-x}\mathbb{P}\{u^\gamma S_{u^\beta} > xN(u)\}dx - \left(1 - e^{-A_\infty}\right)\mathbb{P}\{u^\gamma S_{u^\beta} > A_\infty N(u)\}\right) \\ &= (1-\epsilon)\int_{A_0}^{A_\infty} \left(1 - e^{-x}\right) dF_{u^\gamma S_{u^\beta}}(xN(u)) \\ &\leq \int_{A_0}^{A_\infty} \mathbb{P}\{\sup_{t\in[0,xN(u)]} (Q(t)) > u\} dF_{u^\gamma S_{u^\beta}}(xN(u)) \end{aligned}$$

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$$\begin{aligned}
&\le (1+\epsilon)\int_{A_0}^{A_\infty}\left(1-e^{-x}\right)dF_{u^\gamma S_{u^\beta}}(xN(u))\\
&= (1+\epsilon)\left(\int_{A_0}^{A_\infty} e^{-x}\mathbb{P}\{u^\gamma S_{u^\beta} > xN(u)\}dx + \left(1-e^{-A_0}\right)\mathbb{P}\{u^\gamma S_{u^\beta} > A_0 N(u)\}\right).
\end{aligned}$$

Next, by Lemma 3.3.1 we obtain

$$\lim_{u\to\infty}\sup_{x\in[A_0,A_\infty]}\left|\frac{\mathbb{P}\{S_{u^\beta} > xu^{-\gamma}N(u)\}}{u^\beta\mathbb{P}\{T > xu^{-\gamma}N(u)\}} - 1\right| = 0.$$

Hence, as $u\to\infty$,

$$\int_{A_0}^{A_\infty} e^{-x}\mathbb{P}\{u^\gamma S_{u^\beta} > xN(u)\}dx \sim u^\beta\int_{A_0}^{A_\infty} e^{-x}\mathbb{P}\{T > xu^{-\gamma}N(u)\}dx.$$

Then, Theorem 1.5.2 in [16] gives, as $u\to\infty$,

$$u^\beta\int_{A_0}^{A_\infty} e^{-x}\mathbb{P}\{T > xu^{-\gamma}N(u)\}dx \sim u^\beta\mathbb{P}\{T > u^{-\gamma}N(u)\}\int_{A_0}^{A_\infty} e^{-x}x^{-\alpha}dx.$$

Thus, for each $\epsilon > 0$

$$\liminf_{u\to\infty}\frac{\mathcal{P}_2(u)}{u^\beta\mathbb{P}\{T > u^{-\gamma}N(u)\}} \ge (1-\epsilon)\left[\int_{A_0}^{A_\infty} e^{-x}x^{-\alpha}dx - \left(1-e^{-A_\infty}\right)A_\infty^{-\alpha}\right]$$

and

$$\limsup_{u\to\infty}\frac{\mathcal{P}_2(u)}{u^\beta\mathbb{P}\{T > u^{-\gamma}N(u)\}} \le (1+\epsilon)\left[\int_{A_0}^{A_\infty} e^{-x}x^{-\alpha}dx + \left(1-e^{-A_0}\right)A_0^{-\alpha}\right].$$

Asymptotics of $\mathcal{P}_3(u)$. We have the following upper bound, as $u\to\infty$,

$$\mathcal{P}_3(u) \le \mathbb{P}\{u^\gamma S_{u^\beta} > A_\infty N(u)\} \sim u^\beta\mathbb{P}\{T > A_\infty u^{-\gamma}N(u)\} \sim A_\infty^{-\alpha}u^\beta\mathbb{P}\{T > u^{-\gamma}N(u)\}.$$

Passing $A_0\to 0, A_\infty\to\infty$ and $\epsilon\to 0$, we obtain, as $u\to\infty$,

$$\mathcal{P}_2(u) \sim \Gamma(1-\alpha)\,u^\beta\mathbb{P}\{T > u^{-\gamma}N(u)\},\ \mathcal{P}_1(u) = o(\mathcal{P}_2(u)) \text{ and } \mathcal{P}_3(u) = o(\mathcal{P}_2(u)).$$

This completes the proof. □

Proof of Theorem 3.2.7 Let $S_{u^\beta} = \sum_{k=1}^{u^\beta} T_k$ and $0 < A_0 < A_\infty$. We have

$$p(u^\gamma S_{u^\beta}; u) \ge p(A_\infty N(u); u)\mathbb{P}\{u^\gamma S_{u^\beta} > A_\infty N(u)\} =: p(A_\infty N(u); u)\mathcal{P}_1(u)$$

and

$$p(u^\gamma S_{u^\beta}; u) \le \int_0^{A_0 N(u)} p(x; u)dF_{u^\gamma S_{u^\beta}}(x) + \mathbb{P}\{u^\gamma S_{u^\beta} > A_0 N(u)\} =: \mathcal{P}_2(u) + \mathcal{P}_3(u).$$

Asymptotics of $p(A_\infty N(u);u)$. Lemma 3.3.2 implies, as $u\to\infty$,

$$p(A_\infty N(u);u)\sim 1-e^{-A_\infty}.$$

Asymptotics of $\mathcal{P}_1(u)$ and $\mathcal{P}_3(u)$. Lemma 3.3.1 for $t(u)$ satisfying Assumption **D4** gives, as $u\to\infty$

$$\begin{aligned}\mathcal{P}_1(u) &\sim u^\beta\mathbb{P}\{T>A_\infty u^{-\gamma}N(u)\}\sim u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\},\\ \mathcal{P}_3(u) &\sim u^\beta\mathbb{P}\{T>A_0 u^{-\gamma}N(u)\}\sim u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\}.\end{aligned}$$

Asymptotics of $\mathcal{P}_2(u)$. By the stationarity of process $\{Q(t):t\geq 0\}$ we have

$$\begin{aligned}\mathcal{P}_2(u) &:= \int_0^{A_0N(u)}\mathbb{P}\{\sup_{t\in[0,x]}(Q(t))>u\}dF_{u^\gamma S_{u^\beta}}(x)\\ &\leq \mathbb{P}\{\sup_{t\in[0,1]}(Q(t))>u\}\left[\int_0^{A_0N(u)}xdF_{u^\gamma S_{u^\beta}}(x)+1\right]\\ &= \mathbb{P}\{\sup_{t\in[0,1]}(Q(t))>u\}\left[\int_0^{A_0N(u)}\mathbb{P}\{u^\gamma S_{u^\beta}>x\}dx-A_0N(u)\mathbb{P}\{u^\gamma S_{u^\beta}>A_0N(u)\}+1\right]\\ &\leq \mathbb{P}\{\sup_{t\in[0,1]}(Q(t))>u\}\left[\int_0^{A_0N(u)}\mathbb{P}\{u^\gamma S_{u^\beta}>x\}dx+1\right].\end{aligned}$$

Further, for $t(u)$ following from Lemma 3.3.1 and the corresponding $C>0$, we have

$$\begin{aligned}&\int_0^{A_0N(u)}\mathbb{P}\{u^\gamma S_{u^\beta}>x\}dx=\int_0^{t(u)}\mathbb{P}\{u^\gamma S_{u^\beta}>x\}dx+\int_{t(u)}^{A_0N(u)}\mathbb{P}\{u^\gamma S_{u^\beta}>x\}dx\\ &\leq t(u)+C\int_{t(u)}^{A_0N(u)}u^\beta\mathbb{P}\{T>xu^{-\gamma}\}dx\leq t(u)+C\int_0^{A_0N(u)}u^\beta\mathbb{P}\{T>xu^{-\gamma}\}dx\\ &=t(u)+Cu^{\beta+\gamma}\int_0^{A_0u^{-\gamma}N(u)}\mathbb{P}\{T>x\}dx.\end{aligned}$$

By Karamata's theorem (see, e.g., Proposition 1.5.8 in [16]) we obtain, as $u\to\infty$,

$$\int_0^{A_0u^{-\gamma}N(u)}\mathbb{P}\{T>x\}dx\sim A_0u^{-\gamma}N(u)\mathbb{P}\{T>A_0u^{-\gamma}N(u)\}.$$

Proposition 3.1.1 implies the following asymptotical upper bound for $\mathcal{P}_2(u)$, as $u\to\infty$,

$$\mathcal{P}_2(u)\leq A_0u^\beta\mathbb{P}\{T>A_0u^{-\gamma}N(u)\}(1+o(1))\sim A_0u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\}.$$

Hence, as $u\to\infty$,

$$(1-e^{-A_\infty})u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\}\leq\mathbb{P}\{\sup_{t\in[0,u^\gamma S_{u^\beta}]}(Q(t))>u\}\leq(1+A_0)\,u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\}(1+o(1)).$$

Passing $A_0\to 0$ and $A_\infty\to\infty$ we obtain that, as $u\to\infty$,

$$\mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\sim u^\beta\mathbb{P}\{T>u^{-\gamma}N(u)\}.$$

This completes the proof. □

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3.3.4 Auxiliary lemmas

The following auxiliary lemmas play an important role in the analysis of the asymptotics of (3.4) when T satisfies **D2** or **D3**. In this section, we use the notation introduced above.

The result below provides a uniform comparison of the asymptotic behavior of the tail distribution functions of the random variables $\sum_{k=1}^{u^\beta} T_k$ and T.

Lemma 3.3.1 *Suppose that T satisfies **D2** or **D3**–**D4**. Then*

$$\lim_{u\to\infty} \sup_{s\geq t(u)} \left| \frac{\mathbb{P}\{\sum_{k=1}^{u^\beta} T_k > s\}}{u^\beta \mathbb{P}\{T > s\}} - 1 \right| = 0,$$

*for some function $t(u)$ (in the case **D3**, the existence of $t(u)$ is guaranteed in **D4**).*

As demonstrated below, the asymptotics of (3.4) for rapidly increasing intervals converges to the tail distribution function of an exponential random variable.

Lemma 3.3.2 *Let $0 < A_0 < A_\infty < \infty$. Then*

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{p(xN(u);u)}{1-e^{-x}} - 1 \right| = 0.$$

Proof of Lemma 3.3.1 Observe that for the case $\beta > 0$ the thesis follows from Theorem 3.3 in [26], while for the case $\beta = 0$ holds directly.

Note that □

Proof of Lemma 3.3.2 Let $0 < A_0 < A_\infty < \infty$. Following [118], we have

$$p(xN(u);u) = \mathbb{P}\{ \sup_{s\in[0,x\frac{N(u)}{u}],\tau\geq 0} (Z_u(s,\tau)) > u^{1-H} \},$$

where

$$Z_u(s,\tau) := \frac{B_H(u(s+\tau)) - B_H(su)}{\tau^H u^H \nu(\tau)},$$

with $\nu(u) = \tau^{-H} + c\tau^{1-H}$. Note that the distribution of the field $\{Z_u(s,\tau) : s,\tau \geq 0\}$ does not depend on u. For the sake brevity, write $Z(s,\tau) = Z_u(s,\tau)$. Furthermore, the field $Z(s,\tau)$ is stationary in s, but not in τ.

Let us summarize the strategy of the proof, that consists of the following six steps that require their own proofs.

<u>Step 1.</u> Let $\tau_0 = \frac{H}{c(1-H)}$ and $\tau^*(u) = \frac{\log(Au^{1-H})}{Au^{1-H}}$.

We show that the domain $\{(s,\tau) : s \in [0, x\frac{N(u)}{u}], \tau \geq 0\}$ can be restricted to a narrow strip around the line (s, τ_0), and the rest of the area does not affect the asymptotics:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\mathbb{P}\{\sup\limits_{s\in[0,x\frac{N(u)}{u}],\tau\geq 0} (Z(s,\tau)) \leq u^{1-H}\}}{\mathbb{P}\{\sup\limits_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|\leq\tau^*(u)} (Z(s,\tau)) \leq u^{1-H}\}} - 1 \right| = 0. \tag{3.5}$$

<u>Step 2.</u> Let $\delta > 0$, $J(\tau_0) = \{\tau : |\tau - \tau_0| \leq \tau^*(u)\}$, $K(u) = \lfloor \frac{xN(u)}{n(u)u} \rfloor$, $n(u) = Au^{1-H}$.

We reduce the domain to the union of the separated intervals I_k:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\mathbb{P}\{\sup\limits_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|\leq\tau^*(u)} (Z(s,\tau)) \leq u^{1-H}\}}{\mathbb{P}\{\sup\limits_{(s,\tau)\in \bigcup\limits_{k\leq K(u)} I_k} (Z(s,\tau)) \leq u^{1-H}\}} - 1 \right| = 0, \tag{3.6}$$

where

$$I_k = \Big[(k-1)n(u), kn(u) - \delta\Big) \times J(\tau_0) \text{ and } I_k^* = \Big[kn(u) - \delta, kn(u)\Big) \times J(\tau_0) \text{ for } k = 1, \ldots, K(u)$$

and

$$I_{K(u)+1} = \left[K(u)n(u), x\frac{N(u)}{u}\right] \times J(\tau_0).$$

<u>Step 3.</u> Let $q = q(u) = du^{-\frac{1-H}{H}}$, $d = d(u) = 1/\log\left(\log\left(N(u)\right)\right)$. We define the grid points

$$s_{k,l} = (k-1)n(u) + lq \text{ and } \tau_j = \tau_0 + jq$$

for $j \in \mathbb{Z}, l \geq 0, k \geq 1$. Let

$$\mathcal{R} = \{(s_{k,l}, \tau_j) : j \in \mathbb{Z}, l \geq 0, k \geq 1\}.$$

We approximate the supremum of the process $Z(s,\tau)$ on $\bigcup\limits_{k\leq K(u)} I_k$ by the maximum on the grid points:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\mathbb{P}\{\sup\limits_{(s,\tau)\in \bigcup\limits_{k\leq K(u)} I_k} (Z(s,\tau)) \leq u^{1-H}\}}{\mathbb{P}\{\sup\limits_{(s,\tau)\in \bigcup\limits_{k\leq K(u)} I_k\cap\mathcal{R}} (Z(s,\tau)) \leq u^{1-H}\}} - 1 \right| = 0. \tag{3.7}$$

Step 4. We show that the maxima of $Z(s,\tau)$ on the sets $\{I_k \cap \mathcal{R} : k \le K(u)\}$ are asymptotically independent:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\mathbb{P}\{\sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\}}{\prod_{k=1}^{K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\}} - 1 \right| = 0. \tag{3.8}$$

Step 5. As in Step 3, we prove that the suprema of $Z(s,\tau)$ on the set I_k are asymptotically equivalent to the maxima of $Z(s,\tau)$ on $I_k \cap \mathcal{R}$:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\prod_{k=1}^{K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\}}{\prod_{k=1}^{K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le u^{1-H}\}} - 1 \right| = 0. \tag{3.9}$$

Step 6. We show that the product of the cumulative distribution functions of the supremum of $Z(s,\tau)$ on I_k's asymptotically behaves as a constant that depends on x:

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\prod_{k=1}^{K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le u^{1-H}\}}{\exp(-x)} - 1 \right| = 0. \tag{3.10}$$

With all the above 6 steps proven, we can conclude that

$$\lim_{u\to\infty} \sup_{x\in[A_0,A_\infty]} \left| \frac{\mathbb{P}\{\sup_{t\in[0,xN(u)]} (Q(t)) > u\}}{1-\exp(-x)} - 1 \right| = 0.$$

Therefore, observe that if $f(u) = o(1)$, then $f(u) = o\left(\mathbb{P}\{\sup_{t\in[0,xN(u)]} (Q(t)) > u\}\right)$, as $u \to \infty$.

Proof of Step 1. Observe that

$$\begin{aligned}
\mathcal{P}_1(u) &:= \mathbb{P}\{\sup_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|\le\tau^*(u)} (Z(s,\tau)) > u^{1-H}\} \\
&\le \mathbb{P}\{\sup_{s\in[0,x\frac{N(u)}{u}],\tau\ge 0} (Z(s,\tau)) > u^{1-H}\} \\
&\le \mathbb{P}\{\sup_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|\le\tau^*(u)} (Z(s,\tau)) > u^{1-H}\} + \mathbb{P}\{\sup_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|>\tau^*(u)} (Z(s,\tau)) > u^{1-H}\} \\
&=: \mathcal{P}_1(u) + \mathcal{P}_2(u).
\end{aligned}$$

It is sufficient to show that $\mathcal{P}_2(u) = o\left(\mathcal{P}_1(u)\right)$, as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$. Lemma 2 in [118] implies that there exists a constant $C > 0$ such that for any u

$$\mathcal{P}_2(u) \le Cx\frac{N(u)}{u}\left(Au^{1-H}\right)^{\frac{2}{H}} e^{-\frac{1}{2}A^2u^{2(1-H)}-\frac{B}{2A}\log^2\left(Au^{1-H}\right)} \le CA_\infty u^{2(1-H)} e^{-\frac{B}{2A}\log^2\left(Au^{1-H}\right)}.$$

Hence we obtain that, as $u \to \infty$,

$$\mathcal{P}_2(u) = O\left(\frac{A_\infty u^{2(1-H)}}{e^{\frac{B}{2A}\log^2(Au^{1-H})}}\right) = o(1),$$

uniformly in $x \in [A_0, A_\infty]$.

Proof of Step 2. Note that

$$\begin{aligned}
I_1(u) &:= \mathbb{P}\{\sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k} (AZ(s,\tau)) > Au^{1-H}\} \\
&\le \mathbb{P}\{\sup_{s\in[0,x\frac{N(u)}{u}],|\tau-\tau_0|\le\tau^*(u)} (Z(s,\tau)) > u^{1-H}\} \\
&\le \mathbb{P}\{\sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k} (AZ(s,\tau)) > Au^{1-H}\} + \mathbb{P}\{\sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k^*} (AZ(s,\tau)) > Au^{1-H}\} \\
&+ \mathbb{P}\{\sup_{(s,\tau)\in I_{K(u)+1}} (AZ(s,\tau)) > Au^{1-H}\} =: I_1(u) + I_2(u) + I_3(u).
\end{aligned}$$

It is sufficient to show that $I_2(u)$, $I_3(u)$ are asymptotically negligible comparing to $I_1(u)$, as $u \to \infty$, uniformly for $x \in [A_0, A_\infty]$.

Applying Corrolary 2 in [79] we obtain asymptotic upper bounds, as $u \to \infty$,

$$\begin{aligned}
I_2(u) &\le \sum_{k\le K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k^*} (AZ(s,\tau)) > Au^{1-H}\} = K(u)\mathbb{P}\{\sup_{(s,\tau)\in[0,\delta)\times J(\tau_0)} (AZ(s,\tau)) > Au^{1-H}\} \\
&\le K(u)C\delta u^{2\frac{(1-H)^2}{H}} e^{-\frac{1}{2}A^2u^{2(1-H)}} = C\frac{xN(u)}{n(u)u}\delta u^{2\frac{(1-H)^2}{H}} e^{-\frac{1}{2}A^2u^{2(1-H)}} \\
&\le \delta\tilde{C}\frac{A_\infty}{n(u)} = O\left(\frac{A_\infty\delta}{n(u)}\right) = o(1)
\end{aligned}$$

and

$$\begin{aligned}
I_3(u) &\le \left|\left[K(u)n(u), x\frac{N(u)}{u}\right]\right| C\left(Au^{1-H}\right)^{\frac{2}{H}-1}\Psi\left(Au^{1-H}\right) \\
&\le n(u)C\left(Au^{1-H}\right)^{\frac{2}{H}-1}\Psi\left(Au^{1-H}\right) = O\left(\frac{n(u)u}{N(u)}\right) = o(1)
\end{aligned}$$

uniformly for $x \in [A_0, A_\infty]$.

Proof of Step 3. Let $w = u^{1-H}$ and $\gamma = d^H$. Then

$$\begin{aligned}
0 &\le \mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le w\} - \mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le w\} \\
&\le \mathbb{P}\{w-\gamma/w < \sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le w\} + \mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le w-\gamma/w, \sup_{(s,\tau)\in I_k} (Z(s,\tau)) > w\} \\
&=: R_1(u) + R_2(u).
\end{aligned}$$

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Let $A = [0,1] \times J(\tau_0)$ and recall that $\varphi_1(\cdot)$ denotes the density function of a standard normal random variable. By the stationarity of $Z(s,\tau)$ with respect to s and Lemma 3 in [57] we have

$$\begin{aligned}
R_1(u) &\le \left|\left[(k-1)n(u), kn(u)-\delta\right)\right| \left(\mathbb{P}\{ \sup_{(s,\tau)\in A\cap\mathcal{R}} (Z(s,\tau)) > w - \gamma/w\} - \mathbb{P}\{ \sup_{(s,\tau)\in A\cap\mathcal{R}} (Z(s,\tau)) > w\} \right) \\
&\le Cn(u) \left((w-\gamma/w)^{\frac{2(1-H)}{H}} \varphi_1(A(w-\gamma/w)) - w^{\frac{2(1-H)}{H}} \varphi_1(Aw) \right) \\
&\le \tilde{C} n(u) w^{\frac{2(1-H)}{H}} \left(e^{-\frac{1}{2}A^2(w-\gamma/w)^2} - e^{-\frac{1}{2}A^2 w^2} \right) = \tilde{C} A^2 n(u) w^{\frac{2(1-H)}{H}} e^{-\frac{1}{2}A^2w^2} \gamma \\
&= \tilde{C} A^2 d^H n(u) u^{\frac{2(1-H)^2}{H}} e^{-\frac{1}{2}A^2 u^{2(1-H)}}.
\end{aligned}$$

Using the stationarity of $Z(s,\tau)$ with respect to s and Lemma 2 in [57] we obtain that there exist constants $K_1, K_2, u_0 > 0$ such that for any $u \ge u_0$

$$\begin{aligned}
R_2(u) &\le \left|\left[(k-1)n(u), kn(u)-\delta\right)\right| K_1 u^{\frac{2(1-H)^2}{H}} e^{-\frac{1}{2}A^2u^{2(1-H)}} A^2 \gamma e^{-(A^2\gamma)^{-2}/K_2} \\
&\le K_1 A^{\frac{2}{H}} d^H n(u) u^{\frac{2(1-H)^2}{H}} e^{-\frac{1}{2}A^2u^{2(1-H)}}.
\end{aligned}$$

Hence, as $u \to \infty$,

$$\mathbb{P}\{ \sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le w\} - \mathbb{P}\{ \sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le w\} = O\left(2d^H n(u) \varphi_1(Au^{1-H}) u^{\frac{2(1-H)^2}{H}} \right) \quad (3.11)$$

and

$$\begin{aligned}
0 &\le \mathbb{P}\{ \sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\} - \mathbb{P}\{ \sup_{(s,\tau)\in \bigcup_{k\le K(u)} I_k} (Z(s,\tau)) \le u^{1-H}\} \\
&\le \sum_{k\le K(u)} \left(\mathbb{P}\{ \sup_{(s,\tau)\in I_k\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\} - \mathbb{P}\{ \sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le u^{1-H}\} \right) \\
&= K(u) \left(\mathbb{P}\{ \sup_{(s,\tau)\in I_1\cap\mathcal{R}} (Z(s,\tau)) \le u^{1-H}\} - \mathbb{P}\{ \sup_{(s,\tau)\in I_1} (Z(s,\tau)) \le u^{1-H}\} \right) \\
&\le O\left(K(u) 2d^H n(u) \varphi_1(Au^{1-H}) u^{\frac{2(1-H)^2}{H}} \right) = O\left(2xd^H N(u) \varphi_1(Au^{1-H}) u^{\frac{2(1-H)^2}{H}-1} \right) \\
&\le O\left(2A_\infty d^H N(u) \varphi_1(Au^{1-H}) u^h \right) = O(A_\infty d^H) = o(1)
\end{aligned}$$

uniformly in $x \in [A_0, A_\infty]$, as $u \to \infty$. Thus (3.7) holds.

Proof of Step 4. Let $r(s,\tau,s',\tau')$ denote correlation function of $Z(s,\tau)$

$$r(s,\tau,s',\tau') = \mathbb{E}\{Z(s,\tau)Z(s',\tau')\nu(\tau)\nu(\tau')\}.$$

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Applying Berman's comparison lemma (see, e.g. [103]) we get that

$$
\begin{aligned}
&\mathbb{P}\{\sup_{(s,\tau)\in\bigcup\limits_{k\le K(u)} I_k\cap\mathcal{R}}(Z(s,\tau))\le u^{1-H}\}-\prod_{k=1}^{K(u)}\mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}}(Z(s,\tau))\le u^{1-H}\}\\
&= \mathbb{P}\{\forall_{(s,\tau)\in\bigcup\limits_{k\le K(u)} I_k\cap\mathcal{R}}: Z(s,\tau)\nu(\tau)\le u^{1-H}\nu(\tau)\}-\prod_{k=1}^{K(u)}\mathbb{P}\{\forall_{(s,\tau)\in I_k\cap\mathcal{R}}: Z(s,\tau)\nu(\tau)\le u^{1-H}\nu(\tau)\}\\
&\le \sum_{k\ne k'}\sum_{\substack{(s_{k,l},\tau_j)\in I_k\cap\mathcal{R}\\(s_{k',l'},\tau_j')\in I_{k'}\cap\mathcal{R}}}|r(s_{k,l},\tau_j,s_{k',l'},\tau_j')|\exp\left(-\frac{(\nu^2(\tau_j)+\nu^2(\tau_j'))u^{2(1-H)}}{1+r(s_{k,l},\tau_j,s_{k',l'},\tau_j')}\right)\\
&\le \sum_{k\ne k'}\sum_{\substack{(s_{k,l},\tau_j)\in I_k\cap\mathcal{R}\\(s_{k',l'},\tau_j')\in I_{k'}\cap\mathcal{R}}}|r(s_{k,l},\tau_j,s_{k',l'},\tau_j')|\exp\left(-\frac{\nu^2(\tau_0)u^{2(1-H)}}{1+r(s_{k,l},\tau_j,s_{k',l'},\tau_j')}\right)=:\widetilde{S}_N.
\end{aligned}
$$

Next, we shall extend Lemma 8 in [79] to uniform convergence, i.e. we shall show that

$$\widetilde{S}_N=o(1),$$

as $u\to\infty$, uniformly for $x\in[A_0,A_\infty]$.

Let $\beta=\frac{1-\rho}{1+\rho}$ and $\tilde{N}=xN(u)/u$. We divide the sum into two partial sums $\widetilde{S}_{N,1}$ and $\widetilde{S}_{N,2}$ with $|s_{k,l}-s_{k',l'}|<\tilde{N}^\beta$ and $|s_{k,l}-s_{k',l'}|\ge\tilde{N}^\beta$, respectively.

Sum $\widetilde{S}_{N,1}$. Note that the sum $\widetilde{S}_{N,1}$ consists of $\tilde{N}^{1+\beta}(2\tau^*(u))^2/q^4$ pairs of points

$$(s_{k,l},\tau_j),(s_{k',l'},\tau_j')\in\bigcup_{k\le K(u)} I_k.$$

Furthermore, it holds

$$r(s_{k,l},\tau_0,s_{k',l'},\tau_0)\le\rho<1 \text{ for } |s_{k,l}-s_{k',l'}|>\delta>0.$$

Thus, we have the following upper bound for $\widetilde{S}_{N,1}$.

$$
\begin{aligned}
\widetilde{S}_{N,1} &\le \rho\frac{\tilde{N}^{1+\beta}(2\tau^*(u))^2}{q^4}\exp\left(-\frac{A^2u^{2(1-H)}}{1+\rho}\right)\\
&\le 4\rho e^{(1+\beta)\log(x)}\exp\left((1+\beta)\log(N(u)/u)+2\log(\tau^*(u)/q^2)-\frac{2(1+o(1))\log(N(u))}{1+\rho}\right)\\
&\le 4\rho A_\infty^{1+\beta}\exp\left(-\log(N(u))\left(\frac{2(1+o(1))}{1+\rho}-(1+\beta)\left(1-\frac{\log(u)}{\log(N(u))}\right)-2\frac{\log(\tau^*(u)/q^2)}{\log(N(u))}\right)\right)\\
&= O\left(A_\infty^{1+\beta}\exp\left(-\left(\frac{2}{1+\rho}-(1+\beta)\right)\log(N(u))\right)\right)=o(1)
\end{aligned}
$$

uniformly in $x\in[A_0,A_\infty]$, since $\log(\tau^*(u)/q^2)=o(\log(N(u)))$, $\log(u)=O(\log(\log(N(u))))=o(\log(N(u)))$ and $1+\beta<\frac{2}{1+\rho}$.

Sum $\widetilde{S}_{N,2}$. Observe that the sum $\widetilde{S}_{N,2}$ has $(2\tilde{N}\tau^*(u))^2/q^4$ components. Further, using properties of $r(s,\tau,s',\tau')$ we obtain

$$\sup_{|s_{k,l}-s_{k',l'}|\geq \tilde{N}^{\beta}} |r(s_{k,l},\tau_0,s_{k',l'},\tau_0)| \leq C\tilde{N}^{\beta\lambda}$$

with $\lambda = 2(H-1) < 0$. Thus, we have the following upper bound for $\widetilde{S}_{N,2}$:

$$\begin{aligned}
\widetilde{S}_{N,2} &\leq C\tilde{N}^{\beta\lambda}\frac{(2\tilde{N}\tau^*(u))^2}{q^4}\exp\left(-\frac{A^2u^{2(1-H)}}{1+C\tilde{N}^{\beta\lambda}}\right)\\
&\leq 4C\exp\left(\beta\lambda\log(\tilde{N}) + 2\log(\tilde{N}) + 2\log(\tau^*(u)/q^2) - \frac{2(1+o(1))\log(N(u))}{1+C\tilde{N}^{\beta\lambda}}\right)\\
&\leq 4C\exp\left(\log(\tilde{N})\,(\beta\lambda+o(1))\right)\\
&\leq 4CA_\infty^{\beta\lambda}\exp\left(\log(N(u)/u)\,(\beta\lambda+o(1))\right)\\
&= O\left(A_\infty^{\beta\lambda}\,(N(u)/u)^{\beta\lambda}\right) = o(1)
\end{aligned}$$

uniformly in $x\in[A_0,A_\infty]$, since $\lambda<0$. If $H=\frac{1}{2}$, the sum $\widetilde{S}_{N,2}=0$.

Hence we obtain that, as $u\to\infty$,

$$\widetilde{S}_N = o(1),$$

uniformly for $x\in[A_0,A_\infty]$.

Thus the equation (3.8) holds.

Proof of Step 5. Using (3.11) and that for $a_k, b_k \in (0,1), k=1,\dots,N$ it holds

$$|\prod_{k\leq N} a_k - \prod_{k\leq N} b_k| \leq \sum_{k\leq N}|a_k-b_k|,$$

we obtain (3.9) as follows:

$$\begin{aligned}
0 &\leq \prod_{k\leq K(u)}\mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}}(Z(s,\tau))\leq u^{1-H}\} - \prod_{k\leq K(u)}\mathbb{P}\{\sup_{(s,\tau)\in I_k}(Z(s,\tau))\leq u^{1-H}\}\\
&\leq \sum_{k\leq K(u)}\left(\mathbb{P}\{\sup_{(s,\tau)\in I_k\cap\mathcal{R}}(Z(s,\tau))\leq u^{1-H}\} - \mathbb{P}\{\sup_{(s,\tau)\in I_k}(Z(s,\tau))\leq u^{1-H}\}\right)\\
&= K(u)\left(\mathbb{P}\{\sup_{(s,\tau)\in I_1\cap\mathcal{R}}(Z(s,\tau))\leq u^{1-H}\} - \mathbb{P}\{\sup_{(s,\tau)\in I_1}(Z(s,\tau))\leq u^{1-H}\}\right)\\
&= O\left(A_\infty d^H\right) = o(1),
\end{aligned}$$

as $u \to \infty$, uniformly in $x \in [A_0, A_\infty]$.

Proof of Step 6. Applying Corollary 2 in [79], we have as $u \to \infty$,

$$
\begin{aligned}
& K(u)\mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) > u^{1-H}\} \\
& \sim K(u) c_2 n(u) u^{h+1} \exp\left(-\frac{1}{2}A^2 u^{2(1-H)}\right) = \lfloor \frac{xN(u)}{n(u)u} \rfloor c_2 n(u) u^{h+1} \exp\left(-\frac{1}{2}A^2 u^{2(1-H)}\right) \\
& = \lfloor \frac{x\frac{1}{c_2}u^{-h}e^{\frac{1}{2}A^2u^{2(1-H)}}}{n(u)u} \rfloor c_2 n(u) u^{h+1} \exp\left(-\frac{1}{2}A^2 u^{2(1-H)}\right) \sim x
\end{aligned}
$$

uniformly for $x \in [A_0, A_\infty]$. The above implies (3.10) as follows:

$$
\begin{aligned}
& \prod_{k=1}^{K(u)} \mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) \le u^{1-H}\} \sim \left(1 - \mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) > u^{1-H}\}\right)^{K(u)} \\
& \sim \exp\left(-K(u)\mathbb{P}\{\sup_{(s,\tau)\in I_k} (Z(s,\tau)) > u^{1-H}\}\right) \sim \exp(-x)
\end{aligned}
$$

uniformly for $x \in [A_0, A_\infty]$, which completes the proof. □

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Chapter 4

Tail asymptotics for functionals of stationary Lévy queues

4.1 Introduction

Consider an infinite-buffer single-node fluid queue with a Lévy input $\{X(t) : t \in \mathbb{R}\}$ and a constant service rate $c > 0$. As in Chapter 2, we study the stationary buffer content process $\{Q(t) : t \geq 0\}$ defined as the stationary solution to the *Skorokhod problem* given in [124, 125] (see also [49][Chapter 2]). We recall that a pair (Q, L) with $Q \equiv \{Q(t) : t \geq 0\}$, $L \equiv \{L(t) : t \geq 0\}$ is a solution to the Skorokhod problem for the process $\{X(t) : t \in \mathbb{R}\}$ and drift c if the following conditions hold:

S0: The process $\{L(t) : t \geq 0\}$ is non-decreasing, right-continuous and $L(0) = 0$,

S1: The workload process $\{Q(t) : t \geq 0\}$, defined through

$$Q(0) := x \text{ and } Q(t) := Q(0) + (X(t) - ct) + L(t)$$

is non-negative for all $t \geq 0$,

S2: $L(t)$ can only increase when $Q(t) = 0$, that is,

$$\int_0^T Q(t)dL(t) = 0, \text{ for all } T > 0.$$

Additionally, we suppose that $\mathbb{E}\{X(1)\} < c$, which guarantees

$$\lim_{t\to\infty}(X(t) - ct) = -\infty \text{ a.s.}$$

and the existence of the stationary solution to the *Skorokhod problem* (**S0**–**S2**).

Following [49][Section 2.4], the stationary buffer content process $\{Q(t) : t \geq 0\}$ defined as the stationary solution to the *Skorokhod problem* (**S0**–**S2**) has the following representation:

$$Q(t) = \sup_{-\infty < s \leq t} \left(X(t) - X(s) - c(t-s)\right).$$

In recent years, a wide range of asymptotic results for stationary queues driven by Lévy inputs have been established in the literature, including logarithmic and exact-tail estimates in the light-tailed, intermediate and heavy-tailed regimes [5, 9, 18, 49, 84, 95].

In the following, we focus on the light-tailed scenario, that is the case where the *Cramér condition* is satisfied; see e.g. [28]. Specifically, following e.g. [9, 15, 28], we assume that:

A1: (*Cramér condition*). There exists $\omega > 0$ such that

$$\mathbb{E}\{e^{\omega(X(1)-c)}\} = 1 \text{ and } \mathbb{E}\{e^{\omega(X(1)-c)}|X(1)-c|\} \in (0, \infty),$$

A2: $\{X(t) - ct : t \geq 0\}$ has non-monotone paths and either 0 is regular for $\mathbb{R}_+ \setminus \{0\}$ or the Lévy measure of $\{X(t) - ct : t \geq 0\}$ is non-lattice.

We recall that 0 is regular for a Lévy process $\{Y(t) : t \geq 0\}$ if $\mathbb{P}\{\inf\{t > 0 : Y(t) > 0\} = 0\} = 1$, and the Lévy measure of $\{Y(t) : t \geq 0\}$ is non-lattice if its support is not contained in any set $\{a + bk : k \in \mathbb{Z}\}$ with $a \in \mathbb{R}$, $b > 0$.

It is known that, under **A1**–**A2**,

$$\mathbb{E}\{e^{\omega(X(T)-cT))}|X(T) - cT|\} < \infty,$$

and, as $u \to \infty$,

$$\mathbb{P}\{Q(0) > u\} \sim \frac{1}{\omega k(0,0)} \frac{1}{l'(0, -\omega)} e^{-\omega u} =: C_\omega e^{-\omega u}, \tag{4.1}$$

with

$$k(\vartheta, \alpha) := \exp\left(-\int_{(0,\infty)} \int_{(0,\infty)} \frac{1}{t}\left(e^{-t} - e^{-\vartheta t - \alpha x}\right) \mathbb{P}\{X(t) \in dx\} dt\right) \text{ and } l(0, \beta - \omega) := \frac{1}{k(0, \beta - \omega)};$$

see, e.g., [49][Theorem 8.2].

In this chapter, we study the tail probability

$$p^{\Theta}(E; u) := \mathbb{P}\{\Theta(\{Q(t) : t \in E\}) > u\},$$

where $\Theta : D(E) \to \mathbb{R}$ is a functional defined on the space $D(E)$ of the real-valued càdlàg functions on a compact set $E \subset [0, \infty)$, satisfying:

F1: $\Theta(f) \le \sup_{t \in E}(f(t))$ for any $f \in D(E)$,

F2: $\Theta(af + b) = a\Theta(f) + b$ for any $f \in D(E)$ and $a > 0$, $b \in \mathbb{R}$.

Note that **F1**–**F2** cover the following important examples:

$$\Theta = \sup, \ \inf, \ \lambda \sup + (1-\lambda) \inf \ \text{ with } \lambda \in [0,1].$$

We shall analyze the asymptotic behavior of $p^{\Theta}(E; u)$ with $E = [0, \mathcal{T}_u]$, as $u \to \infty$, for two regimes depending on the form of the time horizon $\mathcal{T}_u$:

(a) *Fixed-interval case.* If $\mathcal{T}_u \equiv T > 0$, we analyze functionals Θ satisfying **F1**–**F2**,

(b) *Growing-interval case.* If $\mathcal{T}_u = n(u) \to \infty$, as $u \to \infty$, we focus only on the supremum functional.

Note that the functional Θ and the probability $p^{\Theta}(E; u)$ are both well-defined in the above cases. For notational convenience, we introduce the shorthand notation

$$p^{\Theta}(\mathcal{T}_u; u) \equiv p^{\Theta}([0, \mathcal{T}_u]; u).$$

4.2 Main results

In the following we shall suppose that the Lévy process $\{X(t) : t \in \mathbb{R}\}$ satisfies **A1**–**A2**. For notational convenience, we introduce $X_c(t) := X(t) - ct$. Furthermore, for a stochastic process $\{Y(t) : t \ge 0\}$ and a functional Θ satisfying **F1**, we define the *generalized Pickands constant*

$$\mathcal{H}_Y^{\Theta} = \lim_{T \to \infty} \frac{1}{T} \mathcal{H}_Y^{\Theta}[0, T] \text{ with } \mathcal{H}_Y^{\Theta}[0, T] = \mathbb{E}\{e^{\Theta(\{Y(t): t \in [0,T]\})}\}, \tag{4.2}$$

provided that the above limit exists, is positive and finite.

Theorem 4.2.1 (Fixed-interval case) *Suppose that the Lévy process $\{X(t) : t \in \mathbb{R}\}$ satisfies **A1**–**A2** and the functional Θ defined on $D[0, T]$, with $T > 0$, satisfies **F1**–**F2**. Then, as $u \to \infty$,*

$$p^{\Theta}(T; u) \sim \mathcal{H}_{\omega X_c}^{\Theta}[0, T] \, \mathbb{P}\{Q(0) > u\}.$$

Note that the combination of Theorem 4.2.1 with (4.1) gives that, as $u \to \infty$,

$$p^{\Theta}(T;u) \sim \mathcal{H}^{\Theta}_{\omega X_c}[0,T]\, C_\omega e^{-\omega u},$$

where $C_\omega > 0$ is given in (4.1).

Theorem 4.2.1 generalizes Theorem 3.3(i) in Baurdoux et al. [9], where the special case $\Theta \equiv \sup$ was considered. We note that the proof of Theorem 4.2.1 is based on a different technique than used in [9] and allows one to interpret the asymptotics of $p^{\Theta}(T;u)$ as the product of the expected value and the tail distribution function of two independent random variables.

Before proceeding to the proof of Theorem 4.2.1, which is presented in Section 4.3.1, we provide a heuristic explanation of the above asymptotics. Recall that, according to **S1**, the stationary buffer content process satisfies

$$Q(t) = Q(0) + (X(t) - ct) + L(t), \tag{4.3}$$

where $L(t) = \max\left\{-\inf_{0 \le s \le t}(X(s) - cs) - Q(0), 0\right\}$; see e.g. [49][Exercise 2.9]. In this context, $\{L(t) : t \ge 0\}$ represents the cumulative reflection required to ensure that $\{Q(t) : t \ge 0\}$ remains non-negative. Specifically, according to **S2**, $\{L(t) : t \ge 0\}$ only increases if the trajectory of the process $\{Q(t) : t \ge 0\}$ would fall below zero. Given that $Q(0) = \sup_{-\infty < s \le 0}(-(X(s) - cs))$, it follows that $Q(0)$ and $\{X(t) - ct : t \ge 0\}$ are independent.

By showing that $L(t)$ is asymptotically negligible compared to $Q(0)$, as $u \to \infty$, we shall deduce that the probability $p^{\Theta}(T;u)$ can be approximated by the product of two independent non-negative random variables

$$p^{\Theta}(T;u) \sim \mathbb{P}\{Q(0) + \Theta\left(\{X(t) - ct : t \in [0,T]\}\right) > u\} = \mathbb{P}\{e^{Q(0)} \cdot e^{\Theta(\{X(t)-ct : t \in [0,T]\})} > e^u\}.$$

Then, the observations that

$$\mathbb{P}\{e^{Q(0)} > u\} \sim C_\omega u^{-\omega}, \text{ as } u \to \infty, \text{ and } \mathcal{H}^{\Theta}_{\omega X_c}[0,T] = \mathbb{E}\{e^{\omega\,\Theta(\{X(t)-ct : t \in [0,T]\})}\} \in (0,\infty),$$

enable the application of Breiman's lemma (stated below) to determine the asymptotic behavior of $p^{\Theta}(T;u)$.

Lemma 4.2.2 (Breiman's lemma; [19]) *Let ξ and η be independent non-negative random variables such that $\mathbb{P}\{\xi > u\} \sim Cu^{-\alpha}$ as $u \to \infty$ for some $C > 0$ and $\mathbb{E}\{\eta^\alpha\} < \infty$. Then $\xi \cdot \eta$ is regularly varying with parameter α. Moreover, as $u \to \infty$,*

$$\mathbb{P}\{\xi \cdot \eta > u\} \sim \mathbb{E}\{\eta^\alpha\} \cdot \mathbb{P}\{\xi > u\}.$$

Next, let us proceed to the case when $\mathcal{T}_u = n(u) \to \infty$, as $u \to \infty$.

Theorem 4.2.3 (Growing-interval case) *Suppose that the Lévy process $\{X(t) : t \in \mathbb{R}\}$ satisfies **A1**–**A2** and $n(u)$ is a function such that $n(u) \to \infty$ and $n(u) = o(e^{\beta u})$, with $\beta \in (0, 1/2)$, as $u \to \infty$. Then, as $u \to \infty$,*

$$p^{\sup}(n(u); u) \sim \mathcal{H}^{\sup}_{\omega X_c} n(u)\mathbb{P}\{Q(0) > u\},$$

provided that the constant $\mathcal{H}^{\sup}_{\omega X_c}$ defined in (4.2) is positive and finite.

Theorem 4.2.3 provides an analogue of the results obtained by Piterbarg for stationary queues driven by fractional Brownian motions in [118][Theorem 7]; see also [46].
Note that, according to Remark 2 in [30] (see also Theorem 3),

$$\mathcal{H}^{\sup}_{\omega X_c} \in (0, \infty),$$

provided that there exists $\varepsilon > 0$ such that $\mathbb{E}\{e^{\theta|X(1)-c|}\} < \infty$ holds for all $\theta \in (-1-\varepsilon, 2+\varepsilon)$.

4.3 Proofs

4.3.1 Proofs of Theorems 4.2.1 and 4.2.3

Proof of Theorem 4.2.1 Let $\tau_0 := \inf\{t \in [0, T] : Q(t) = 0\}$. According to (4.3) and **F1**–**F2**, for any $u > 0$, we have that

$$\begin{aligned} p^\Theta(T; u) &\geq \mathbb{P}\{\Theta(\{Q(t) : t \in [0, T]\}) > u, \tau_0 > T\} \\ &= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0, T]\}) > u, \tau_0 > T\} \\ &= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0, T]\}) > u\} \\ &\quad -\mathbb{P}\{Q(0) + \Theta(\{(X(t) - ct) : t \in [0, T]\}) > u, \tau_0 \leq T\} \end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\} \\
&\quad -\mathbb{P}\{\Theta(\{Q(0) + (X(t) - ct) : t \in [0,T]\}) > u, \tau_0 \le T\} \\
&\ge \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\} \\
&\quad -\mathbb{P}\{\sup_{t\in[0,T]} (Q(0) + (X(t) - ct)) > u, \tau_0 \le T\} \\
&\ge \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\} - \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \tau_0 \le T\}
\end{aligned}
$$

and

$$
\begin{aligned}
p^{\Theta}(T;u) &= \mathbb{P}\{\Theta(\{Q(t) : t \in [0,T]\}) > u, \tau_0 > T\} + \mathbb{P}\{\Theta(\{Q(t) : t \in [0,T]\}) > u, \tau_0 \le T\} \\
&= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u, \tau_0 > T\} \\
&\quad +\mathbb{P}\{\Theta(\{Q(t) : t \in [0,T]\}) > u, \tau_0 \le T\} \\
&\le \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\} + \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \tau_0 \le T\}.
\end{aligned}
$$

Hence, for any $u > 0$, we obtain that

$$
\mathcal{P}_1(T;u) - \mathcal{P}_2(T;u) \le p^{\Theta}(T;u) \le \mathcal{P}_1(T;u) + \mathcal{P}_2(T;u),
$$

where

$$
\begin{aligned}
\mathcal{P}_1(T;u) &:= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\}, \\
\mathcal{P}_2(T;u) &:= \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \tau_0 \le T\}.
\end{aligned}
$$

The idea of the rest of the proof is to find the exact asymptotics of $\mathcal{P}_1(T;u)$ and to prove that $\mathcal{P}_2(T;u)$ is asymptotically negligible, as $u \to \infty$, in comparison to $\mathcal{P}_1(T;u)$.

Asymptotics of $\mathcal{P}_1(T;u)$. For $v = \exp(u)$, we have that

$$
\begin{aligned}
\mathcal{P}_1(T;u) &= \mathbb{P}\{Q(0) + \Theta(\{X(t) - ct : t \in [0,T]\}) > u\} \\
&= \mathbb{P}\{\exp(Q(0)) \cdot \exp(\Theta(\{X(t) - ct : t \in [0,T]\})) > v\},
\end{aligned}
$$

where $\exp(Q(0))$ and $\exp(\Theta(\{X(t) - ct : t \in [0,T]\}))$ are mutually independent.

We show that the assumptions of Lemma 4.2.2 are satisfied. Note that $\exp(Q(0)) \in \mathcal{RV}(\omega)$, as $u \to \infty$, since, for C_ω defined in (4.1), it holds

$$
\mathbb{P}\{\exp(Q(0)) > u\} = \mathbb{P}\{Q(0) > \log(u)\} \sim C_\omega e^{-\omega \log(u)} = C_\omega u^{-\omega}, \text{ as } u \to \infty.
$$

Let

$$C(x) := 1/\mathbb{P}\{\inf_{t\in[0,T]}(X(t)-ct) > -\frac{\log(x)}{\omega}\} \text{ and } M := \max\{\inf\{x>1 : C(x)>0\}, 2\}.$$

Next, we show that the ω–th moment of $\exp(\Theta(\{X(t)-ct : t\in[0,T]\}))$ is finite.

Using **F1**, for $X_c(t) = X(t)-ct$, we obtain that

$$\begin{aligned}
\mathcal{H}^{\Theta}_{\omega X_c}[0,T] &\le \mathcal{H}^{\sup}_{\omega X_c}[0,T] = \mathbb{E}\{\exp\left(\omega \sup_{t\in[0,T]}(X_c(t))\right)\} \\
&= \int_0^\infty \mathbb{P}\{\exp\left(\omega \sup_{t\in[0,T]}(X_c(t))\right) > x\}dx = \int_0^\infty \mathbb{P}\{\sup_{t\in[0,T]}(X_c(t)) > \frac{\log(x)}{\omega}\}dx \\
&\le M + \int_M^\infty \mathbb{P}\{\sup_{t\in[0,T]}(X_c(t)) > \frac{\log(x)}{\omega}\}dx.
\end{aligned}$$

By Lemma 9.1 in [49], we have that

$$\mathbb{P}\{\sup_{t\in[0,T]}(X_c(t)) > \frac{\log(x)}{\omega}\} \le \frac{\mathbb{P}\{X_c(T) > \frac{\log(x)}{\omega} - \frac{\log(M)}{\omega}\}}{\mathbb{P}\{\inf_{t\in[0,T]}(X_c(t)) > -\frac{\log(M)}{\omega}\}}.$$

Hence, it holds

$$\begin{aligned}
\mathcal{H}^{\Theta}_{\omega X_c}[0,T] &\le M + \int_M^\infty \frac{\mathbb{P}\{X_c(T) > \frac{\log(x)}{\omega} - \frac{\log(M)}{\omega}\}}{\mathbb{P}\{\inf_{t\in[0,T]}(X_c(t)) > -\frac{\log(M)}{\omega}\}}dx \\
&\le M + C(M)\int_0^\infty \mathbb{P}\{X_c(T) > \frac{\log(x/M)}{\omega}\}dx \\
&= M\left(1 + C(M)\int_0^\infty \mathbb{P}\{X_c(T) > \frac{\log(y)}{\omega}\}dy\right) \\
&= M\left(1 + C(M)\int_0^\infty \mathbb{P}\{\exp(\omega X_c(T)) > y\}dy\right) \\
&= M\left(1 + C(M)\mathbb{E}\{e^{\omega(X(T)-cT)}\}\right) \\
&= M(1 + C(M)) < \infty.
\end{aligned}$$

Applying Lemma 4.2.2 we have that, as $u\to\infty$,

$$\mathcal{P}_1(T;u) \sim \mathbb{E}\{\exp(\omega\,\Theta(\{X(t)-ct : t\in[0,T]\}))\}\mathbb{P}\{Q(0)>u\} = \mathcal{H}^{\Theta}_{\omega X_c}[0,T]\mathbb{P}\{Q(0)>u\}.$$

Asymptotics of $\mathcal{P}_2(T;u)$. Let $\tau_u := \inf\{t\in[0,T] : Q(t)>u\}$. Note that, for $u>0$,

$$\mathcal{P}_2(T;u) = \mathbb{P}\{\tau_u \le T, \tau_0 \le T\} = \mathbb{P}\{\tau_u \le \tau_0 \le T\} + \mathbb{P}\{\tau_0 < \tau_u \le T\} =: \mathcal{P}_{2,1}(T;u) + \mathcal{P}_{2,2}(T;u).$$

Observe that

$$
\begin{aligned}
\mathcal{P}_{2,1}(T;u) &= \int_{[0,T]} \mathbb{P}\{\sup_{t\in[s,\tau_0]}(Q(s)-Q(t))>u,\tau_0\le T\}dF_{\tau_u}(s)\\
&\le \int_{[0,T]} \mathbb{P}\{\sup_{t\in[s,T+s]}((X(s)-X(t))-c(s-t))>u\}dF_{\tau_u}(s)\\
&= \int_{[0,T]} \mathbb{P}\{\sup_{t\in[0,T]}((\widetilde{X}(s)-\widetilde{X}(t))-c(s-t))>u\}dF_{\tau_u}(s)\\
&= \mathbb{P}\{\sup_{0\le s\le t\le T}((\widetilde{X}(s)-\widetilde{X}(t))-c(s-t))>u,\tau_u\le T\}\\
&= \mathbb{P}\{\tau_u\le T\}\mathbb{P}\{\sup_{0\le s\le t\le T}((\widetilde{X}(s)-\widetilde{X}(t))-c(s-t))>u\}\\
&\le \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\mathbb{P}\{\sup_{0\le s\le T}(\widetilde{X}(s)-cs)+\sup_{0\le t\le T}(-(\widetilde{X}(t)-ct))>u\}\\
&\le \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\mathbb{P}\{2\sup_{0\le t\le T}|\widetilde{X}(t)-ct|>u\}\\
&=: p^{\sup}(T;u)\widetilde{\mathcal{P}}_{2,1}(T;u),
\end{aligned}
$$

where $\{\widetilde{X}(t):t\ge 0\}$ is a copy of $\{X(t):t\ge 0\}$, independent of the process $\{Q(t):t\ge 0\}$. Further, Assumption **A1** gives that, as $u\to\infty$,

$$
\widetilde{\mathcal{P}}_{2,1}(T;u)=o\left(1\right).
$$

Thus, we obtain, as $u\to\infty$,

$$
\mathcal{P}_{2,1}(T;u)=o\left(\mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\right).
$$

Futhermore, for $\mu:=\mathbb{E}\{X(1)-c\}<0$ and $Z_\mu(t)=\sup_{s\in[0,t]}((X(t)-X(s))-\mu(t-s))$, we obtain that

$$
\begin{aligned}
\mathcal{P}_{2,2}(T;u) &\le \mathbb{P}\{\sup_{t\in[\tau_0,T]}(Q(t)-Q(\tau_0))>u\}\\
&\le \mathbb{P}\{\sup_{0\le s\le t\le T}((X(t)-X(s))-c\,(t-s))>u\}\\
&= \mathbb{P}\{\sup_{0\le s\le t\le T}((X(t)-X(s))-\mu\,(t-s)+(\mu-c)\,(t-s))>u\}\\
&\le \mathbb{P}\{\sup_{0\le s\le t\le T}((X(t)-X(s))-\mu\,(t-s))>u+\max\{\mu-c,0\}T\}\\
&= \mathbb{P}\{\sup_{0\le t\le T}(Z_\mu(t))>\tilde{u}\}\\
&\le \mathbb{P}\{\sup_{0\le t\le T}(Z_\mu(t))\,e^{\omega\sup_{0\le t\le T}(Z_\mu(t))}>\tilde{u}e^{\omega\tilde{u}}\}\\
&= \mathbb{P}\{f(\sup_{0\le t\le T}(Z_\mu(t)))>\tilde{u}e^{\omega\tilde{u}}\}=\mathbb{P}\{\sup_{0\le t\le T}f(Z_\mu(t))>\tilde{u}e^{\omega\tilde{u}}\}=:\widetilde{\mathcal{P}}_{2,2}(T;u),
\end{aligned}
$$

with $\tilde{u}=u+\max\{\mu-c,0\}T$ and $f(x)=x\exp(\omega x)$.

Observe that, for a natural filtration $\{\mathcal{F}(t):t\ge 0\}$ generated by $\{X(t):t\ge 0\}$ and $0\le h\le t$, it holds

$$
\begin{aligned}
\mathbb{E}\{Z_\mu(t)|\mathcal{F}(h)\} &= \mathbb{E}\{(X(t)-\mu t)+\sup_{s\in[0,t]}(-(X(s)-\mu s))|\mathcal{F}(h)\}\\
&\geq (X(h)-\mu h)+\mathbb{E}\{(X(t)-X(h))-\mu(t-h)\}+\mathbb{E}\{\sup_{s\in[0,h]}(-(X(s)-\mu s))|\mathcal{F}(h)\}\\
&= (X(h)-\mu h)+\sup_{s\in[0,h]}(-(X(s)-\mu s))+\mathbb{E}\{(X(t)-X(h))-\mu(t-h)\}\\
&= Z_\mu(h)+\mathbb{E}\{(X(t)-X(h))-\mu(t-h)\}=Z_\mu(h).
\end{aligned}
$$

Hence $\{Z_\mu(t):t\geq 0\}$ is a submartingale with respect to the natural filtration. Since function $f(x)$ is convex for $\omega>0$, $x>0$, then $\{e^{\omega Z_\mu(t)}:t\geq 0\}$ is a submartingale too. Thus, using Doob's Maximal Inequality (see e.g. [63]), we obtain that

$$
\begin{aligned}
\widetilde{\mathcal{P}}_{2,2}(T;u) &= \mathbb{P}\{\sup_{0\leq s\leq T} f(Z_\mu(t))>\tilde{u}e^{\omega\tilde{u}}\}\leq \frac{1}{\tilde{u}}e^{-\omega\tilde{u}}\mathbb{E}\{\max\{f(Z_\mu(T)),0\}\}\\
&= \frac{1}{\tilde{u}}e^{-\omega\tilde{u}}\mathbb{E}\{f(\sup_{s\in[0,T]}((X(T)-X(s))-\mu(T-s)))\}\\
&= \frac{1}{\tilde{u}}e^{-\omega\tilde{u}}\mathbb{E}\{f(\sup_{t\in[0,T]}(X(t)-\mu t))\}=\frac{1}{\tilde{u}}e^{-\omega\tilde{u}}\mathbb{E}\{f(\sup_{t\in[0,T]}(X_\mu(t)))\},
\end{aligned}
$$

with $X_\mu(t)=X(t)-\mu t$. Let $f^{-1}(x)$ denotes the inverse function of $f(x)$ (note that $f(x)$ is strictly increasing), with Lemma 9.1 in [49], we have that

$$
\begin{aligned}
&\mathbb{E}\{f(\sup_{t\in[0,T]}(X_\mu(t)))\}=\int_0^\infty \mathbb{P}\{f(\sup_{t\in[0,T]}(X_\mu(t)))>x\}dx=\int_0^\infty \mathbb{P}\{\sup_{t\in[0,T]}(X_\mu(t))>f^{-1}(x)\}dx\\
&=\int_0^M \mathbb{P}\{\sup_{t\in[0,T]}(X_\mu(t))>f^{-1}(x)\}dx+\int_M^\infty \mathbb{P}\{\sup_{t\in[0,T]}(X_\mu(t))>f^{-1}(x)\}dx\\
&\leq M+\int_M^\infty \frac{\mathbb{P}\{X_\mu(T)>f^{-1}(x)-M\}}{\mathbb{P}\{\inf_{t\in[0,T]}(X_\mu(t))>-M\}}dx\leq M+C(M)\int_M^\infty \mathbb{P}\{X_\mu(T)>f^{-1}(x)-M\}dx\\
&=M+C(M)\mathcal{E}_{X_\mu},
\end{aligned}
$$

where

$$
C(x):=1/\mathbb{P}\{\inf_{t\in[0,T]}(X_\mu(t))>-x\},\ M:=\max\{\inf\{x>1:C(x)>0\},2\}
$$

and

$$
\mathcal{E}_{X_\mu}:=\mathbb{P}\{X_\mu(T)>f^{-1}(x)-M\}dx.
$$

Further, we obtain that

$$
\mathcal{E}_{X_\mu} = \int_M^\infty \mathbb{P}\{X_\mu(T)>f^{-1}(x)-M\}dx=\int_M^\infty\int_{f^{-1}(x)-M}^\infty dF_{X_\mu(T)}(y)dx
$$

$$
\begin{aligned}
&= \int_{f^{-1}(M)-M}^{\infty}\int_{M}^{f(y+M)} dx dF_{X_\mu(T)}(y) = \int_{f^{-1}(M)-M}^{\infty}(f(y+M)-M)dF_{X(T)}(y)\\
&\le \int_{f^{-1}(M)-M}^{\infty} f(y+M)dF_{X_\mu(T)}(y) \le \int_{-\infty}^{\infty} f(y+M)dF_{X_\mu(T)}(y)\\
&= \mathbb{E}\{f(X_\mu(T)+M)\} = \mathbb{E}\{(X_\mu(T)+M)e^{\omega(X_\mu(T)+M)}\}\\
&= e^{\omega M}\mathbb{E}\{X_\mu(T)e^{\omega X_\mu(T)}\} + Me^{\omega M}\mathbb{E}\{e^{\omega X_\mu(T)}\}\\
&\le e^{\omega M}\mathbb{E}\{|X_\mu(T)|e^{\omega X_\mu(T)}\} + Me^{\omega M}\mathbb{E}\{e^{\omega X_\mu(T)}\} < \infty.
\end{aligned}
$$

Hence, we have that, as $u \to \infty$,

$$
\mathcal{P}_{2,2}(T;u) \le \mathbb{P}\{\sup_{0\le s\le t\le T}((X(t)-X(s))-c(t-s)) > u\} = o(e^{-\omega\tilde{u}}) = o(e^{-\omega u}).
$$

Collecting together the exact asymptotics of $\mathcal{P}_1(T;u)$ and the asymptotic negligibility of $\mathcal{P}_2(T;u)$ with respect to $\mathcal{P}_1(T;u)$, we get, as $u \to \infty$,

$$
p^{\Theta}(T;u) \sim \mathcal{P}_1(T;u) \sim \mathcal{H}^{\Theta}_{\omega X_c}[0,T]\mathbb{P}\{Q(0) > u\},
$$

which completes the proof. □

Proof of Theorem 4.2.3 For given $T > 0$ and $n(u)$, we introduce

$$
I_k := [kT,(k+1)T], \text{ for } k = 0,\dots,\lfloor\frac{n(u)}{T}\rfloor + 1,
$$

where $\lfloor . \rfloor$ denotes integer part of a number.

The stationarity of $\{Q(t) : t \ge 0\}$ gives that, for any $u > 0$,

$$
\begin{aligned}
\Sigma_1(u) - \Sigma_2(u) - \Sigma_3(u) &:= \sum_{k=0}^{\lfloor\frac{n(u)}{T}\rfloor} \mathcal{P}_k(u) - \lfloor\frac{n(u)}{T}\rfloor\mathcal{S}_1(u) - \sum_{k=2}^{\lfloor\frac{n(u)}{T}\rfloor+1}\lfloor\frac{n(u)}{T}\rfloor\mathcal{S}_k(u)\\
&\le p^{\sup}(n(u);u)\\
&\le \sum_{k=0}^{\lfloor\frac{n(u)}{T}\rfloor+1} \mathcal{P}_k(u) + \mathcal{P}_{\lfloor\frac{n(u)}{T}\rfloor+1}(u) =: \Sigma_4(u),
\end{aligned}
$$

with

$$
\mathcal{P}_k(u) := \mathbb{P}\{\sup_{t\in I_k}(Q(t)) > u\}, \text{ for } k = 0,\dots,\lfloor\frac{n(u)}{T}\rfloor + 1,
$$

$$
\mathcal{S}_k(u) := \mathbb{P}\{\sup_{t\in I_0}(Q(t)) > u, \sup_{t\in I_k}(Q(t)) > u\}, \text{ for } k = 1,\dots,\lfloor\frac{n(u)}{T}\rfloor + 1.
$$

We shall prove that $\Sigma_2(u)$, $\Sigma_3(u)$ and $\Sigma_4(u)$ are asymptotically negligible, comparing to $\Sigma_1(u)$ as $u \to \infty$ and then $T \to \infty$.

Asymptotics of $\Sigma_1(u)$ and $\Sigma_4(u)$. By the stationarity of $\{Q(t) : t \geq 0\}$ and applying Theorem 4.2.1 for $k = 0, \ldots, \lfloor \frac{n(u)}{T} \rfloor + 1$, we have that, as $u \to \infty$,

$$\mathcal{P}_k(u) = \mathcal{P}_1(u) = p^{\sup}(T; u) \sim \mathcal{H}^{\sup}_{\omega X_c}[0, T]\, \mathbb{P}\{Q(0) > u\},$$

with $X_c(t) = X(t) - ct$.

Then, we obtain that, as $u \to \infty$,

$$\Sigma_1(u) \sim \Sigma_4(u) \sim \lfloor \frac{n(u)}{T} \rfloor \mathcal{H}^{\sup}_{X_c}[0, T]\mathbb{P}\{Q(0) > u\} \sim \frac{\mathcal{H}^{\sup}_{\omega X_c}[0, T]}{T} n(u)\mathbb{P}\{Q(0) > u\}.$$

Further, due to the assumption that $\mathcal{H}^{\sup}_{\omega X_c} = \lim_{T\to\infty} \frac{\mathcal{H}^{\sup}_{\omega X_c}[0,T]}{T} \in (0, \infty)$, we get that

$$\lim_{T\to\infty} \lim_{u\to\infty} \frac{\Sigma_1(u)}{\mathcal{H}^{\sup}_{\omega X_c} n(u)\mathbb{P}\{Q(0) > u\}} = \lim_{T\to\infty} \lim_{u\to\infty} \frac{\Sigma_4(u)}{\mathcal{H}^{\sup}_{\omega X_c} n(u)\mathbb{P}\{Q(0) > u\}} = 1.$$

Asymptotics of $\Sigma_2(u)$. Observe that

$$\begin{aligned}
\mathcal{S}_1(u) &\leq \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \sup_{t\in[T,2T]} (Q(t)) > u\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \sup_{t\in[T,T+\sqrt{T}]} (Q(t)) > u\} + \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]} (Q(t)) > u\} \\
&=: \mathcal{S}_{1,1}(u) + \mathcal{S}_{1,2}(u).
\end{aligned}$$

Stationarity of $\{Q(t) : t \geq 0\}$ and Theorem 4.2.1 gives us, as $u \to \infty$,

$$\mathcal{S}_{1,1}(u) \leq \mathbb{P}\{\sup_{t\in[T,T+\sqrt{T}]} (Q(t)) > u\} = \mathbb{P}\{\sup_{t\in[0,\sqrt{T}]} (Q(t)) > u\} \sim \mathcal{H}^{\sup}_{X_c}\left[0, \sqrt{T}\right] \mathbb{P}\{Q(0) > u\}.$$

Further, we have that

$$\mathcal{S}_{1,2}(u) \leq \mathcal{S}_{1,2,1}(u) + \mathcal{S}_{1,2,2}(u),$$

with

$$\begin{aligned}
\mathcal{S}_{1,2,1}(u) &:= \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]} (Q(t)) > u,\, Q(T) \geq u + \frac{1}{8}c\sqrt{T}\}, \\
\mathcal{S}_{1,2,2}(u) &:= \mathbb{P}\{\sup_{t\in[0,T]} (Q(t)) > u, \sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]} (Q(t)) > u,\, Q(T) < u + \frac{1}{8}c\sqrt{T}\}.
\end{aligned}$$

Applying Theorem 4.2.1 (see also [49][Theorem 8.2]) we have that, as $u \to \infty$,

$$\mathcal{S}_{1,2,1}(u) \leq \mathbb{P}\{Q(T) \geq u + \frac{1}{8}c\sqrt{T}\} \sim e^{-\frac{\omega c}{8}\sqrt{T}}\mathbb{P}\{Q(0) > u\}.$$

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Furthermore, we obtain

$$
\begin{aligned}
\mathcal{S}_{1,2,2}(u) &\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u, \sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]}(Q(t)-Q(T))>-\frac{1}{8}c\sqrt{T},\, Q(T)<u+\frac{1}{8}c\sqrt{T}\}\\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u, \sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]}((X(t)-X(T))-c(t-T))>-\frac{1}{8}c\sqrt{T}\}\\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\mathbb{P}\{\sup_{t\in[T+\sqrt{T},2T+\sqrt{T}]}((X(t)-X(T))-\frac{c}{2}(t-T))>\frac{3}{8}c\sqrt{T}\}\\
&= \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\mathbb{P}\{\sup_{t\in[\sqrt{T},T+\sqrt{T}]}(X(t)-\frac{c}{2}t)>\frac{3}{8}c\sqrt{T}\}\\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\mathbb{P}\{\sup_{t\geq 0}(X(t)-\frac{c}{2}t)>\frac{3}{8}c\sqrt{T}\}.
\end{aligned}
$$

By combination of Theorem 4.2.1 with (4.1), we obtain, as $u\to\infty$,

$$
\mathbb{P}\{\sup_{t\in[0,T]}(Q(t))>u\}\sim \mathcal{H}^{\sup}_{\omega X_c}[0,T]\,\mathbb{P}\{Q(0)>u\},
$$

and, as $T\to\infty$,

$$
\mathbb{P}\{\sup_{t\geq 0}(X(t)-\frac{c}{2}t)>\frac{3}{8}c\sqrt{T}\}=\mathbb{P}\{Q_{X_{\frac{c}{2}}}(0)>\frac{3c}{8}\sqrt{T}\}\sim C_{\widetilde{\omega}}e^{-\widetilde{\omega}\frac{3c}{8}\sqrt{T}},
$$

for $\widetilde{\omega}, C_{\widetilde{\omega}}>0$.

Hence, we have, for some $C>0$,

$$
\begin{aligned}
\mathcal{S}_1(u) &\leq \mathcal{S}_{1,1}(u)+\mathcal{S}_{1,2,1}(u)+\mathcal{S}_{1,2,2}(u)\\
&\leq C\left(\mathcal{H}^{\sup}_{\omega X_c}\left[0,\sqrt{T}\right]+e^{-\frac{\omega c}{8}\sqrt{T}}+C_{\widetilde{\omega}}e^{-\frac{\widetilde{\omega}3c}{8}\sqrt{T}}\mathcal{H}^{\sup}_{\omega X_c}[0,T]\right)\mathbb{P}\{Q(0)>u\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{T\to\infty}\lim_{u\to\infty}\frac{\Sigma_2(u)}{\Sigma_1(u)} &\leq \lim_{T\to\infty}\lim_{u\to\infty} C\frac{\frac{n(u)}{T}\left(\mathcal{H}^{\sup}_{\omega X_c}\left[0,\sqrt{T}\right]+e^{-\frac{\omega c}{8}\sqrt{T}}+C_{\widetilde{\omega}}e^{-\frac{\widetilde{\omega}3c}{8}\sqrt{T}}\mathcal{H}^{\sup}_{\omega X_c}[0,T]\right)\mathbb{P}\{Q(0)>u\}}{\frac{\mathcal{H}^{\sup}_{\omega X_c}[0,T]}{T}n(u)\mathbb{P}\{Q(0)>u\}}\\
&= \lim_{T\to\infty} C\frac{\frac{1}{\sqrt{T}}\frac{\mathcal{H}^{\sup}_{\omega X_c}[0,\sqrt{T}]}{\sqrt{T}}+\frac{1}{T}e^{-\frac{\omega c}{8}\sqrt{T}}+C_{\widetilde{\omega}}e^{-\frac{\widetilde{\omega}3c}{8}\sqrt{T}}\frac{\mathcal{H}^{\sup}_{\omega X_c}[0,T]}{T}}{\frac{\mathcal{H}^{\sup}_{\omega X_c}[0,T]}{T}}\\
&= \lim_{T\to\infty} C\frac{\frac{1}{\sqrt{T}}\mathcal{H}^{\sup}_{\omega X_c}+\frac{1}{T}e^{-\frac{\omega c}{8}\sqrt{T}}+C_{\widetilde{\omega}}e^{-\frac{\widetilde{\omega}3c}{8}\sqrt{T}}\mathcal{H}^{\sup}_{\omega X_c}}{\mathcal{H}^{\sup}_{\omega X_c}}=0.
\end{aligned}
$$

Asymptotics of $\Sigma_3(u)$. For $k=2,\ldots,\lfloor\frac{n(u)}{T}\rfloor+1$ we have

$$
\mathcal{S}_k(u) = \mathcal{S}_{k,1}(u)+\mathcal{S}_{k,2}(u),
$$

with

$$\mathcal{S}_{k,1}(u) := \mathbb{P}\{\sup_{t\in I_0}(Q(t)) > u, \sup_{t\in I_k}(Q(t)) > u, Q(T) \geq u + \frac{c}{8}kT\},$$
$$\mathcal{S}_{k,2}(u) := \mathbb{P}\{\sup_{t\in I_0}(Q(t)) > u, \sup_{t\in I_k}(Q(t)) > u, Q(T) < u + \frac{c}{8}kT\}.$$

By applying the stationarity of $\{Q(t) : t \geq 0\}$ and Theorem 4.2.1 we have that, as $u \to \infty$,

$$\mathcal{S}_{k,1}(u) \leq \mathbb{P}\{Q(T) \geq u + \frac{c}{8}kT\} = \mathbb{P}\{Q(0) \geq u + \frac{c}{8}kT\} \sim e^{-\frac{\omega c}{8}kT}\mathbb{P}\{Q(0) \geq u\}.$$

Furthermore, we obtain that

$$\begin{aligned}
\mathcal{S}_{k,2}(u) &\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u, \sup_{t\in[kT,(k+1)T]}(Q(t) - Q(T)) > -\frac{c}{8}kT, Q(T) < u + \frac{c}{8}kT\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u, \sup_{t\in[kT,(k+1)T]}((X(t) - X(T)) - c(t-T)) > -\frac{c}{8}kT\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u, \sup_{t\in[kT,(k+1)T]}((X(t) - X(T)) - \frac{c}{2}(t-T)) > (3k-4)\frac{c}{8}T\} \\
&= \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u\}\mathbb{P}\{\sup_{t\in[kT,(k+1)T]}((X(t) - X(T)) - \frac{c}{2}(t-T)) > (3k-4)\frac{c}{8}T\} \\
&= \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u\}\mathbb{P}\{\sup_{t\in[(k-1)T,kT]}(X(t) - \frac{c}{2}t) > (3k-4)\frac{c}{8}T\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u\}\mathbb{P}\{\sup_{t\geq 0}(X(t) - \frac{c}{2}t) > (3k-4)\frac{c}{8}T\}.
\end{aligned}$$

By Theorem 4.2.1 we have that, as $u \to \infty$,

$$\mathbb{P}\{\sup_{t\in[0,T]}(Q(t)) > u\} \sim \mathcal{H}^{\sup}_{\omega X_c}[0,T]\,\mathbb{P}\{Q(0) > u\}$$

and, as $T \to \infty$,

$$\mathbb{P}\{\sup_{t\geq 0}(X(t) - \frac{c}{2}t) > (3k-4)\frac{c}{8}T\} = \mathbb{P}\{Q_{\frac{c}{2}}(0) > (3k-4)\frac{c}{8}T\} \sim C_{\widetilde{\omega}}e^{-(3k-4)\frac{\widetilde{\omega}c}{8}T},$$

for $\widetilde{\omega}, C_{\widetilde{\omega}} > 0$. Hence, we obtain

$$\mathcal{S}_k(u) \leq C\left(e^{-\frac{\omega c}{8}kT} + \mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}e^{-(3k-4)\frac{\widetilde{\omega}c}{8}T}\right)\mathbb{P}\{Q(0) > u\},$$

where $C > 0$ does not depend on k. Then

$$\begin{aligned}
\Sigma_3(u) &\leq \sum_{k=2}^{\lfloor\frac{n(u)}{T}\rfloor+1}\lfloor\frac{n(u)}{T}\rfloor\mathcal{S}_k(u) \\
&\leq \sum_{k=2}^{\lfloor\frac{n(u)}{T}\rfloor+1} C\lfloor\frac{n(u)}{T}\rfloor\mathbb{P}\{Q(0) > u\}\left(e^{-\frac{\omega c}{8}kT} + \mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}e^{-(3k-4)\frac{\widetilde{\omega}c}{8}T}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\lfloor\frac{n(u)}{T}\rfloor\mathbb{P}\{Q(0)>u\}\sum_{k=2}^{\infty}\left(e^{-\frac{\omega c}{8}kT}+\mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}e^{-(3k-4)\frac{\widetilde{\omega}c}{8}T}\right)\\
&= C\lfloor\frac{n(u)}{T}\rfloor\mathbb{P}\{Q(0)>u\}\left(\sum_{k=2}^{\infty}e^{-\frac{\omega c}{8}kT}+\mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}e^{\frac{\widetilde{\omega}c}{2}T}\sum_{k=2}^{\infty}e^{-\frac{3\widetilde{\omega}c}{8}kT}\right)\\
&= C\lfloor\frac{n(u)}{T}\rfloor\mathbb{P}\{Q(0)>u\}\left(\frac{e^{-\frac{\omega c}{4}T}}{1-e^{-\frac{\omega c}{8}T}}+\mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}\frac{e^{-\frac{\widetilde{\omega}c}{4}T}}{1-e^{-\frac{3\widetilde{\omega}c}{8}T}}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{T\to\infty}\lim_{u\to\infty}\frac{\Sigma_3(u)}{\Sigma_1(u)} &\leq \lim_{T\to\infty}\lim_{u\to\infty}C\frac{\lfloor\frac{n(u)}{T}\rfloor\mathbb{P}\{Q(0)>u\}\left(\frac{e^{-\frac{\omega c}{8}T}}{1-e^{-\frac{\omega c}{8}T}}+\mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}\frac{e^{-\frac{\widetilde{\omega}c}{4}T}}{1-e^{-\frac{3\widetilde{\omega}c}{8}T}}\right)}{\lfloor\frac{n(u)}{T}\rfloor\mathcal{H}^{\sup}_{\omega X_c}[0,T]\mathbb{P}\{Q(0)>u\}}\\
&= \lim_{T\to\infty}C\frac{\frac{e^{-\frac{\omega c}{8}T}}{1-e^{-\frac{\omega c}{8}T}}+\mathcal{H}^{\sup}_{\omega X_c}[0,T]\,C_{\widetilde{\omega}}\frac{e^{-\frac{\widetilde{\omega}c}{4}T}}{1-e^{-\frac{3\widetilde{\omega}c}{8}T}}}{\mathcal{H}^{\sup}_{\omega X_c}[0,T]}=0.
\end{aligned}
$$

This completes the proof. □

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Chapter 5

An extension of Breiman's Lemma

5.1 Introduction

Products of random variables with heavy tailed distibution are of fundamental interest in both theoretical and applied probability. The seminal work of Breiman [19] provides a key technique for studying the asymptotics of such products – the so-called Breiman's lemma – and gives the asymptotic behavior, as $u \to \infty$, of the tail distribution of the product of two non-negative independent random variables X and Y

$$\mathbb{P}\{XY > u\} \sim \mathbb{E}\{Y^{\alpha}\}\mathbb{P}\{X > u\},$$

where X is a regularly varying with index $\alpha > 0$ and $\mathbb{E}\{Y^{\alpha+\varepsilon}\} \in (0, \infty)$ for some $\varepsilon > 0$.

The above asymptotic result implies that the heavy tail of X dominates the asymptotic behavior of the product XY, while the random variable Y with the lighter tail distribution contributes only through its α–th moment. Under a slightly different conditions, such as subexponentiality or long-tailed classes, extensions of this result have been established; see e.g. [27]. Denisov & Zwart [60] further weakened the assumptions on X, Y, and applied Breiman's lemma to random difference equations (stationary solutions of $Y = AY + X$), showing that the regularly varying tail of e^X implies the heavy tail of e^Y.

Further extensions, to the case of dependent random variables or to the multivariate settings have also been studied [8, 22, 23, 29, 64, 68, 77, 85, 88, 92, 126, 130, 131]; see also [72, 73, 78]. In particular, Jiang & Tang [92] studied the behavior of the tail distribution of the product XY, where (X, Y) has the generalized Farlie–Gumbel–Morgenstern distribution, and Yang & Wang [131] considered the product under the bivariate Sarmanov dependence structure. In related papers, Chen et al. [23] and Yang & Sun [130] obtained a Breiman-type result for $\mathbb{P}\{X\Theta > u\}$ for a wide class of copulas.

At the same time, multidimensional extensions of Breiman's lemma were investigated [8, 29, 68, 77, 85]. In particular, Basrak et al. [8] established Breiman-type asymptotic behavior, as $u \to \infty$, of

$$u\mathbb{P}\{\boldsymbol{M}\boldsymbol{X} \in a_u \cdot\} \xrightarrow{v} \mathbb{E}\{\mu(\boldsymbol{M}^{-1}(\cdot))\},$$

where $\xrightarrow{v}$ denotes vague convergence on $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, with $\overline{\mathbb{R}} = [-\infty, \infty]$, $\boldsymbol{X}$ is a d–dimensional random vector such that

$$u\mathbb{P}\{\frac{\boldsymbol{X}}{a_u} \in \cdot\} \xrightarrow{v} \mu(\cdot), \text{ as } u \to \infty,$$

for some normalization function $a_u \geq 0$, a measure μ on $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$, and a random matrix $\boldsymbol{M}$ of size $q \times d$ such that $\mathbb{E}\{\|\boldsymbol{M}\|^{\alpha+\varepsilon}\} < \infty$ for some $\varepsilon > 0$, where $\alpha > 0$ denotes the homogeneity index of μ. In a related paper, Fougeres & Mercadier [68] allowed a wider class of dependency structures and weakened the conditions on $\mathbf{X}$, $\mathbf{M}$ and a_u under the following assumptions.

A1: It holds, as $u \to \infty$,

$$u\mathbb{P}\{\left(\frac{\boldsymbol{X}}{a_u}, \boldsymbol{M}\right) \in \cdot\} \xrightarrow{v} (\nu \times G)(\cdot),$$

where ν is a Radon measure on $\overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$ not concentrated at ∞, and homogeneous with index $\alpha > 0$ and G is a probability measure on $\overline{\mathbb{R}}^{qd}$.

A2: For some $\delta > 0$,

$$\lim_{\epsilon \to 0} \limsup_{u \to \infty} u\mathbb{E}\{\left(\frac{\|\boldsymbol{X}\|\|\boldsymbol{M}\|}{a_u}\right)^\delta \mathbb{I}\{\|\boldsymbol{X}\| \leq \epsilon a_u\}\} = 0.$$

A3: For $\alpha > 0$, the homogeneity index of the measure ν,

$$\int_{\overline{\mathbb{R}}^{qd}} \|\boldsymbol{M}\|^\alpha G(d\boldsymbol{M}) < \infty.$$

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Note that the Assumption **A1** implies that $\boldsymbol{X}$ has a multivariate regularly varying distribution with index $\alpha > 0$; see e.g. [121][Chapter 6] and [68].

The following proposition can be found in [68][Theorem 6].

Proposition 5.1.1 *Let $\boldsymbol{X} \in \mathbb{R}^d$ be a random vector and $\boldsymbol{M} \in \mathbb{R}^{q\times d}$ be a random matrix, and a_u be a function satysfying* **A1**–**A3**. *Then, as $u \to \infty$,*

$$u\mathbb{P}\{\boldsymbol{M}\boldsymbol{X} \in a_u \cdot\} \stackrel{v}{\to} \nu_G(\cdot),$$

where ν_G is the measure on $\overline{\mathbb{R}}^q \setminus \{\mathbf{0}\}$ defined by

$$\nu_G(\cdot) = (\nu \otimes G)(\{(\boldsymbol{x}, \boldsymbol{M}) : \boldsymbol{M}\boldsymbol{x} \in \cdot\}) = \int_{\overline{\mathbb{R}}^{qd}} \nu(\boldsymbol{M}^{-1}\cdot)G(d\boldsymbol{M}) = \mathbb{E}\{\nu(\boldsymbol{M}^{*-1}\cdot)\},$$

where $\boldsymbol{M}^*$ is a random matrix with distribution G.

We extend this result to the threshold-driven case by formulating a version of Breiman's lemma in which both the random variables and the matrices depend on the threshold u. More precisely, we study the families of random vectors $\{\boldsymbol{X}_{u,\tau_u} : u \geq 0\}$ and matrices $\{\boldsymbol{M}_{u,\tau_u} : u \geq 0\}$ parameterized by u and $\tau_u \in \mathcal{K}_u$ with some index set $\mathcal{K}_u$, and we introduce a normalization function b_{u,τ_u}. Our main object of study is the uniform with respect to the $\tau_u \in \mathcal{K}_u$ asymptotic behavior of

$$b_{u,\tau_u}\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}\cdot\},$$

as $u \to \infty$.

This extension allows to derive exact asymptotics for a wide range of problems that could not be tracted by the previous multivariate Breiman-type results. Notably, it provides an independent proof of the celebrated Pickands lemma (see e.g. [114] and Lemma D.1 in [117]) and establishes the uniform Pickands lemma for homogeneous functionals of Gaussian fields (Theorem 2.1 in [37]).

5.2 Main results

We begin this section with some notation. Let $\tilde{\mathbb{E}}_d$ denote $[-\infty, \infty]^d$ or $[0, \infty]^d$, and $\mathbb{E}_d = \tilde{\mathbb{E}}_d \backslash \{\mathbf{0}\}$. We slightly abuse the notation and write $\|\cdot\|$ for both vector and matrix norms; the exact interpretation depends on the context. For an invertable matrix $\boldsymbol{M} \in \mathbb{R}^{q\times d}$ and $K \subset \mathbb{R}^q$, we define

$$\boldsymbol{M}^{-1}K = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{M}\boldsymbol{x} \in K\}.$$

For $E = \mathbb{E}_d$ or $E = \mathbb{E}_d \times \tilde{\mathbb{E}}_d$, let $\mathcal{B}_E$ denote the Borel σ–algebra on E. A subset $K \subset E$ is relatively compact if its closure in E is compact; equivalently, every sequence of elements of K contains a convergent subsequence in K. Consequently, any relatively compact set $K \subset E$ is separated from zero; that is, there exists $\kappa > 0$ such that for any $\boldsymbol{x} \in K$, it holds $\|\boldsymbol{x}\| > \kappa$.

We say that a sequence of measures $\{\nu_n : n \in \mathbb{N}\}$ converges vaguely to a measure ν on $\mathcal{B}_E$ if

$$\int_E f(\boldsymbol{x}) d\nu_n(\boldsymbol{x}) \to \int_E f(\boldsymbol{x}) d\nu(\boldsymbol{x}), \text{ as } n \to \infty,$$

for any continuous function $f(\cdot)$ with compact support E; equivalently, $\nu_n(K) \to \nu(K)$, as $n \to \infty$, for any relatively compact set $K \subset E$ such that $\nu(\partial K) = 0$. Recall that a Radon measure on $\mathcal{B}_E$ is a measure that is finite on any compact subset of E. Finally, we say that a measure ν is homogeneous with index $\alpha > 0$, if $\nu(tK) = t^{-\alpha}\nu(K)$, for any $t > 0$ and a relatively compact set $K \in \mathcal{B}_E$.

In order to extend the framework of Fougeres & Mercadier [68] to allow explicit dependence on u, we impose the following assumptions to ensure the vague convergence of $\{\boldsymbol{M}_{u,\tau_u} \boldsymbol{X}_{u,\tau_u} a^{-1}_{u,\tau_u} : u \geq 0\}$, as $u \to \infty$, where the random families $\{\boldsymbol{X}_{u,\tau_u} \geq 0\}$ and $\{\boldsymbol{M}_{u,\tau_u} : u \geq 0\}$ are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A1: It holds, as $u \to \infty$,

$$b_{u,\tau_u} \mathbb{P}\{\left(\frac{\boldsymbol{X}_{u,\tau_u}}{a_{u,\tau_u}}, \boldsymbol{M}_{u,\tau_u}\right) \in \cdot\} \xrightarrow{v} (\nu \times G)(\cdot),$$

uniformly with respect to $\tau_u \in \mathcal{K}_u$; where ν is homogenous with index $\alpha > 0$ Radon measure on $\mathbb{E}_d$ not concentrated at ∞, and G is a probability measure on $\tilde{\mathbb{E}}_{qd}$.

A2: For some $\delta > 0$,

$$\lim_{\epsilon \to 0} \limsup_{u \to \infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u} \mathbb{E}\{\left(\frac{\|\boldsymbol{X}_{u,\tau_u}\| \|\boldsymbol{M}_{u,\tau_u}\|}{a_{u,\tau_u}}\right)^\delta \mathbb{I}\{\|\boldsymbol{X}_{u,\tau_u}\| \leq \epsilon a_{u,\tau_u}\}\} = 0.$$

A3: For $\alpha > 0$ denoting the homogeneity index of the measure ν,

$$\int_{\tilde{\mathbb{E}}_{qd}} \|\boldsymbol{M}\|^\alpha G(d\boldsymbol{M}) < \infty.$$

Theorem 5.2.1 *Let* $\{\boldsymbol{X}_{u,\tau_u} : u \geq 0\}$ *and* $\{\boldsymbol{M}_{u,\tau_u} : u \geq 0\}$ *be families of random vectors in* $\mathbb{R}^d$ *and random matrices in* $\mathbb{R}^{q\times d}$*, respectively, and* $a_{u,\tau_u}, b_{u,\tau_u}$ *be functions satysfying* ***A1****–****A3****. Then, as* $u \to \infty$,

$$b_{u,\tau_u}\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}\cdot\} \xrightarrow{v} \nu_G(\cdot),$$

uniformly with respect to $\tau_u \in \mathcal{K}_u$*; where* ν_G *is the measure defined on* $\mathbb{E}_q$ *by*

$$\nu_G(\cdot) = (\nu \otimes G)(\{(\boldsymbol{x}, \boldsymbol{M}) : \boldsymbol{M}\boldsymbol{x} \in \cdot\}) = \int_{\tilde{\mathbb{E}}_{qd}} \nu(\boldsymbol{M}^{-1}\cdot)G(d\boldsymbol{M}) = \mathbb{E}\{\nu(\boldsymbol{M}^{*-1}\cdot)\},$$

where $\boldsymbol{M}^*$ *is a random matrix with distribution* G.

The complete proof of Theorem 5.2.1 is given in Section 5.4.1.

5.3 Applications

The extension of Breiman's lemma presented in Theorem 5.2.1 allows to obtain asymptotic behavior for many interesting applied probability problems, including:

(a) *Uniform Pickands Lemma for homogeneous functionals of Gaussian fields*, which was analyzed in [37][Theorem 2.1].

(b) *Supremum of self-standardized Gaussian processes*, which are related to the *Gamma bridges* and Gaussian processes with random variance.

5.3.1 Uniform Pickands Lemma for homogeneous functionals of Gaussian fields.

We demonstrate that the extension of Breiman's lemma presented in Theorem 5.2.1, provides an alternative proof for the extremes of a broad class of Gaussian processes. Specifically, this result allows us to calculate the tail asymptotics stated in Theorem 2.1 of [37] and the celebrated Pickands lemma ([114] and Lemma D.1 in [117]) and generalizes these results to a broader class of sets depending on the threshold u than the half-line $[u, \infty)$. To this end, we first recall the notation and framework introduced in [37].

Let $E \subset \mathbb{R}^d$ be a compact set and write $C(E)$ for the set of real-valued continuous functions defined on E. Let $\Theta : C(E) \to \mathbb{R}$ be a real-valued continuous functional satisfying:

F1: There exists $c > 0$ such that $\Theta(f) \le c \sup_{t\in E} f(t)$ for any $f \in C(E)$,

F2: $\Theta(af + b) = a\Theta(f) + b$ for any $f \in C(E)$ and $a > 0, b \in \mathbb{R}$.

We shall consider a family of centered Gaussian random fields $\{\xi_{u,\tau_u}(t) : t \in E\}$ given by

$$\xi_{u,\tau_u}(t) = \frac{Z_{u,\tau_u}(t)}{1 + h_{u,\tau_u}(t)}, t \in E, \tau_u \in \mathcal{K}_u,$$

with $\{Z_{u,\tau_u}(t) : t \in E\}$ a centered Gaussian random field with unit variance and continuous trajectories a.s., and $h_{u,\tau_u} \in C_0(E)$, where $C_0(E)$ is the Banach space of all continuous functions $f(\cdot)$ on E such that $f(0) = 0$, equipped with the sup-norm. In order to avoid trivialities, the thresholds g_{u,τ_u} will be chosen such that

$$\lim_{u\to\infty} \mathbb{P}\{\Theta(\xi_{u,\tau_u}) > g_{u,\tau_u}\} = 0.$$

C0: The positive constants g_{u,τ_u} are such that

$$\lim_{u\to\infty} \inf_{\tau_u\in\mathcal{K}_u} g_{u,\tau_u} = \infty.$$

C1: There exists $h \in C_0(E)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u\in\mathcal{K}_u, t\in E} |g^2_{u,\tau_u} h_{u,\tau_u}(t) - h(t)| = 0.$$

C2: There exists $\theta_{u,\tau_u}(s,t)$ such that

$$\lim_{u\to\infty} \sup_{\tau_u\in\mathcal{K}_u} \sup_{s\neq t\in E} \left| g^2_{u,\tau_u} \frac{\mathbb{V}ar\{Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)\}}{2\theta_{u,\tau_u}(s,t)} - 1 \right| = 0$$

and for some centered Gaussian random field $\eta(t), t \in \mathbb{R}^d$ with continuous trajectories and $\eta(0) = 0$

$$\lim_{u\to\infty} \sup_{\tau_u\in\mathcal{K}_u} |\theta_{u,\tau_u}(s,t) - \mathbb{V}ar\{\eta(t) - \eta(s)\}| = 0, \text{ for all } s, t \in E.$$

C3: There exists $a > 0$ such that

$$\limsup_{u\to\infty} \sup_{\tau_u\in\mathcal{K}_u} \sup_{s\neq t, s,t\in E} \frac{\theta_{u,\tau_u}(s,t)}{\sum_{i=1}^d |s_i - t_i|^a} < \infty$$

and

$$\lim_{\epsilon\downarrow 0} \limsup_{u\to\infty} \sup_{\tau_u\in\mathcal{K}_u} \sup_{\|t-s\|<\epsilon, s,t\in E} g^2_{u,\tau_u} \mathbb{E}\{[Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)] Z_{u,\tau_u}(0)\} = 0.$$

Given $h \in C_0(E)$ and the functional Θ satisfying **F1**–**F2**, for η introduced in **C2**, we define the following constant

$$\mathcal{H}_{\eta,h}^{\Theta}(E) = \mathbb{E}\{e^{\Theta(\{\eta_h(t):t\in E\})}\} \text{ with } \eta_h(t) = \sqrt{2}\eta(t) - \mathbb{V}ar\{\eta(t)\} - h(t), \tag{5.1}$$

which by **F1** is finite.

Let us recall that $\Psi(\cdot)$ denotes the tail distribution function of a standard normal random variable.

Theorem 5.3.1 *Under assumptions **C0**–**C3** and **F1**–**F2**, if further for all $\tau_u \in \mathcal{K}_u$ and sufficiently large u, it holds $\mathbb{P}\{\Theta(\{\xi_{u,\tau_u}(t) : t \in E\}) > g_{u,\tau_u}\} > 0$. Then, for every relatively compact $K \in \mathcal{B}_{(0,\infty]}$ such that $\nu(\partial K) = 0$, it holds*

$$\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \left| \frac{\mathbb{P}\{\Theta\left(\{\xi_{u,\tau_u}(t) : t \in E\}\right) \in g_{u,\tau_u} + \frac{1}{g_{u,\tau_u}} \log(K)\}}{\mathcal{H}_{\eta,h}^{\Theta}(E)\nu(K)\Psi(g_{u,\tau_u})} - 1 \right| = 0,$$

where $\nu(K) = \int_K \frac{1}{x^2} dx$ and $\log(K) = \{\log(x) : x \in K\}$.

Theorem 5.3.1 generalizes Theorem 2.1 in [37], where the special case $K = [1, \infty)$ was considered. Indeed, by putting $K = [1, \infty)$ in Theorem 5.3.1, we obtain that $\nu(K) = 1$ and

$$\mathbb{P}\{\Theta\left(\{\xi_{u,\tau_u}(t) : t \in E\}\right) \in g_{u,\tau_u} + \frac{1}{g_{u,\tau_u}} \log(K)\} = \mathbb{P}\{\Theta\left(\{\xi_{u,\tau_u}(t) : t \in E\}\right) > g_{u,\tau_u}\}.$$

Additionally, note that Theorem 5.3.1 provides an alternative proof for the celebrated Pickands and Piterbarg lemmas, as discussed below. Before stating these results, let us recall that $\{B_H(t) : t \geq 0\}$ denotes the fractional Brownian motion with Hurst index $H \in (0, 1]$.

Corollary 5.3.2 (Pickands lemma) *Suppose that $\{X(t) : t \geq 0\}$ is a continuous centered Gaussian stationary process with covariance function $r(t)$ that satisfies*

$$r(t) < 1 \text{ for } t > 0 \quad \text{and} \quad r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0,$$

where $\alpha \in (0, 2]$. Then, for every relatively compact $K \in \mathcal{B}_{(0,\infty]}$ such that $\nu(\partial K) = 0$, it holds

$$\mathbb{P}\{\sup_{t\in[0,Tu^{-2/\alpha}]} (X(t)) \in u + \frac{1}{u} \log(K)\} \sim \mathcal{H}_{B_{\alpha/2},0}^{\sup}([0, T]) \int_K \frac{1}{x^2} dx\, \Psi(u),$$

as $u \to \infty$.

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Corollary 5.3.3 (Piterbarg lemma) *Suppose that $\{X(t) : t \geq 0\}$ is a continuous centered Gaussian stationary process with covariance function $r(t)$ that satisfies*

$$r(t) < 1 \text{ for } t > 0 \quad \text{and} \quad r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0,$$

where $\alpha \in (0, 2]$ and let $\beta \geq \alpha$, $b > 0$. Then, for every relatively compact $K \in \mathcal{B}_{(0,\infty]}$ such that $\nu(\partial K) = 0$, it holds, as $u \to \infty$,

$$\mathbb{P}\{ \sup_{t\in[0,Tu^{-2/\alpha}]} \left(\frac{X(t)}{1 + bt^\beta} \right) \in u + \frac{1}{u} \log(K)\} \sim \mathcal{P}_{\alpha,\beta}([0, T]) \int_K \frac{1}{x^2} dx\, \Psi(u),$$

where

$$\mathcal{P}_{\alpha,\beta}([0, T]) = \mathbb{E}\{e^{\sup_{t\in[0,T]} (\sqrt{2}B_{\alpha/2}(t) - (1+b\mathbb{I}\{\alpha=\beta\})t^\alpha)}\}.$$

Remark 5.3.4 *Corollary 5.3.2 leads straightforwardly to an extension of the classical Pickands theorem, see e.g. Theorem D.2 in [117], i.e. for a continuous centered Gaussian stationary process $\{X(t) : t \geq 0\}$ with covariance function $r(t)$ that satisfies*

$$r(t) < 1 \text{ for } t > 0 \quad \text{and} \quad r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0,$$

where $\alpha \in (0, 2]$, $T > 0$, and for every relatively compact $K \in \mathcal{B}_{(0,\infty]}$ such that $\nu(\partial K) = 0$, we obtain, as $u \to \infty$,

$$\mathbb{P}\{ \sup_{t\in[0,T]} (X(t)) \in u + \frac{1}{u} \log(K)\} \sim \mathcal{H}_\alpha T \int_K \frac{1}{x^2} dx\, u^{2/\alpha} \Psi(u),$$

where

$$\mathcal{H}_\alpha = \lim_{\Delta\to\infty} \frac{1}{\Delta} \mathcal{H}^{\sup}_{B_{\alpha/2},0}([0, T]) = \lim_{\Delta\to\infty} \frac{1}{\Delta} \mathbb{E}\{e^{\sup_{t\in[0,\Delta]} (\sqrt{2}B_{\alpha/2}(t) - t^\alpha)}\} \in (0, \infty).$$

Analogously, Corollary 5.3.3 gives, for $\alpha \in (0, 2]$, $\beta \geq \alpha$, $b > 0$ and every relatively compact $K \in \mathcal{B}_{(0,\infty]}$ such that $\nu(\partial K) = 0$, as $u \to \infty$,

$$\mathbb{P}\{ \sup_{t\in[0,T]} \left(\frac{X(t)}{1 + bt^\beta} \right) \in u + \frac{1}{u} \log(K)\} \sim \begin{cases} \frac{\mathcal{H}_\alpha \Gamma(1/\beta)}{\beta b^{1/\beta}} \int_K \frac{1}{x^2} dx\, u^{2/\alpha - 2/\beta} \Psi(u), & \text{if } \beta > \alpha \\ \mathcal{P}^b_\alpha \int_K \frac{1}{x^2} dx\, \Psi(u), & \text{if } \beta = \alpha \end{cases},$$

where

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

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and

$$\mathcal{P}_\alpha^b = \lim_{\Delta\to\infty} \mathcal{P}_{\alpha,\beta}([0,\Delta]) = \lim_{\Delta\to\infty} \mathbb{E}\{e^{\sup_{t\in[0,\Delta]}(\sqrt{2}B_{\alpha/2}(t)-(1+b)t^\alpha)}\} \in (0,\infty).$$

Since the proofs of the above asymptotics are identical to those given in [117][Proof of Theorem D.2] and [119][Proof of Theorem 10.1] (see also [96]), we shall not provide them here.

5.3.2 Supremum of self-standardized Gaussian processes

For a given stochastic process $\{Y(t) : t \geq 0\}$, we investigate the *self-standardized process*

$$\left\{\frac{Y(t)}{Y(T)} : t \in [0,T]\right\}.$$

The probabilistic properties of the *self-standardized Gamma process* (also known as the *Gamma bridge*), defined as

$$\left\{\frac{\gamma(t)}{\gamma(T)} : t \in [0,T]\right\},$$

where $\{\gamma(t) : t \in [0,T]\}$ is the *standard Gamma process*, have been widely analyzed in recent literature from both a theoretical point of view (see, e.g., [65, 66]) and in terms of its applicability to ruin theory and financial mathematics (see e.g. [20, 74, 122, 132]).

We focus on self-standardized processes that are driven by Gaussian processes $\{Y(t) : t \geq 0\}$. This framework is related to a Gaussian process with random variance and expectated value

$$\{\xi(t)\cdot(\eta - \zeta t^\beta) : t \geq 0\},$$

where η and ζ are non-negative random variables that are independent of the Gaussian process $\{\xi(t) : t \geq 0\}$; see e.g. [75, 80, 116]. Let us observe that in the setting considered in this chapter, we analyze a model with dependent random variance, defined as follows

$$\{Y(t)\cdot\eta : t \in [0,T]\}, \text{ where } \eta = \frac{1}{Y(T)}.$$

We study the asymptotic behavior, as $u \to \infty$, of the self-standardized Gaussian processes under two regimes

G1: $\{Y(t) : t \in [0,T]\}$ is a centered Gaussian process with variance function $\sigma_Y^2(t)$ and covariance function $R_Y(s,t)$, with $T > 0$.

G2: $\{Y(t) : t \in [0, \delta(u)]\}$ is a centered Gaussian process with stationary increments a variance function $\sigma_Y^2(t)$ satysfying $\sigma_Y \in \mathcal{R}V_0(\lambda)$, for $\lambda \in (0,1)$, with $\delta(u) = \inf\{t > 0 : t/\sigma_Y(t) = 1/u\}$, which was analyzed in the queueing theory in the light-traffic regime [58].

Theorem 5.3.5 *Suppose that* ***G1*** *holds and* $\mu \in \mathbb{R}$. *Then, for every relatively compact* $K \in \mathcal{B}_{(0,\infty]}$ *such that* $\nu(\partial K) = 0$, *it holds, as* $u \to \infty$,

$$\mathbb{P}\{\sup_{t\in[0,T]} \left(\frac{Y(t)+\mu t}{Y(T)+\mu T}\right) \in uK\} \sim \frac{e^{-\frac{(\mu T)^2}{2\sigma_Y^2(T)}}}{\sqrt{2\pi\sigma_Y^2(T)}} \left(\mathbb{E}\{\sup_{t\in[0,T]}(Z_\mu(t))\} - \mathbb{E}\{\inf_{t\in[0,T]}(Z_\mu(t))\}\right) \int_K \frac{1}{x^2}dx\, \frac{1}{u},$$

where $Z_\mu(t) = (Y(t) + \mu t | Y(T) + \mu T = 0)$.

Theorem 5.3.6 *Suppose that* ***G2*** *holds. Then, for every relatively compact* $K \in \mathcal{B}_{(0,\infty]}$ *such that* $\nu(\partial K) = 0$, *it holds, as* $u \to \infty$,

$$\mathbb{P}\{\sup_{t\in[0,\delta(u)]} \left(\frac{Y(t)}{Y(\delta(u))}\right) \in uK\} \sim \sqrt{\frac{2}{\pi}}\mathbb{E}\{\sup_{t\in[0,1]}(B_\lambda(t)|B_\lambda(1)=0)\} \int_K \frac{1}{x^2}dx\, \frac{1}{u},$$

where $\delta(u) := \inf\{t > 0 : t/\sigma_Y(t) = 1/u\}$.

5.4 Proofs

5.4.1 Proof of Theorem 5.2.1

Proof of Theorem 5.2.1 The main idea of the proof is similar to the proof of Theorem 6 in [68], with certain modifications which we analyze in detail below.

Let $K \in \mathcal{B}_{\mathbb{E}_q}$ be relatively compact such that $\nu_G(\partial K) = 0$. For $s > 0$, it holds that

$$\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K\}$$
$$= \mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K, \|\boldsymbol{M}_{u,\tau_u}\| \le s\} + \mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K, \|\boldsymbol{M}_{u,\tau_u}\| > s\}.$$

Since K is relatively compact, there exists $\kappa > 0$ such that for all $\boldsymbol{x} \in K$ it holds $\|\boldsymbol{x}\| \ge \kappa$. Consequently, if $\|\boldsymbol{M}_{u,\tau_u}\| \le s$ and $\boldsymbol{M}_{u,\tau_u}\boldsymbol{x} \in K$, then $\|\boldsymbol{x}\| \ge s^{-1}\kappa$. Note that

$$\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K, \|\boldsymbol{M}_{u,\tau_u}\| \le s\} = \mathbb{P}\{\left(\frac{\boldsymbol{X}_{u,\tau_u}}{a_{u,\tau_u}}, \boldsymbol{M}_{u,\tau_u}\right) \in L\},$$

where $L = \{(\boldsymbol{x}, \boldsymbol{m}) : \boldsymbol{m}\boldsymbol{x} \in K, \|\boldsymbol{m}\| \le s\} \in \mathcal{B}_{\mathbb{E}_d \times \tilde{\mathbb{E}}_{qd}}$ is relatively compact. Assumption **A1** implies that

$$\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \left| \frac{b_{u,\tau_u} \mathbb{P}\{\boldsymbol{M}_{u,\tau_u} \boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u} K, \|\boldsymbol{M}_{u,\tau_u}\| \le s\}}{\mathbb{E}\{\nu(\boldsymbol{M}^{*-1}K)\mathbb{I}\{\|\boldsymbol{M}^*\| \le s\}\}} - 1 \right| = 0.$$

According to **A1** and **A3**, together with Lebesgue's monotone convergence theorem, we obtain that

$$\lim_{s\to\infty} \mathbb{E}\{\nu(\boldsymbol{M}^{*-1}K)\mathbb{I}\{\|\boldsymbol{M}^*\| \le s\}\} = \mathbb{E}\{\nu(\boldsymbol{M}^{*-1}K)\}.$$

Due to the relatively compactness of K, there exists $\epsilon > 0$, we have that

$$\begin{aligned}
\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K, \|\boldsymbol{M}_{u,\tau_u}\| > s\} &\le \mathbb{P}\{\|\boldsymbol{M}_{u,\tau_u}\|\|\boldsymbol{X}_{u,\tau_u}\| > \kappa a_{u,\tau_u}, \|\boldsymbol{M}_{u,\tau_u}\| > s\} \\
&\le \mathbb{P}\{\|\boldsymbol{X}_{u,\tau_u}\| > \epsilon a_{u,\tau_u}, \|\boldsymbol{M}_{u,\tau_u}\| > s\} + \mathbb{P}\{\frac{\|\boldsymbol{M}_{u,\tau_u}\|\|\boldsymbol{X}_{u,\tau_u}\|}{a_{u,\tau_u}} > \kappa, \frac{\|\boldsymbol{X}_{u,\tau_u}\|}{a_{u,\tau_u}} \le \epsilon\}.
\end{aligned}$$

By condition **A1**, it follows that

$$\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \left| \frac{b_{u,\tau_u}\mathbb{P}\{\|\boldsymbol{X}_{u,\tau_u}\| > \epsilon a_{u,\tau_u}, \|\boldsymbol{M}_{u,\tau_u}\| > s\}}{\epsilon^{-\alpha}\nu(\{\|\boldsymbol{x}\| > 1\})\mathbb{P}\{\|\boldsymbol{M}^*\| > s\}} - 1 \right| = 0,$$

where $\epsilon^{-\alpha}\nu(\{\|\boldsymbol{x}\| > 1\})\mathbb{P}\{\|\boldsymbol{M}^*\| > s\} = o(s)$, as $s \to \infty$.

Further, applying Markov's inequality and Assumption **A2**, we have that

$$\begin{aligned}
&\limsup_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{P}\{\frac{\|\boldsymbol{M}_{u,\tau_u}\|\|\boldsymbol{X}_{u,\tau_u}\|}{a_{u,\tau_u}} > \kappa, \frac{\|\boldsymbol{X}_{u,\tau_u}\|}{a_{u,\tau_u}} \le \epsilon\} \\
&\quad \le \kappa^{-\delta} \limsup_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{E}\{\left(\frac{\|\boldsymbol{M}_{u,\tau_u}\|\|\boldsymbol{X}_{u,\tau_u}\|}{a_{u,\tau_u}}\right)^{\delta} \mathbb{I}\{\|\boldsymbol{X}_{u,\tau_u}\| \le \epsilon a_{u,\tau_u}\}\} = 0.
\end{aligned}$$

Thus, we obtain that

$$\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \left| \frac{b_{u,\tau_u}\mathbb{P}\{\boldsymbol{M}_{u,\tau_u}\boldsymbol{X}_{u,\tau_u} \in a_{u,\tau_u}K\}}{\mathbb{E}\{\nu(\boldsymbol{M}^{*-1}K)\}} - 1 \right| = 0.$$

This completes the proof. □

5.4.2 Proof of Theorem 5.3.1

Note that, for any $u > 0$ and $K \in \mathcal{B}_{(0,\infty]}$, the following holds

$$\mathbb{P}\{\Theta\left(\{\xi_{u,\tau_u}(t) : t \in E\}\right) \in g_{u,\tau_u} + \frac{1}{g_{u,\tau_u}}\log(K)\} = \mathbb{P}\{M_{u,\tau_u}X_{u,\tau_u} \in e^{g^2_{u,\tau_u}}K\}$$

where $X_{u,\tau_u} := e^{g_{u,\tau_u}\xi_{u,\tau_u}(0)}$ and $M_{u,\tau_u} := e^{\Theta(\{\zeta_{u,\tau_u}(t):t\in E\})}$, with

$$\zeta_{u,\tau_u}(t) := g_{u,\tau_u}[\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(0)].$$

We begin the proof by analyzing the convergence property of the processes

$$\{\zeta_{u,\tau_u}(t) : t \in E\} \text{ and } \left\{\chi_{u,\tau_u}(t) := \left(\zeta_{u,\tau_u}(t) | g_{u,\tau_u}\xi_{u,\tau_u}(0) = g_{u,\tau_u}^2 + x\right) : t \in E\right\}, \text{ with } x \in \mathbb{R}.$$

Lemma 5.4.1 *Assume that **C0–C3** hold.*

(i) Then, as $u \to \infty$, uniformly for $\tau_u \in \mathcal{K}_u$,

$$\Theta(\{\zeta_{u,\tau_u}(t) : t \in E\}) \xrightarrow{d} \Theta(\{\sqrt{2}\eta(t) : t \in E\}). \tag{5.2}$$

(ii) Then, as $u \to \infty$, uniformly for $\tau_u \in \mathcal{K}_u$, and uniformly for $x \in [-w, w]$, for any $w > 0$,

$$\Theta(\{\chi_{u,\tau_u}(t) : t \in E\}) \xrightarrow{d} \Theta(\{\eta_h(t) : t \in E\}), \tag{5.3}$$

where $\{\eta_h(t) \equiv \sqrt{2}\eta(t) - \mathbb{V}ar\{\eta(t)\} - h(t) : t \in E\}$.

Proof of Lemma 5.4.1 Let

$$R_{\xi_{u,\tau_u}}(s,t) := \mathbb{C}ov(\xi_{u,\tau_u}(s), \xi_{u,\tau_u}(t)) = \frac{r_{u,\tau_u}(s,t)}{[1 + h_{u,\tau_u}(s)][1 + h_{u,\tau_u}(t)]}$$

with

$$r_{u,\tau_u}(s,t) = \mathbb{C}orr(Z_{u,\tau_u}(s), Z_{u,\tau_u}(t)).$$

(i) We have that

$$\zeta_{u,\tau_u}(0) = 0 \text{ a.s. and } \mathbb{E}\{\zeta_{u,\tau_u}(t)\} = 0, \tag{5.4}$$

and, as $u \to \infty$, uniformly for $\tau_u \in \mathcal{K}_u$,

$$\begin{aligned}
&\mathbb{V}ar\{\zeta_{u,\tau_u}(t) - \zeta_{u,\tau_u}(s)\} = \mathbb{V}ar\{g_{u,\tau_u}[\xi_{u,\tau_u}(t) - \xi_{u,\tau_u}(s)]\} \qquad (5.5)\\
&= g_{u,\tau_u}^2 \left[\mathbb{V}ar\{\xi_{u,\tau_u}(t)\} - 2\mathbb{C}orr(\xi_{u,\tau_u}(t), \xi_{u,\tau_u}(s)) + \mathbb{V}ar\{\xi_{u,\tau_u}(s)\}\right]\\
&= g_{u,\tau_u}^2 \left[\frac{1}{[1 + h_u(t)]^2} - 2\frac{r_{u,\tau_u}(s,t)}{[1 + h_u(s)][1 + h_u(t)]} + \frac{1}{[1 + h_u(s)]^2}\right]\\
&= g_{u,\tau_u}^2 \frac{[1 + h_u(s)]^2 - 2r_{u,\tau_u}(s,t)[1 + h_u(s)][1 + h_u(t)] + [1 + h_u(t)]^2}{[1 + h_u(s)]^2[1 + h_u(t)]^2}\\
&= \frac{g_{u,\tau_u}^2 \left\{2[1 - r_{u,\tau_u}(s,t)][1 + (h_u(s) + h_u(t))] + [h_u^2(s) - 2r_{u,\tau_u}(s,t)h_u(s)h_u(t) + h_u^2(t)]\right\}}{[1 + h_u(s)]^2[1 + h_u(t)]^2}\\
&= \frac{2g_{u,\tau_u}^2[1 - r_{u,\tau_u}(s,t)]}{[1 + h_u(s)]^2[1 + h_u(t)]^2}
\end{aligned}$$

$$+\frac{g_{u,\tau_u}^2\left\{2[1-r_{u,\tau_u}(s,t)][h_u(s)+h_u(t)]+[h_u^2(s)-2r_{u,\tau_u}(s,t)h_u(s)h_u(t)+h_u^2(t)]\right\}}{[1+h_u(s)]^2[1+h_u(t)]^2}$$
$$=\frac{2g_{u,\tau_u}^2[1-r_{u,\tau_u}(s,t)]}{[1+h_u(s)]^2[1+h_u(t)]^2}+o(1).$$

According to assumptions **C1**–**C3**, we obtain that

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u,s,t\in E}\left|\frac{\mathbb{V}ar\{\zeta_{u,\tau_u}(t)-\zeta_{u,\tau_u}(s)\}}{g_{u,\tau_u}^2\mathbb{V}ar\{Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s)\}}-1\right|=0 \tag{5.6}$$

and

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u,s,t\in E}\left|\frac{\mathbb{V}ar\{\zeta_{u,\tau_u}(t)-\zeta_{u,\tau_u}(s)\}}{2\mathbb{V}ar\{\eta(t)-\eta(s)\}}-1\right|=0. \tag{5.7}$$

Hence, by Lemma 4.1 in [133], as $u\to\infty$, the finite dimensional distributions of $\{\zeta_{u,\tau_u}(t):t\in E\}$ converge uniformly with respect to $\tau_u\in\mathcal{K}_u$ to those of $\{\sqrt{2}\eta(t):t\in E\}$. Since condition **C3** and (5.4) hold, Proposition 9.7 in [119] implies the uniform tightness of $\{\zeta_{u,\tau_u}(t):t\in E\}$. Hence, $\{\zeta_{u,\tau_u}(t):t\in E\}$ weakly converges to $\{\sqrt{2}\eta(t):t\in E\}$ as $u\to\infty$, uniformly for $\tau_u\in\mathcal{K}_u$. Further, due to $\lim_{u\to\infty}\sup_{t\in E,\tau_u\in\mathcal{K}_u}h_{u,\tau_u}(t)=0$, we obtain that, as $u\to\infty$, $\{\zeta_{u,\tau_u}(t):t\in E\}$ converges weakly to $\{\sqrt{2}\eta(t):t\in E\}$, uniformly for $\tau_u\in\mathcal{K}_u$. Consequently, the continuity of the functional $\Theta(\cdot)$ together with the continous mapping theorem imply that, as $u\to\infty$, uniformly for $\tau_u\in\mathcal{K}_u$,

$$\Theta(\{\zeta_{u,\tau_u}(t):t\in E\})\xrightarrow{d}\Theta(\{\sqrt{2}\eta(t):t\in E\}).$$

(ii) Note that

$$\chi_{u,\tau_u}(t)=\zeta_{u,\tau_u}(t)-\frac{\mathbb{C}ov(\zeta_{u,\tau_u}(t),g_{u,\tau_u}\xi_{u,\tau_u}(0))}{\mathbb{V}ar\{g_{u,\tau_u}\xi_{u,\tau_u}(0)\}}g_{u,\tau_u}\xi_{u,\tau_u}(0)+\frac{\mathbb{C}ov(\zeta_{u,\tau_u}(t),g_{u,\tau_u}\xi_{u,\tau_u}(0))}{\mathbb{V}ar\{g_{u,\tau_u}\xi_{u,\tau_u}(0)\}}\left(g_{u,\tau_u}^2+x\right)$$
$$=g_{u,\tau_u}[\xi_{u,\tau_u}(t)-\xi_{u,\tau_u}(0)]+g_{u,\tau_u}[1-R_{\xi_{u,\tau_u}}(t,0)]\xi_{u,\tau_u}(0)-[1-R_{\xi_{u,\tau_u}}(t,0)]\left(g_{u,\tau_u}^2+x\right)$$
$$=g_{u,\tau_u}[\xi_{u,\tau_u}(t)-R_{\xi_{u,\tau_u}}(t,0)\xi_{u,\tau_u}(0)]-[1-R_{\xi_{u,\tau_u}}(t,0)]\left(g_{u,\tau_u}^2+x\right)$$
$$=\frac{1}{1+h_u(t)}\left\{g_{u,\tau_u}[Z_{u,\tau_u}(t)-r_{u,\tau_u}(t,0)Z_{u,\tau_u}(0)]-(g_{u,\tau_u}^2+x)[(1-r_{u,\tau_u}(t,0))+h_u(t)]\right\}.$$

Hence $\chi_{u,\tau_u}(0)=0$ a.s. Note that

$$\mathbb{E}\{\chi_{u,\tau_u}(t)\}=-\frac{g_{u,\tau_u}^2[(1-r_{u,\tau_u}(t,0))+h_u(t)]}{1+h_u(t)}-\frac{x[(1-r_{u,\tau_u}(t,0))+h_u(t)]}{1+h_u(t)},$$
$$\mathbb{C}ov(\chi_{u,\tau_u}(s),\chi_{u,\tau_u}(t))=\frac{g_{u,\tau_u}^2[r_{u,\tau_u}(s,t)-r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0)]}{[1+h_u(s)][1+h_u(t)]},$$

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$$\mathbb{V}ar\{\chi_{u,\tau_u}(t)\} = \frac{g^2_{u,\tau_u}[1-r^2_{u,\tau_u}(t,0)]}{[1+h_u(t)]^2} = \frac{g^2_{u,\tau_u}[1-r_{u,\tau_u}(t,0)][1+r_{u,\tau_u}(t,0)]}{[1+h_u(t)]^2}.$$

Hence

$$\begin{aligned}
&\frac{[1+h_u(s)]^2[1+h_u(t)]^2}{g^2_{u,\tau_u}}\mathbb{V}ar\{\chi_{u,\tau_u}(t)-\chi_{u,\tau_u}(s)\}\\
&= \frac{[1+h_u(s)]^2[1+h_u(t)]^2}{g^2_{u,\tau_u}}\left\{\mathbb{V}ar\{\chi_{u,\tau_u}(t)\} - 2\mathbb{C}ov(\chi_{u,\tau_u}(t),\chi_{u,\tau_u}(s)) + \mathbb{V}ar\{\chi_{u,\tau_u}(s)\}\right\}\\
&= [1-r^2_{u,\tau_u}(t,0)][1+2h_u(s)+h^2_u(s)]\\
&- 2[r_{u,\tau_u}(s,t)-r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0)][1+h_u(s)+h_u(t)+h_u(s)h_u(t)]\\
&+ [1-r^2_{u,\tau_u}(s,0)][1+2h_u(t)+h^2_u(t)]\\
&= 1+2h_u(s)-2h_u(s)r^2_{u,\tau_u}(t,0)+h^2_u(s)[1-r^2_{u,\tau_u}(t,0)]-r^2_{u,\tau_u}(t,0)\\
&- 2\Big\{r_{u,\tau_u}(s,t)+r_{u,\tau_u}(s,t)h_u(s)+r_{u,\tau_u}(s,t)h_u(t)+r_{u,\tau_u}(s,t)h_u(s)h_u(t)\\
&\quad - r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0) - h_u(s)r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0) - h_u(t)r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0)\\
&\quad - h_u(s)h_u(t)r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0)\Big\}\\
&+ 1+2h_u(t)-2h_u(t)r^2_{u,\tau_u}(s,0)+h^2_u(t)[1-r^2_{u,\tau_u}(s,0)]-r^2_{u,\tau_u}(s,0)\\
&= 2[1-r_{u,\tau_u}(s,t)]\\
&+ 2h_u(s)[1-r_{u,\tau_u}(s,t)]+2h_u(t)[1-r_{u,\tau_u}(s,t)]\\
&- 2h_u(s)r_{u,\tau_u}(t,0)[(1-r_{u,\tau_u}(s,0))-(1-r_{u,\tau_u}(t,0))]\\
&- 2h_u(t)r_{u,\tau_u}(s,0)[(1-r_{u,\tau_u}(t,0))-(1-r_{u,\tau_u}(s,0))]\\
&+ h^2_u(s)[1-r^2_{u,\tau_u}(t,0)]+h^2_u(t)[1-r^2_{u,\tau_u}(s,0)]\\
&\quad + 2h_u(s)h_u(t)r_{u,\tau_u}(s,0)r_{u,\tau_u}(t,0) - 2r_{u,\tau_u}(s,t)h_u(s)h_u(t)\\
&- [(1-r_{u,\tau_u}(s,0))-(1-r_{u,\tau_u}(t,0))]^2\\
&= 2[1-r_{u,\tau_u}(s,t)] + o(\frac{1}{g^2_{u,\tau_u}}),
\end{aligned}$$

as $u\to\infty$, uniformly for $\tau_u\in\mathcal{K}_u$.

According to assumptions **C1**–**C3**, we obtain that

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u,t\in E}\left|\frac{g^2_{u,\tau_u}h_{u,\tau_u}(t)+g^2_{u,\tau_u}[1-r_{u,\tau_u}(t,0)]}{\mathbb{V}ar\{\eta(t)\}+h(t)}-1\right| = 0, \tag{5.8}$$

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u,s,t\in E}\left|\frac{\mathbb{V}ar\{\chi_{u,\tau_u}(t)-\chi_{u,\tau_u}(s)\}}{g^2_{u,\tau_u}\mathbb{V}ar\{Z_{u,\tau_u}(t)-Z_{u,\tau_u}(s)\}}-1\right| = 0$$

and

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u,s,t\in E}\left|\frac{\mathbb{V}ar\{\chi_{u,\tau_u}(t)-\chi_{u,\tau_u}(s)\}}{2\mathbb{V}ar\{\eta(t)-\eta(s)\}}-1\right|=0.$$

Hence, by Lemma 4.1 in [133], as $u\to\infty$, the finite dimensional distributions of $\{\chi_{u,\tau_u}(t):t\in E\}$ converge to those of $\{\eta_h(t)\equiv\sqrt{2}\eta(t)-\mathbb{V}ar\{\eta(t)\}-h(t):t\in E\}$, the convergence being uniformly for $\tau_u\in\mathcal{K}_u$. Since condition **C3** and (5.8) hold, Proposition 9.7 in [119] implies the uniform tightness of $\{\chi_{u,\tau_u}(t):t\in E\}$. Hence, $\{\chi_{u,\tau_u}(t):t\in E\}$ weakly converges to $\{\eta_h(t):t\in E\}$ as $u\to\infty$, uniformly for $\tau_u\in\mathcal{K}_u$. Further, due to $\lim_{u\to\infty}\sup_{t\in E,\tau_u\in\mathcal{K}_u}h_{u,\tau_u}(t)=0$, we obtain that, as $u\to\infty$, $\{\chi_{u,\tau_u}(t):t\in E\}$ converges weakly to $\{\eta_h(t):t\in E\}$, uniformly for $\tau_u\in\mathcal{K}_u$. Consequently, the continuity of the functional Θ together with the continous mapping theorem imply that, as $u\to\infty$, uniformly for $\tau_u\in\mathcal{K}_u$,

$$\Theta(\{\chi_{u,\tau_u}(t):t\in E\})\xrightarrow{d}\Theta(\{\eta_h(t):t\in E\}).$$

This completes the proof. □

Proof of Theorem 5.3.1 Recall $\tilde{\mathbb{E}}\equiv\tilde{\mathbb{E}}_1=[0,\infty]$ and let define $a_{u,\tau_u}:=e^{g^2_{u,\tau_u}}$, $b_{u,\tau_u}:=\sqrt{2\pi}g_{u,\tau_u}e^{\frac{g^2_{u,\tau_u}}{2}}$. We show that the assumptions of Theorem 5.2.1 are satisfied.

Verification of **A1**. Let $K=K_X\times K_M$, where $K_X=[A,B]$, $K_M=[C,D]$ with $B\geq A>0$, $D\geq C\geq 0$. We have

$$\begin{aligned}
b_{u,\tau_u}\mathbb{P}\{\left(\frac{X_{u,\tau_u}}{a_{u,\tau_u}},M_{u,\tau_u}\right)\in K\}&=b_{u,\tau_u}\mathbb{P}\{\frac{X_{u,\tau_u}}{a_{u,\tau_u}}\in K_X,M_{u,\tau_u}\in K_M\}\\
&=b_{u,\tau_u}\mathbb{P}\{g_{u,\tau_u}\xi_{u,\tau_u}(0)\in g^2_{u,\tau_u}+\log(K_X),\Theta(\{\zeta_{u,\tau_u}(t):t\in E\})\in\log(K_M)\}\\
&=b_{u,\tau_u}\int_{\log(C)}^{\log(D)}\mathbb{P}\{\Theta(\{\zeta_{u,\tau_u}(t):t\in E\})\in\log(K_M)|\vartheta_u=g^2_{u,\tau_u}+x\}dF_{\vartheta_u}(g^2_{u,\tau_u}+x)\\
&=b_{u,\tau_u}\int_{\log(C)}^{\log(D)}\mathbb{P}\{\Theta(\{(\zeta_{u,\tau_u}(t)|g_{u,\tau_u}\xi_{u,\tau_u}(0)=g^2_{u,\tau_u}+x):t\in E\})\in\log(K_M)\}dF_{\vartheta_u}(g^2_{u,\tau_u}+x),
\end{aligned}$$

with $\vartheta_u:=g_{u,\tau_u}\xi_{u,\tau_u}(0)$.

Lemma 5.4.1 (ii) gives that

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u}\sup_{x\in[\log(C),\log(D)]}\left|\frac{\mathbb{P}\{\Theta(\{(\zeta_{u,\tau_u}(t)|g_{u,\tau_u}\xi_{u,\tau_u}(0)=g^2_{u,\tau_u}+x):t\in E\})\in\log(K_M)\}}{\mathbb{P}\{\Theta(\{\eta_h(t):t\in E\})\in\log(K_M)\}}-1\right|=0.$$

Further, applying Lebesgue's dominated convergence theorem we obtain

$$\lim_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u}b_{u,\tau_u}\mathbb{P}\{\left(\frac{X_{u,\tau_u}}{a_{u,\tau_u}},M_{u,\tau_u}\right)\in K\}$$

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$$
\begin{aligned}
&= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u} \int_{\log(C)}^{\log(D)} \mathbb{P}\{\Theta(\{\eta_h(t) : t \in E\}) \in \log(K_M)\} dF_{g_{u,\tau_u}\xi_{u,\tau_u}(0)}(g_{u,\tau_u}^2 + x) \\
&= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u} \mathbb{P}\{g_{u,\tau_u}\xi_{u,\tau_u}(0) \in g_{u,\tau_u}^2 + \log(K_X)\}\mathbb{P}\{\Theta(\{\eta_h(t) : t \in E\}) \in \log(K_M)\} \\
&= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u} \mathbb{P}\{e^{g_{u,\tau_u}\xi_{u,\tau_u}(0)} \in e^{g_{u,\tau_u}^2} K_X\}\mathbb{P}\{e^{\Theta(\{\eta_h(t):t\in E\})} \in K_M\} \\
&= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \nu_u(K_X)G(K_M),
\end{aligned}
$$

where

$$
\nu_u(K_X) := b_{u,\tau_u}\mathbb{P}\{\frac{X_{u,\tau_u}}{a_{u,\tau_u}} \in K_X\} \text{ and } G(K_M) := \mathbb{P}\{e^{\Theta(\{\eta_h(t):t\in E\})} \in K_M\}.
$$

Note that for each $B \geq 0$, $G(\{B\}) = 0$ and $G(\{\infty\}) = 0$.

Next, we analyze uniform convergence of $\nu_u(K_X)$ with respect to $\tau_u \in \mathcal{K}_u$, as $u \to \infty$. Observe that, for $A > 0$, we have that

$$
\begin{aligned}
&\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \nu_u([A,\infty]) = \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{P}\{X_{u,\tau_u} \geq A a_{u,\tau_u}\} \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \sqrt{2\pi} g_{u,\tau_u} e^{\frac{g_{u,\tau_u}^2}{2}} \mathbb{P}\{e^{g_{u,\tau_u}\xi_{u,\tau_u}(0)} \geq A e^{g_{u,\tau_u}^2}\} \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \sqrt{2\pi} g_{u,\tau_u} e^{\frac{g_{u,\tau_u}^2}{2}} \mathbb{P}\{Z_{u,\tau_u}(0) \geq \frac{\log(A e^{g_{u,\tau_u}^2})}{g_{u,\tau_u}}\} \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \sqrt{2\pi} g_{u,\tau_u} e^{\frac{g_{u,\tau_u}^2}{2}} \Psi\left(\frac{\log(A) + g_{u,\tau_u}^2}{g_{u,\tau_u}}\right) \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \sqrt{2\pi} g_{u,\tau_u} e^{\frac{g_{u,\tau_u}^2}{2}} \frac{1}{\sqrt{2\pi}} \frac{g_{u,\tau_u}}{\log(A) + g_{u,\tau_u}^2} e^{-\frac{\left[\log(A)+g_{u,\tau_u}^2\right]^2}{2g_{u,\tau_u}^2}} \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} A^{-1}\sqrt{2\pi} g_{u,\tau_u} e^{\frac{g_{u,\tau_u}^2}{2}} \frac{1}{\sqrt{2\pi}} \frac{1}{g_{u,\tau_u}} e^{-\frac{g_{u,\tau_u}^2}{2}} = A^{-1} = \nu([A,\infty]).
\end{aligned}
$$

Hence for each $A > 0$, $\nu(\{A\}) = 0$ and $\nu(\{\infty\}) = 0$, and for $K_X = [A, B]$, it holds

$$
\begin{aligned}
&\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \nu_u(K_x) = \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{P}\{\frac{X_{u,\tau_u}}{a_{u,\tau_u}} \in K_X\} \\
&\quad= \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{P}\{\frac{X_{u,\tau_u}}{a_{u,\tau_u}} \geq A\} - \lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} b_{u,\tau_u}\mathbb{P}\{\frac{X_{u,\tau_u}}{a_{u,\tau_u}} \geq B\} \\
&\quad= A^{-1} - B^{-1} = \nu([A,\infty]) - \nu([B,\infty]) = \nu([A,B]).
\end{aligned}
$$

Thus, for every relatively compact $K \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}\times\tilde{\mathbb{E}}}$ such that $(\nu \times G)(\partial K) = 0$, it holds

$$
\lim_{u\to\infty} \sup_{\tau_u \in \mathcal{K}_u} \left| \frac{b_{u,\tau_u}\mathbb{P}\{\left(\frac{X_{u,\tau_u}}{a_{u,\tau_u}}, M_{u,\tau_u}\right) \in K\}}{(\nu \times G)(K)} - 1 \right| = 0,
$$

where $\nu(A) = \int_A \frac{1}{x^2}dx$ for every relatively compact $A \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}}$ such that $\nu(\partial A) = 0$, $\nu(\cdot)$ is homogenous with index $\alpha = 1$, and $G(B) = \mathbb{P}\{e^{\Theta(\{\eta_h(t):t\in E\})} \in B\}$ for $B \in \mathcal{B}_{\tilde{\mathbb{E}}}$.

Verification of **A2**. For $\delta = 1$, we obtain that

$$b_{u,\tau_u}\mathbb{E}\{\left(\frac{X_{u,\tau_u}M_{u,\tau_u}}{a_{u,\tau_u}}\right)^\delta \mathbb{I}\{X_{u,\tau_u} < \epsilon a_{u,\tau_u}\}\} = b_{u,\tau_u}\mathbb{E}\{\left(\frac{e^{\Theta(\{\xi_{u,\tau_u}(t):t\in E\})}}{a_{u,\tau_u}}\right)\mathbb{I}\{X_{u,\tau_u} < \epsilon a_{u,\tau_u}\}\}$$
$$= \frac{b_{u,\tau_u}}{a_{u,\tau_u}}\mathbb{E}\{e^{\Theta(\{\delta\xi_{u,\tau_u}(t):t\in E\})}\mathbb{I}\{X_{u,\tau_u} < \epsilon a_{u,\tau_u}\}\} \le \frac{b_{u,\tau_u}}{a_{u,\tau_u}}\mathbb{E}\{e^{\Theta(\{\delta\xi_{u,\tau_u}(t):t\in E\})}\}.$$

The combination of (5.5), (5.6), (5.7), **F1** with the Sudakov-Fernique inequality (Theorem 2.2.3 in [1]) implies, for all sufficiently large $u > 0$,

$$m := \mathbb{E}\{\Theta\left(\{\xi_{u,\tau_u}(t) : t \in E\}\right)\} \le \mathbb{E}\{\sup_{t\in E}(\xi_{u,\tau_u}(t))\} \le C\mathbb{E}\{\sup_{t\in E}(\sqrt{2}\eta(t))\} < \infty.$$

Further, Borell-TIS inequality (Theorem 2.1.1 in [1]) gives that

$$\mathbb{P}\{\sup_{t\in E}(\xi_{u,\tau_u}(t)) > x\} \le \exp\left(-\frac{(x-m)^2}{2C}\right) \text{ for all } x \ge m,$$

with $C = \sup_{t\in E} \mathbb{V}ar\{\xi_{u,\tau_u}(t)\} = \sup_{t\in E} \frac{1}{[1+h_{u,\tau_u}(t)]^2} < \infty$, where the finiteness of C follows from **C1**. Thus, we obtain that

$$\mathbb{E}\{e^{\Theta(\{\xi_{u,\tau_u}(t):t\in E\})}\} \le \mathbb{E}\{e^{\sup_{t\in E}(\xi_{u,\tau_u}(t))}\} = \int_{-\infty}^{\infty} e^x\mathbb{P}\{e^{\sup_{t\in E}(\xi_{u,\tau_u}(t))} > x\}dx$$
$$= \int_{-\infty}^{m} e^x\mathbb{P}\{e^{\sup_{t\in E}(\xi_{u,\tau_u}(t))} > x\}dx + \int_{m}^{\infty} e^x\mathbb{P}\{e^{\sup_{t\in E}(\xi_{u,\tau_u}(t))} > x\}dx \le e^m + \int_m^\infty e^x e^{-\frac{(x-m)^2}{2C}}dx$$
$$\le e^m + \int_{-\infty}^{\infty} e^x e^{-\frac{(x-m)^2}{2C}}dx = e^m + \sqrt{2\pi C}e^{\frac{C}{2}+m} < \infty.$$

Hence

$$b_{u,\tau_u}\mathbb{E}\{\left(\frac{X_{u,\tau_u}M_{u,\tau_u}}{a_{u,\tau_u}}\right)^\delta \mathbb{I}\{X_{u,\tau_u} < \epsilon a_{u,\tau_u}\}\} \le \widetilde{C}\frac{b_{u,\tau_u}}{a_{u,\tau_u}} = \widetilde{C}\frac{\sqrt{2\pi}g_{u,\tau_u}e^{\frac{g^2_{u,\tau_u}}{2}}}{e^{g^2_{u,\tau_u}}} = \widetilde{C}\sqrt{2\pi}g_{u,\tau_u}e^{-\frac{1}{2}g^2_{u,\tau_u}},$$

where $\widetilde{C} > 0$.

Putting $u \to \infty$ and $\epsilon \to 0$, we obtain that

$$\lim_{\epsilon\to 0}\limsup_{u\to\infty}\sup_{\tau_u\in\mathcal{K}_u} b_{u,\tau_u}\mathbb{E}\{\left(\frac{X_{u,\tau_u}M_{u,\tau_u}}{a_{u,\tau_u}}\right)^\delta \mathbb{I}\{X_{u,\tau_u} < \epsilon a_{u,\tau_u}\}\} = \lim_{\epsilon\to 0} 0 = 0.$$

Verification of **A3**. Condition **F1** gives that

$$\int_0^\infty x dG(x) = \mathbb{E}\{e^{\Theta(\{\eta_h(t):t\in E\})}\} < \infty.$$

Thus, Theorem 5.2.1 implies the thesis. □

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5.4.3 Proofs of Theorems 5.3.5 and 5.3.6

Proof of Theorem 5.3.5 Let $a_u \equiv a_{u,\tau_u} := \frac{1}{\sqrt{2\pi\sigma_Y^2(T)}} e^{-\frac{(\mu T)^2}{2\sigma_Y^2(T)}} u$, $b_u \equiv b_{u,\tau_u} = u$ and $Y_\mu(t) = Y(t) + \mu(t)$. Note that, for any $u > 0$ and $K \in \mathcal{B}_{(0,\infty]}$, it holds

$$\begin{aligned}
&\mathbb{P}\{\sup_{t\in[0,T]} \left(\frac{Y_\mu(t)}{Y_\mu(T)}\right) \in a_u K\} \\
&= \mathbb{P}\{\frac{\sup_{t\in[0,T]}(Y_\mu(t))}{Y_\mu(T)} \in a_u K, Y_\mu(T) > 0\} + \mathbb{P}\{\frac{\sup_{t\in[0,T]}(-Y_\mu(t))}{-Y_\mu(T)} \in a_u K, Y_\mu(T) < 0\} \\
&= \mathbb{P}\{\frac{\sup_{t\in[0,T]}(Y_\mu(t))}{Y_\mu(T)} \in a_u K\} + \mathbb{P}\{\frac{\sup_{t\in[0,T]}(-Y_\mu(t))}{-Y_\mu(T)} \in a_u K\} \\
&= \mathbb{P}\{M_+ X_+ \in a_u K\} + \mathbb{P}\{M_- X_- \in a_u K\},
\end{aligned}$$

where $M_\pm = \sup_{t\in[0,T]} (\pm Y_\mu(t))$ and $X_\pm = \frac{1}{\pm Y_\mu(T)}$.

Recall $\tilde{\mathbb{E}} \equiv \tilde{\mathbb{E}}_1 = [0,\infty]$. We show that the assumptions of Theorem 5.2.1 are satisfied for $M \equiv M_+$ and $X \equiv X_+$. The argument for M_-, X_- follows in a similar way.

Verification of **A1**. Let $K = [A,\infty] \times [B,\infty]$ for $A > 0$, $B \geq 0$, then

$$\begin{aligned}
b_u \mathbb{P}\{\left(\frac{X}{a_u}, M\right) \in K\} &= b_u \mathbb{P}\{\frac{X}{a_u} \geq A, M \geq B\} = b_u \mathbb{P}\{\frac{1}{Y_\mu(T)} \geq A a_u, \sup_{t\in[0,T]} (Y_\mu(t)) \geq B\} \\
&= b_u \mathbb{P}\{0 < Y_\mu(T) \leq A^{-1} a_u^{-1}, \sup_{t\in[0,T]} (Y_\mu(t)) \geq B\} \\
&= b_u \int_0^{\frac{1}{A a_u}} \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = x\} dF_{Y_\mu(T)}(x).
\end{aligned}$$

Let $C_R := \sup_{t\in[0,T]} \left|\frac{R_Y(t,T)}{R_Y(T,T)}\right| < \infty$. By gaussianity of the process $\{Y(t) : t \geq 0\}$, we have that, for $x > 0$,

$$\begin{aligned}
&\mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B + C_R x | Y_\mu(T) = 0\} = \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t) - \frac{R_Y(t,T)}{R_Y(T,T)} Y_\mu(T)) - C_R x \geq B\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = x\} = \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t) - \frac{R_Y(t,T)}{R_Y(T,T)} Y_\mu(T) + \frac{R_Y(t,T)}{R_Y(T,T)} x) \geq B\} \\
&\leq \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t) + \frac{R_Y(t,T)}{R_Y(T,T)} Y_\mu(T)) + C_R x \geq B\} = \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B - C_R x | Y_\mu(T) = 0\}.
\end{aligned}$$

Hence, it holds that

$$\lim_{u\to\infty} \sup_{x\in[0,\frac{1}{A a_u}]} \left| \frac{\mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = x\}}{\mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = 0\}} - 1 \right| = 0.$$

Applying Lebesgue's dominated convergence theorem, we obtain, as $u \to \infty$, for $K = [A, \infty] \times [B, \infty]$,

$$\begin{aligned}\lim_{u\to\infty} b_u \mathbb{P}\{\left(\frac{X}{a_u}, M\right) \in K\} &= \lim_{u\to\infty} b_u \mathbb{P}\{\frac{1}{Y_\mu(T)} \geq A a_u, \sup_{t\in[0,T]} (Y_\mu(t)) \geq B\} \\ &= \lim_{u\to\infty} b_u \int_0^{\frac{1}{Aa_u}} \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = x\} dF_{Y_\mu(T)}(x) \\ &= \lim_{u\to\infty} b_u \int_0^{\frac{1}{Aa_u}} \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = 0\} dF_{Y_\mu(T)}(x) \\ &= \lim_{u\to\infty} b_u \mathbb{P}\{\frac{1}{Y_\mu(T)} \geq A a_u\} \cdot \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = 0\} \\ &= \lim_{u\to\infty} \nu_u([A, \infty]) G([B, \infty]),\end{aligned}$$

where

$$\nu_u([A, \infty]) := b_u \mathbb{P}\{\frac{1}{Y_\mu(T)} \in a_u[A, \infty]\} \text{ and } G([B, \infty]) := \mathbb{P}\{\sup_{t\in[0,T]} (Y_\mu(t)) \geq B | Y_\mu(T) = 0\}.$$

Note that, for each $B \geq 0$, $G(\{B\}) = 0$ and $G(\{\infty\}) = 0$.

Further, observe that, for $A > 0$, we have that

$$\begin{aligned}\lim_{u\to\infty} \nu_u([A, \infty]) &= \lim_{u\to\infty} b_u \mathbb{P}\{X \geq A a_u\} = \lim_{u\to\infty} b_u \mathbb{P}\{\frac{1}{Y_\mu(T)} \geq A a_u\} \\ &= \lim_{u\to\infty} A^{-1} a_u^{-1} u \frac{\mathbb{P}\{0 < Y_\mu(T) \leq A^{-1} a_u^{-1}\}}{A^{-1} a_u^{-1}} = A^{-1} \sqrt{2\pi\sigma_Y^2(T)} e^{\frac{(\mu T)^2}{2\sigma_Y^2(T)}} \lim_{u\to\infty} \frac{\mathbb{P}\{0 < Y_\mu(T) \leq A^{-1} a_u^{-1}\}}{A^{-1} a_u^{-1}} \\ &= A^{-1} \sqrt{2\pi\sigma_Y^2(T)} e^{\frac{(\mu T)^2}{2\sigma_Y^2(T)}} \frac{1}{\sqrt{2\pi\sigma_Y^2(T)}} e^{-\frac{(\mu T)^2}{2\sigma_Y^2(T)}} = A^{-1} = \nu([A, \infty]).\end{aligned}$$

Hence for each $A > 0$, $\nu(\{A\}) = 0$ and $\nu(\{\infty\}) = 0$.

Thus, for every relatively compact $K \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}\times\tilde{\mathbb{E}}}$ such that $(\nu \times G)(\partial K) = 0$, it holds

$$\lim_{u\to\infty} \left| \frac{b_{u,\tau_u} \mathbb{P}\{\left(\frac{X}{a_u}, M\right) \in K\}}{(\nu \times G)(K)} - 1 \right| = 0,$$

where $\nu(A) = \int_A \frac{1}{x^2} dx$ for every relatively compact $A \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}}$ such that $\nu(\partial A) = 0$, $\nu(\cdot)$ is homogenous with index $\alpha = 1$, and $G(B) = \mathbb{P}\{(\sup_{t\in[0,T]} (Y_\mu(t) | Y_\mu(T) = 0) \in B\}$ for $B \in \mathcal{B}_{\tilde{\mathbb{E}}}$.

<u>Verification of **A2**.</u> We obtain that

$$\begin{aligned}&b_u \mathbb{E}\{\left(\frac{M|X|}{a_u}\right)^\delta \mathbb{I}\{|X| < \epsilon a_u\}\} \\ &\quad = b_u \mathbb{E}\{\left(\frac{M|X|}{a_u}\right)^\delta \mathbb{I}\{|X| \leq 1\}\} + b_u \mathbb{E}\{\left(\frac{M|X|}{a_u}\right)^\delta \mathbb{I}\{1 < |X| < \epsilon a_u\}\}\end{aligned}$$

$$\le \frac{b_u}{a_u^\delta}\mathbb{E}\{M^\delta\} + \frac{b_u}{a_u^\delta}\int_{\{1<|x|<\epsilon a_u\}} \mathbb{E}\{M^\delta|X=x\}dF_X(x)$$
$$= \frac{b_u}{a_u^\delta}\mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t))^\delta\} + \frac{b_u}{a_u^\delta}\int_{\{1<|x|<\epsilon a_u\}} \mathbb{E}\{\sup_{t\in[0,T]}\left\{Y_\mu(t)|Y_\mu(T)=\frac{1}{x}\right\}^\delta\}dF_{\frac{1}{Y_\mu(T)}}(x).$$

Let $Z_\mu(t) = Y_\mu(t) - \frac{R_Y(t,T)}{R_Y(T,T)}Y_\mu(T)$ and recall $C_R = \sup_{t\in[0,T]}\left|\frac{R_Y(t,T)}{R_Y(T,T)}\right| < \infty$. Straightforward calculations give, for $\delta = 2$,

$$\mathbb{E}\{\sup_{t\in[0,T]}\left\{Y_\mu(t)|Y_\mu(T)=\frac{1}{x}\right\}^\delta\} = \mathbb{E}\{\sup_{t\in[0,T]}\left\{Y_\mu(t) - \frac{R_Y(t,T)}{R_Y(T,T)}Y_\mu(T) + \frac{R_Y(t,T)}{R_Y(T,T)}\frac{1}{x}\right\}^2\}$$
$$= \mathbb{E}\{\sup_{t\in[0,T]}\left\{Z_\mu(t) + \frac{R_Y(t,T)}{R_Y(T,T)}\frac{1}{x}\right\}^2\}$$
$$= \mathbb{E}\{\sup_{t\in[0,T]}\left(Z_\mu(t)^2 + 2\frac{R_Y(t,T)}{R_Y(T,T)}\frac{1}{x}Z_\mu(t) + \left\{\frac{R_Y(t,T)}{R_Y(T,T)}\frac{1}{x}\right\}^2\right)\}$$
$$\le \mathbb{E}\{\sup_{t\in[0,T]}(Z_\mu(t)^2)\} + 2C_R\frac{1}{x}\mathbb{E}\{\sup_{t\in[0,T]}(Z_\mu(t))\} + C_R^2\frac{1}{x^2}$$
$$\le \mathbb{E}\{\sup_{t\in[0,T]}(Z_\mu(t)^2)\} + 2C_R\frac{1}{|x|}\mathbb{E}\{\sup_{t\in[0,T]}(Z_\mu(t))\} + C_R^2\frac{1}{x^2}.$$

Then

$$\begin{aligned}&\mathbb{E}\{\sup_{t\in[0,T]}\left\{Y_\mu(t)|Y_\mu(T)=\frac{1}{x}\right\}^2\}\\ &\le \mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t)|Y_\mu(T)=0)^2\} + 2C_R\frac{1}{|x|}\mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t)|Y_\mu(T)=0)\} + C_R^2\frac{1}{x^2}.\end{aligned} \tag{5.9}$$

Hence, for $\delta = 2$, we have that

$$\frac{b_u}{a_u^2}\int_{\{1<|x|<\epsilon a_u\}} \mathbb{E}\{\sup_{t\in[0,T]}\left\{Y_\mu(t)|Y_\mu(T)=\frac{1}{x}\right\}^2\}dF_{\frac{1}{Y_\mu(T)}}(x)$$
$$\le \frac{b_u}{a_u^2}\int_{\{1<|x|<\epsilon a_u\}}\left[C_1 + C_2\frac{1}{|x|} + C_3\frac{1}{x^2}\right]dF_{\frac{1}{Y_\mu(T)}}(x)$$
$$\le \frac{b_u}{a_u^2}\int_{\{1<|x|\}}[C_1+C_2+C_3]\,dF_{\frac{1}{Y_\mu(T)}}(x) \le C\frac{b_u}{a_u^2}\mathbb{P}\{\left|\frac{1}{Y_\mu(T)}\right| > 1\} \le C\frac{b_u}{a_u^2},$$

where the finiteness of the above expectated values follows from Borell-TIS inequality (see Theorem 2.1.1 in [1]).

Putting $u\to\infty$ and $\epsilon\to 0$, we obtain that

$$\lim_{\epsilon\to0}\limsup_{u\to\infty} b_u\mathbb{E}\{\left(\frac{M|X|}{a_u}\right)^\delta \mathbb{I}\{|X|<\epsilon a_u\}\} \le \lim_{\epsilon\to0}\limsup_{u\to\infty}\left\{D\frac{b_u}{a_u^2} + C\frac{b_u}{a_u^2}\right\} \le C\lim_{\epsilon\to0}0 = 0,$$

where $D = \mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t))^2\} < \infty$.

Verification of **A3**. Borell-TIS inequality gives

$$\int_0^\infty x dG(x) = \mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t)|Y_\mu(T)=0)\} < \infty.$$

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Thus, assumptions of Theorem 5.2.1 are satisfied for M and X. Hence, we obtain that

$$\lim_{u\to\infty}\left|\frac{\mathbb{P}\{MX\in uK\}}{\frac{1}{u}\frac{1}{\sqrt{2\pi\sigma_Y^2(T)}}e^{-\frac{(\mu T)^2}{2\sigma_Y^2(T)}}\mathbb{E}\{\sup_{t\in[0,T]}(Y_\mu(t)|Y_\mu(T)=0)\}\nu(K)}-1\right|=0.$$

This completes the proof. □

Proof of Theorem 5.3.6 Let $a_u\equiv a_{u,\tau_u}:=\frac{1}{\sqrt{2\pi}}u$ and $b_u\equiv b_{u,\tau_u}=u$. Note that, for any $u>0$ and $K\in\mathcal{B}_{(0,\infty]}$, it holds

$$\begin{aligned}
&\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{Y(\delta(u))}\right)\in a_uK\}=\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)/\sigma_Y(\delta(u))}{Y(\delta(u))/\sigma_Y(\delta(u))}\right)\in a_uK\}\\
&\quad=\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\in a_uK, Y(\delta(u))>0\}\\
&\quad+\mathbb{P}\{\sup_{[0,1]}\left(\frac{-Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{-Y(\delta(u))}\in a_uK, Y(\delta(u))<0\}\\
&\quad=\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\in a_uK\}+\mathbb{P}\{\sup_{[0,1]}\left(\frac{-Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{-Y(\delta(u))}\in a_uK\}\\
&=2\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\in a_uK\}=2\mathbb{P}\{M_uX_u\in a_uK\},
\end{aligned}$$

where $M_u=\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)$ and $X_u=\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}$.

Recall $\tilde{\mathbb{E}}\equiv\tilde{\mathbb{E}}_1=[0,\infty]$. We show that the assumptions of Theorem 5.2.1 are satisfied for M_u and X_u. The argument is analogous to the proof of Theorem 5.3.5. For the sake of brevity, we will only present the main steps of the proof.

<u>Verification of **A1**.</u> Let $K=[A,\infty]\times[B,\infty]$ for $A>0$, $B\geq 0$, then

$$b_u\mathbb{P}\{\left(\frac{X_u}{a_u},M_u\right)\in K\} \;=\; b_u\int_0^{\frac{1}{Aa_u}}\mathbb{P}\{\sup_{t\in[0,T]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=x\}dF_{\frac{Y(\delta(u))}{\delta(u)}}(x).$$

Let denote $C_R:=\sup_{t\in[0,1]}\left|\frac{R_Y(\delta(u)t,\delta(u))}{R_Y(\delta(u),\delta(u))}\right|<\infty$. By gaussianity of the process $\{Y(t):t\geq 0\}$, we have that, for $x>0$,

$$\begin{aligned}
&\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B+C_Rx|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\}\\
&\quad\leq\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=x\}\\
&\quad\leq\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B-C_Rx|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\}.
\end{aligned}$$

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Hence, it holds that

$$\lim_{u\to\infty}\sup_{x\in[0,\frac{1}{Aa_u}]}\left|\frac{\mathbb{P}\{\sup\limits_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=x\}}{\mathbb{P}\{\sup\limits_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\}}-1\right|=0.$$

Further, applying Proposition 4 in [58], we obtain that

$$\lim_{u\to\infty}\sup_{x\in[0,\frac{1}{Aa_u}]}\left|\frac{\mathbb{P}\{\sup\limits_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\}}{\mathbb{P}\{\sup\limits_{t\in[0,1]}(B_\lambda(t))\geq B|B_\lambda(1)=0\}}-1\right|=0,$$

where $\{B_\lambda(t):t\geq 0\}$ is a fractional Brownian motion with Hurst parameter $\lambda\in(0,1)$.

Applying the above with Lebesgue's dominated convergence theorem, we obtain, as $u\to\infty$, for $K=[A,\infty]\times[B,\infty]$,

$$\begin{aligned}
\lim_{u\to\infty} b_u\mathbb{P}\{\left(\frac{X_u}{a_u},M_u\right)\in K\} &= \lim_{u\to\infty} b_u\mathbb{P}\{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\geq Aa_u,\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B\}\\
&= \lim_{u\to\infty} b_u\int_0^{\frac{1}{Aa_u}}\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=x\}dF_{\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}}(x)\\
&= \lim_{u\to\infty} b_u\int_0^{\frac{1}{Aa_u}}\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\geq B|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\}dF_{\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}}(x)\\
&= \lim_{u\to\infty} b_u\int_0^{\frac{1}{Aa_u}}\mathbb{P}\{\sup_{t\in[0,1]}(B_\lambda(t))\geq B|B_\lambda(1)=0\}dF_{\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}}(x)\\
&= \lim_{u\to\infty} b_u\mathbb{P}\{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\geq Aa_u\}\cdot\mathbb{P}\{\sup_{t\in[0,1]}(B_\lambda(t))\geq B|B_\lambda(1)=0\}\\
&= \lim_{u\to\infty}\nu_u([A,\infty])G([B,\infty]),
\end{aligned}$$

where

$$\nu_u([A,\infty]):=b_u\mathbb{P}\{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}\in a_u[A,\infty]\}\text{ and }G([B,\infty])=\mathbb{P}\{\sup_{t\in[0,T]}(B_\lambda(t))\geq B|B_\lambda(1)=0\}.$$

Note that, for each $B\geq 0$, $G(\{B\})=0$ and $G(\{\infty\})=0$.

Further, observe that, for $A>0$, we have that

$$\begin{aligned}
\lim_{u\to\infty}\nu_u([A,\infty]) &= \lim_{u\to\infty} b_u\mathbb{P}\{X_u\geq Aa_u\}=\lim_{u\to\infty} u\frac{1}{Aa_u}\mathbb{P}\{0<\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}\leq\frac{1}{Aa_u}\}\\
&= u\frac{1}{A\frac{1}{\sqrt{2\pi}}u}\frac{1}{\sqrt{2\pi}}=A^{-1}=\nu([A,\infty]).
\end{aligned}$$

Hence for each $A > 0$, $\nu(\{A\}) = 0$ and $\nu(\{\infty\}) = 0$.

Thus, for every relatively compact $K \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}\times\tilde{\mathbb{E}}}$ such that $(\nu \times G)(\partial K) = 0$, it holds

$$\lim_{u\to\infty} \left| \frac{b_u \mathbb{P}\{\left(\frac{X_u}{a_u}, M_u\right) \in K\}}{(\nu \times G)(K)} - 1 \right| = 0,$$

where $\nu(A) = \int_A \frac{1}{x^2} dx$ for every relatively compact $A \in \mathcal{B}_{\tilde{\mathbb{E}}\setminus\{0\}}$ such that $\nu(\partial A) = 0$, $\nu(\cdot)$ is homogenous with index $\alpha = 1$, and $G(B) = \mathbb{P}\{(\sup_{t\in[0,T]} (Y_\mu(t)|Y_\mu(T) = 0) \in B\}$ for $B \in \mathcal{B}_{\tilde{\mathbb{E}}}$.

Verification of **A2**. Before presenting the detailed argument, we shall mention that we abuse slightly the notation in the rest of the proof. Namely, in order to simplify notation, we use C to denote a generic positive constant that does not depend on u and may vary from line to line.

For $\delta > 1$, we obtain that

$$\begin{aligned} &b_u \mathbb{E}\{\left(\frac{M_u|X_u|}{a_u}\right)^\delta \mathbb{I}\{|X_u| < \epsilon a_u\}\} \\ &\quad\le \frac{b_u}{a_u^\delta}\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^\delta\} + \frac{b_u}{a_u^\delta}\int_{\{1<|x|<\epsilon a_u\}} \mathbb{E}\{\sup_{t\in[0,1]}\left\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} \Big| \frac{Y(\delta(u))}{\sigma_Y(\delta(u))} = \frac{1}{x}\right\}^\delta\} dF_{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}}(x). \end{aligned}$$

Let $C_R := \sup_{t\in[0,1]} \left|\frac{R_Y(\delta(u)t,\delta(u))}{R_Y(\delta(u),\delta(u))}\right|$. First, we shall analyze the finiteness of C_R.

Since the process $\{Y(t) : t \ge 0\}$ has stationary increments, for all $t \in [0, 1]$, we have that

$$\begin{aligned} \left|\frac{R_Y(\delta(u)t, \delta(u))}{R_Y(\delta(u), \delta(u))}\right| &= \frac{1}{2}\left|\frac{\sigma_Y^2(\delta(u)t) + \sigma_Y^2(\delta(u)) - \sigma_Y^2(\delta(u)|1-t|)}{\sigma_Y^2(\delta(u))}\right| \\ &\le \frac{1}{2}\max\left\{\frac{\sigma_Y^2(\delta(u)t) + \sigma_Y^2(\delta(u))}{\sigma_Y^2(\delta(u))}, \frac{\sigma_Y^2(\delta(u)|1-t|)}{\sigma_Y^2(\delta(u))}\right\}. \end{aligned}$$

By Potter's bound (Theorem 1.5.6 in [16]), for each $\varepsilon \in (0, \lambda)$ there exists $x_0 \equiv x_0(\varepsilon) \le 1$ such that

$$\frac{\sigma_Y(xt)}{\sigma_Y(x)} \le (1+\varepsilon)t^{\lambda-\varepsilon}, \quad \text{for each } x \le x_0 \text{ and } tx \le x_0.$$

Hence, for sufficiently large $u > 0$ and $|t - s| \le 1$, we obtain that

$$\frac{\sigma_Y^2(\delta(u)|t-s|)}{\sigma_Y^2(\delta(u))} \le C|t-s|^{2(\lambda-\varepsilon)} = C\mathbb{V}ar\{B_{\lambda-\varepsilon}(t) - B_{\lambda-\varepsilon}(s)\}, \tag{5.10}$$

with $C = (1+\varepsilon)^2 > 0$.

Thus, for sufficiently large $u > 0$,

$$C_R = \sup_{t\in[0,1]}\left|\frac{R_Y(\delta(u)t, \delta(u))}{R_Y(\delta(u), \delta(u))}\right| \le \frac{1}{2}\sup_{t\in[0,1]}\max\left\{\frac{\sigma_Y^2(\delta(u)t) + \sigma_Y^2(\delta(u))}{\sigma_Y^2(\delta(u))}, \frac{\sigma_Y^2(\delta(u)|1-t|)}{\sigma_Y^2(\delta(u))}\right\}$$

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$$\leq \frac{1}{2}C \sup_{t\in[0,1]} \max\left\{t^{2(\lambda-\varepsilon)}+1, |1-t|^{2(\lambda-\varepsilon)}\right\} \leq C < \infty.$$

Further, direct calculations follow analogously to (5.9) and give, for $\delta = 2$, sufficiently large $u > 0$,

$$\mathbb{E}\{\sup_{t\in[0,1]} \left\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} | \frac{Y(\delta(u))}{\sigma_Y(\delta(u))} = \frac{1}{x}\right\}^{\delta}\} \leq \mathbb{E}\{\sup_{t\in[0,1]} (Z_u(t))^2\} + 2C_R\frac{1}{|x|}\mathbb{E}\{\sup_{t\in[0,1]} (Z_u(t))\} + C_R^2\frac{1}{x^2} \quad (5.11)$$

where $\{Z_u(t) \equiv \left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} | \frac{Y(\delta(u))}{\sigma_Y(\delta(u))} = 0\right) : t \in [0,1]\}$.

We shall show that the above expectated values are bounded uniformly in u. Observe that

$$Z_u(t) - Z_u(s) = \frac{1}{\sigma_Y(\delta(u))}\left\{[Y(\delta(u)t) - Y(\delta(u)s)] - Y(\delta(u))\frac{R_Y(\delta(u)t, \delta(u)) - R_Y(\delta(u)s, \delta(u))}{\sigma_Y^2(\delta(u))}\right\}.$$

Hence

$$\mathbb{E}\{Z_u(t)\} = \mathbb{E}\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} - \frac{R_Y(\delta(u)t, \delta(u))}{R_Y(\delta(u), \delta(u))}\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}\} = 0$$

and

$$\mathbb{V}ar\{Z_u(t) - Z_u(s)\} \leq \mathbb{V}ar\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} - \frac{Y(\delta(u)s)}{\sigma_Y(\delta(u))}\} = \frac{\sigma_Y^2(\delta(u)|t-s|)}{\sigma_Y^2(\delta(u))}.$$

Inequality (5.10) gives that

$$\mathbb{V}ar\{Z_u(t) - Z_u(s)\} \leq \mathbb{V}ar\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} - \frac{Y(\delta(u)s)}{\sigma_Y(\delta(u))}\} \leq C\mathbb{V}ar\{B_{\lambda-\varepsilon}(t) - B_{\lambda-\varepsilon}(s)\}, \quad (5.12)$$

where $C > 0$ does not depent on u.

The Sudakov-Fernique inequality (Theorem 2.2.3 in [1]) implies, for sufficiently large $u > 0$ and $\varepsilon \in (0, \lambda)$,

$$m_1 := \mathbb{E}\{\sup_{t\in[0,1]} \left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)\} \leq C\mathbb{E}\{\sup_{t\in[0,1]} (B_{\lambda-\varepsilon}(t))\} < \infty \quad (5.13)$$

and

$$m_2 := \mathbb{E}\{\sup_{t\in[0,1]} \left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} | \frac{Y(\delta(u))}{\sigma_Y(\delta(u))} = 0\right)\} \leq C\mathbb{E}\{\sup_{t\in[0,1]} (B_{\lambda-\varepsilon}(t))\} < \infty. \quad (5.14)$$

Further, by (5.10) we have, for all $t \in [0,1]$,

$$\mathbb{V}ar\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))} | \frac{Y(\delta(u))}{\sigma_Y(\delta(u))} = 0\} \leq \mathbb{V}ar\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\} = \frac{\sigma_Y^2(\delta(u)t)}{\sigma_Y^2(\delta(u))} \leq 1 + \varepsilon =: C < \infty.$$

Hence, Borell-TIS inequality (see Theorem 2.1.1 in [1]) implies that

$$\mathbb{P}\{\sup_{t\in[0,1]} \left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right) > x\} \leq \exp\left(-\frac{(x-m_1)^2}{2C}\right) \text{ for all } x \geq m_1$$

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and

$$\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\right)>x\}\le\exp\left(-\frac{(x-m_2)^2}{2C}\right)\text{ for all }x\ge m_2.$$

Thus, we obtain that

$$\begin{aligned}\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^2\}&=\int_0^\infty\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^2>x\}dx\\&=\int_0^\infty 2x\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)>x\}dx\\&\le m_1^2+\int_{m_1}^\infty 2ye^{-\frac{(y-m_2)^2}{2C}}dy\le m_1^2+\int_{-\infty}^\infty 2ye^{-\frac{(y-m_1)^2}{2C}}dy=m_1^2+2m_1<\infty\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\right)^2\}=\int_0^\infty\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\right)^2>x\}dx\\&=\int_0^\infty 2x\mathbb{P}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\right)>x\}dx\\&\le m_2^2+\int_{m_2}^\infty 2ye^{-\frac{(y-m_2)^2}{2C}}dy\le m_2^2+\int_{-\infty}^\infty 2ye^{-\frac{(y-m_2)^2}{2C}}dy=m_2^2+2m_2<\infty.\end{aligned}$$

Finally, we have

$$\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^2\}\le m_1^2+2m_1<\infty \tag{5.15}$$

and

$$\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=0\right)^2\}\le m_2^2+2m_2<\infty. \tag{5.16}$$

The combination of (5.11), (5.14), (5.15) and (5.16) gives

$$\frac{b_u}{a_u^2}\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^2\}\le C\frac{b_u}{a_u^2}$$

and

$$\begin{aligned}&\frac{b_u}{a_u^2}\int_{\{1<|x|<\epsilon a_u\}}\mathbb{E}\{\sup_{t\in[0,1]}\left\{\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}|\frac{Y(\delta(u))}{\sigma_Y(\delta(u))}=\frac{1}{x}\right\}^2\}dF_{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}}(x)\\&\le\frac{b_u}{a_u^2}\int_{\{1<|x|<\epsilon a_u\}}\left[C_1+C_2\frac{1}{|x|}+C_3\frac{1}{x^2}\right]dF_{\frac{\sigma_Y(\delta(u))}{Y(\delta(u))}}(x)\le C\frac{b_u}{a_u^2}.\end{aligned}$$

Thus, putting $u\to\infty$ and $\epsilon\to 0$, we obtain that

$$\lim_{\epsilon\to0}\limsup_{u\to\infty}b_u\mathbb{E}\{\left\{\frac{M_u|X_u|}{a_u}\right\}^2\mathbb{I}\{|X_u|<\epsilon a_u\}\}\le\lim_{\epsilon\to0}\limsup_{u\to\infty}\left\{D\frac{b_u}{a_u^2}+C\frac{b_u}{a_u^2}\right\}\le C\lim_{\epsilon\to0}0=0,$$

where $\mathbb{E}\{\sup_{t\in[0,1]}\left(\frac{Y(\delta(u)t)}{\sigma_Y(\delta(u))}\right)^2\} < D\mathbb{E}\{\sup_{t\in[0,1]}(B_\lambda(t))^2\} = E < \infty$.

Verification of **A3**. Borell-TIS inequality gives

$$\int_0^\infty x dG(x) = \mathbb{E}\{\sup_{t\in[0,1]}(B_\lambda(t)|B_\lambda(1)=0)\} < \infty.$$

Thus, assumptions of Theorem 5.2.1 are satisfied for $\{M_u : u \geq 0\}$ and $\{X_u : u \geq 0\}$. Hence, we obtain that

$$\lim_{u\to\infty}\left|\frac{\mathbb{P}\{\sup_{t\in[0,\delta(u)]}\left(\frac{Y(t)}{\sigma_Y(\delta(u))}\right)\cdot\frac{\sigma_Y(\delta(u))}{Y(\delta(u))} \in uK\}}{\frac{1}{u}\sqrt{\frac{1}{2\pi}}\mathbb{E}\{\sup_{t\in[0,1]}(B_\lambda(t)|B_\lambda(1)=0)\}\nu(K)} - 1\right| = 0.$$

This completes the proof. □

Bibliography

[1] Adler, R. J. and Taylor, J. E. (2007). *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York.

[2] Arendarczyk, M. and Dębicki, K. (2011). Asymptotics of supremum distribution of a Gaussian process over a Weibullian time. *Bernoulli*, 17(1):194–210.

[3] Asmussen, S. (2003). *Applied probability and queues*, volume 2. Springer.

[4] Asmussen, S. and Albrecher, H. (2010). *Ruin probabilities*. Advanced Series on Statistical Science & Applied Probability, 14. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition.

[5] Asmussen, S. and Klüppelberg, C. (1997). Stationary M/G/1 excursions in the presence of heavy tails. *Journal of Applied Probability*, 34(1):208–212.

[6] Avram, F., Palmowski, Z., and Pistorius, M. R. (2008). Exit problem of a two-dimensional risk process from the quadrant: exact and asymptotic results. *The Annals of Applied Probability*, 18(6):2421–2449.

[7] Bai, L., Dębicki, K., Hashorva, E., and Ji, L. (2018). Extremes of threshold-dependent Gaussian processes. *Science China Mathematics*, 61(11):1971–2002.

[8] Basrak, B., Davis, R. A., and Mikosch, T. (2002). Regular variation of GARCH processes. *Stochastic Processes and their Applications*, 99(1):95–115.

[9] Baurdoux, E. J., Palmowski, Z., and Pistorius, M. R. (2017). On future drawdowns of Lévy processes. *Stochastic Processes and their Applications*, 127(8):2679–2698.

[10] Berman, S. M. (1973). Excursions of stationary Gaussian processes above high moving barriers. *The Annals of Probability*, pages 365–387.

[11] Berman, S. M. (1985a). An asymptotic bound for the tail of the distribution of the maximum of a Gaussian process. *Ann. Inst. H. Poincaré Probab. Statist.*, 21(1):47–57.

[12] Berman, S. M. (1985b). Sojourns above a high level for a Gaussian process with a point of maximum variance. *Comm. Pure Appl. Math.*, 38(5):519–528.

[13] Berman, S. M. (1987). Extreme sojourns of a Gaussian process with a point of maximum variance. *Probability theory and related fields*, 74(1):113–124.

[14] Berman, S. M. (1992). *Sojourns and Extremes of Stochastic Processes*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA.

[15] Bertoin, J. and Doney, R. A. (1994). Cramér's estimate for Lévy processes. *Statistics & Probability Letters*, 21(5):363–365.

[16] Bingham, N. H., Goldie, C. M., and Teugels, J. L. (1989). *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

[17] Bisewski, K., Dębicki, K., and Kriukov, N. (2023). Simultaneous ruin probability for multivariate Gaussian risk model. *Stochastic Processes and their Applications*, 160:386–408.

[18] Blanchet, J. and Mandjes, M. (2013). Asymptotics of the area under the graph of a Lévy-driven workload process. *Operations Research Letters*, 41(6):730–736.

[19] Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Theory of Probability & Its Applications*, 10(2):323–331.

[20] Brody, D. C., Hughston, L. P., and Macrina, A. (2008). Dam rain and cumulative gain. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 464(2095):1801–1822.

[21] Cai, N., Chen, N., and Wan, X. (2010). Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options. *Mathematics of Operations Research*, 35(2):412–437.

[22] Chen, Y. (2011). The finite-time ruin probability with dependent insurance and financial risks. *Journal of Applied Probability*, 48(4):1035–1048.

[23] Chen, Y., Chen, D., and Gao, W. (2019). Extensions of Breiman's theorem of product of dependent random variables with applications to ruin theory. *Communications in Mathematics and Statistics*, 7(1):1–23.

[24] Chesney, M., Jeanblanc-Picqué, M., and Yor, M. (1997). Brownian excursions and Parisian barrier options. *Advances in Applied Probability*, 29(1):165–184.

[25] Cheung, E. C. and Zhu, W. (2023). Cumulative Parisian ruin in finite and infinite time horizons for a renewal risk process with exponential claims. *Insurance: Mathematics and Economics*, 111:84–101.

[26] Cline, D. B. H. and Hsing, T. (1998). Large deviation probabilities for sums of random variables with heavy or subexponential tails. *Technical Report, Texas A & M University*.

[27] Cline, D. B. H. and Samorodnitsky, G. (1994). Subexponentiality of the product of independent random variables. *Stochastic Processes and their Applications*, 49(1):75–98.

[28] Cramér, H. (1994). Sur un nouveau théoreme-limite de la théorie des probabilités. In *Collected Works II*, pages 895–913. Springer.

[29] Das, B., Fasen-Hartmann, V., and Klüppelberg, C. (2022). Tail probabilities of random linear functions of regularly varying random vectors. *Extremes*, 25(4):721–758.

[30] Dębicki, K., Engelke, S., and Hashorva, E. (2017a). Generalized Pickands constants and stationary max-stable processes. *Extremes*, 20:493–517.

[31] Dębicki, K. and Hashorva, E. (2019). Approximation of Supremum of Max-Stable Stationary Processes & Pickands Constants. *J. Theoretical Probabilty*.

[32] Dębicki, K., Hashorva, E., and Ji, L. (2014). Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. *Extremes*, 17(3):411–429.

[33] Dębicki, K., Hashorva, E., Ji, L., and Rolski, T. (2018). Extremal behavior of hitting a cone by correlated Brownian motion with drift. *Stochastic Processes and their Applications*, 128(12):4171 – 4206.

[34] Dębicki, K., Hashorva, E., and Kriukov, N. (2022). Pandemic-type Failures in Multivariate Brownian Risk Models. *Extremes*, pages 1–23.

[35] Dębicki, K., Hashorva, E., and Krystecki, K. (2021). Finite-time ruin probability for correlated Brownian motions. *Scandinavian Actuarial Journal*, 2021(10):890–915.

[36] Dębicki, K., Hashorva, E., and Liu, P. (2017b). Extremes of γ-reflected Gaussian process with stationary increments. *ESAIM Probab. Stat.*, 21:495–535.

[37] Dębicki, K., Hashorva, E., and Liu, P. (2017c). Uniform tail approximation of homogenous functionals of Gaussian fields. *Adv. Applied Probab*, 49(4):1037–1066.

[38] Dębicki, K., Hashorva, E., and Liu, P. (2023a). Sojourns of fractional Brownian motion queues: transient asymptotics. *Queueing Systems*, 105(1):139–170.

[39] Dębicki, K., Hashorva, E., Liu, P., and Michna, Z. (2023b). Sojourn times of Gaussian and related random fields. *ALEA*, 20:249–289.

[40] Dębicki, K., Hashorva, E., and Michna, Z. (2020). Simultaneous ruin probability for two-dimensional Brownian risk model. *J. Appl. Probab.*, 57(2):597–612.

[41] Dębicki, K. and Jasnovidov, G. (2024). Extremes of reflecting Gaussian processes on discrete grid. *Journal of Mathematical Analysis and Applications*, 532(1):127952.

[42] Dębicki, K., Ji, L., and Rolski, T. (2019a). Logarithmic Asymptotics for Probability of Component-Wise Ruin in a Two-Dimensional Brownian Model. *Risks*, 7(3).

[43] Dębicki, K., Ji, L., and Rolski, T. (2020a). Exact asymptotics of component-wise extrema of two-dimensional Brownian motion. *Extremes*, 23(4):569–602.

[44] Dębicki, K. and Kosiński, K. (2014). On the infimum attained by the reflected fractional Brownian motion. *Extremes*, 17(3):431–446.

[45] Dębicki, K., Kosiński, K., Mandjes, M., and Rolski, T. (2010). Extremes of multidimensional Gaussian processes. *Stochastic Processes and their Applications*, 120(12):2289 – 2301.

[46] Dębicki, K. and Liu, P. (2016). Extremes of stationary Gaussian storage models. *Extremes*, 19(2):273–302.

[47] Dębicki, K., Liu, P., and Michna, Z. (2020b). Sojourn times of Gaussian processes with trend. *Journal of Theoretical Probability*, 33:2119–2166.

[48] Dębicki, K. and Mandjes, M. (2003). Exact overflow asymptotics for queues with many Gaussian inputs. *Journal of Applied Probability*, 40(3):704–720.

[49] Dębicki, K. and Mandjes, M. (2015). *Queues and Lévy Fluctuation Theory*. Springer International Publishing.

[50] Dębicki, K., Michna, Z., and Peng, X. (2019b). Approximation of sojourn times of Gaussian processes. *Methodology and Computing in Applied Probability*, 21:1183–1213.

[51] Dębicki, K. and Tabiś, K. (2011). Extremes of the time-average of stationary Gaussian processes. *Stochastic Process. Appl.*, 121(9):2049–2063.

[52] Dębicki, K., Hashorva, E., Ji, L., and Tabiś, K. (2015). Extremes of vector-valued Gaussian processes: Exact asymptotics. *Stochastic Process. Appl.*, 125(11):4039–4065.

[53] Dębicki, K., Hashorva, E., and Liu, P. (2017). Extremes of Gaussian processes with regularly varying dependence structure. *Extremes*, 20(2):333–392.

[54] Dębicki, K., Hashorva, E., and Wang, L. (2020). Extremes of vector-valued Gaussian processes. *Stochastic Processes and their Applications.*

[55] Dębicki, K. and Kisowski, P. (2008). A note on upper estimates for Pickands constants. *Statistics & Probability Letters*, 78(14):2046–2051.

[56] Dębicki, K. and Kosiński, K. (2014). On the infimum attained by the reflected fractional Brownian motion. *Extremes*, 17(3):431–446.

[57] Dębicki, K. and Kosiński, K. (2017). An Erdös–Révész type law of the iterated logarithm for reflected fractional Brownian motion. *Extremes*, 20:729–749.

[58] Dębicki, K., Kosiński, K. M., and Mandjes, M. (2012). Gaussian queues in light and heavy traffic. *Queueing Systems*, 71(1):137–149.

[59] Dębicki, K. and Peng, X. (2023). Sojourns of Stationary Gaussian Processes over a Random Interval. *ALEA. Latin American Journal of Probability & Mathematical Statistics*, 20(2).

[60] Denisov, D. and Zwart, B. (2007). On a theorem of Breiman and a class of random difference equations. *Journal of Applied Probability*, 44(4):1031–1046.

[61] Dieker, A. B. (2005). Extremes of Gaussian processes over an infinite horizon. *Stochastic Processes and their Applications*, 115(2):207 – 248.

[62] Donsker, M. D. (1951). An Invariance Principle for Certain Probability Limit Theorems. *Memoirs of the American Mathematical Society*, 6.

[63] Doob, J. L. (1990). *Stochastic processes, Reprint of the 1953 original*. Wiley Classics Library, A Wiley-Interscience Publication. John Wiley & Sons.

[64] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.

[65] Émery, M. and Yor, M. (2004). A parallel between Brownian bridges and gamma bridges. *Publications of the Research Institute for Mathematical Sciences*, 40(3):669–688.

[66] Erraoui, M., Hilbert, A., and Louriki, M. (2020). Bridges with random length: Gamma case. *Journal of Theoretical Probability*, 33:931–953.

[67] Evans, M. R., Majumdar, S. N., and Schehr, G. (2020). Stochastic resetting and applications. *Journal of Physics A: Mathematical and Theoretical*, 53(19):193001.

[68] Fougeres, A.-L. and Mercadier, C. (2012). Risk measures and multivariate extensions of Breiman's theorem. *Journal of Applied Probability*, 49(2):364–384.

[69] Guérin, H. and Renaud, J.-F. (2016). Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view. *Adv. in Appl. Probab.*, 48(1):274–297.

[70] Guérin, H. and Renaud, J.-F. (2017). On the distribution of cumulative Parisian ruin. *Insurance: Mathematics and Economics*, 73:116–123.

[71] Hashorva, E. and Hüsler, J. (2000). Extremes of Gaussian processes with maximal variance near the boundary points. *Methodology and Computing in Applied Probability*, 2(3):255–269.

[72] Hashorva, E. and Pakes, A. G. (2010). Tail asymptotics under beta random scaling. *Journal of mathematical analysis and applications*, 372(2):496–514.

[73] Hashorva, E., Pakes, A. G., and Tang, Q. (2010). Asymptotics of random contractions. *Insurance: Mathematics and Economics*, 47(3):405–414.

[74] Hoyle, E. and Mengütürk, L. A. (2013). Archimedean survival processes. *Journal of Multivariate Analysis*, 115:1–15.

[75] Huesler, J., Piterbarg, V., and Zhang, Y. (2011). Extremes of Gaussian Processes with Random Variance. *Electronic Journal of Probability*, 16:1254 – 1280.

[76] Hugonnier, J.-N. (1999). The Feynman–Kac formula and pricing occupation time derivatives. *International Journal of Theoretical and Applied Finance*, 2(02):153–178.

[77] Hult, E. and Lindskog, F. (2007). Extremal behavior of stochastic integrals driven by regularly varying Lévy processes. *The Annals of Probability*, 35(1):309–339.

[78] Hult, H. and Lindskog, F. (2006). Regular variation for measures on metric spaces. *Publications de l'Institut Mathématique*, 80(94):121–140.

[79] Hüsler, J. and Piterbarg, V. (2004). Limit theorem for maximum of the storage process with fractional Brownian motion as input. *Stochastic Processes and their Applications*, 114(2):231–250.

[80] Hüsler, J., Piterbarg, V., and Rumyantseva, E. (2011). Extremes of Gaussian processes with a smooth random variance. *Stochastic Processes and their Applications*, 121(11):2592–2605.

[81] Hüsler, J. and Piterbarg, V. I. (1999). Extremes of a certain class of Gaussian processes. *Stochastic Process. Appl.*, 83(2):257–271.

[82] Ievlev, P. (2024). Parisian ruin with power-asymmetric variance near the optimal point with application to many-inputs proportional reinsurance. *Stochastic Models*, 40(3):518–535.

[83] Iglehart, L. D. (1969). Diffusion approximations in collective risk theory. *Journal of Applied Probability*, 6(2):285–292.

[84] Ivanovs, J. and Mandjes, M. (2015). Transient analysis of a stationary Lévy-driven queue. *Statistics & Probability Letters*, 107:341–347.

[85] Janssen, A. and Drees, H. (2016). A stochastic volatility model with flexible extremal dependence structure. *Bernoulli*, pages 1448–1490.

[86] Jasnovidov, G. (2020). Approximation of ruin probability and ruin time in discrete Brownian risk models. *Scandinavian Actuarial Journal*, 2020(8):718–735.

[87] Jasnovidov, G. (2021). Simultaneous ruin probability for two-dimensional fractional Brownian motion risk process over discrete grid. *Lithuanian Mathematical Journal*, 61(2):246–260.

[88] Jessen, H. A. and Mikosch, T. (2006). Regularly varying functions. *Publications de L'institut Mathematique*, 80(94):171–192.

[89] Ji, L. (2020). On the cumulative Parisian ruin of multi-dimensional Brownian motion risk models. *Scandinavian Actuarial Journal*, 2020(9):819–842.

[90] Ji, L. and Peng, X. (2022). Extrema of multi-dimensional Gaussian processes over random intervals. *Journal of Applied Probability*, 59(1):81–104.

[91] Ji, L. and Robert, S. (2018). Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stochastic Models*, 34(1):73–97.

[92] Jiang, J. and Tang, Q. (2011). The product of two dependent random variables with regularly varying or rapidly varying tails. *Statistics & Probability Letters*, 81(8):957–961.

[93] Kępczyński, K. (2020). Proportional reinsurance for fractional Brownian risk model. *Silesian Statistical Review*, (18).

[94] Kępczyński, K. (2022). Running supremum of Brownian motion in dimension 2: exact and asymptotic results. *Stochastic Models*, 38(1):116–129.

[95] Kella, O. and Whitt, W. (1992). Useful martingales for stochastic storage processes with Lévy input. *Journal of Applied Probability*, 29(2):396–403.

[96] Konstant, D. G. and Piterbarg, V. I. (1993). Extreme values of the cyclostationary Gaussian random process. *Journal of applied probability*, 30(1):82–97.

[97] Kozik, I. and Piterbarg, V. (2021). High Excursions of Gaussian Nonstationary Processes in Discrete Time. *Journal of Mathematical Sciences*, 253:867–874.

[98] Kriukov, N. (2022). Parisian & cumulative Parisian ruin probability for two-dimensional Brownian risk model. *Stochastics*, 94(4):629–645.

[99] Krystecki, K. (2022). Parisian ruin probability for two-dimensional Brownian risk model. *Statistics & Probability Letters*, 182(109327).

[100] Krystecki, K. (2023). Cumulative Parisian ruin probability for two-dimensional Brownian risk model. *Probability and Mathematical Statistics*, 43.

[101] Kyprianou, A. E. (2006). *Introductory lectures on fluctuations of Lévy processes with applications*. Springer Science & Business Media.

[102] Kyprianou, A. E. (2014). *Fluctuations of Lévy processes with applications: Introductory Lectures*. Springer Science & Business Media.

[103] Leadbetter, M. R., Lindgren, G., and Rootzén, H. (2012). *Extremes and related properties of random sequences and processes*. Springer Science & Business Media.

[104] Lieshout, P. and Mandjes, M. (2007). Tandem Brownian queues. *Mathematical Methods of Operations Research*, 66(2):275–298.

[105] Lifshits, M. (1995). *Gaussian Random Functions*. Mathematics and Its Applications. Springer Netherlands.

[106] Linetsky, V. (1999). Step options. *Mathematical Finance*, 9(1):55–96.

[107] Mandjes, M. and van Uitert, M. (2005). Sample-Path Large Deviations for Tandem and Priority Queues with Gaussian Inputs. *Annals of Applied Probability*, pages 1193–1226.

[108] Michna, Z. (1998). Self-similar processes in collective risk theory. *Journal of Applied Mathematics and Stochastic Analysis*, 11(4):429–448.

[109] Michna, Z. (2020). Ruin Probabilities for Two Collaborating Insurance Companies. *Probability and Mathematical Statistics*, 40.

[110] Norros, I. (1994). A storage model with self-similar input. *Queueing systems*, 16:387–396.

[111] Novikov, S. M. (2025). Sojourns of locally self-similar Gaussian processes. *Stochastics*, 97(1):115–145.

[112] Pang, G., Stuhl, I., and Suhov, Y. (2025). Queueing models with random resetting. *arXiv preprint arXiv:2505.00198*.

[113] Pickands, III, J. (1967). Maxima of stationary Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:190–223.

[114] Pickands, III, J. (1969). Upcrossing probabilities for stationary Gaussian processes. *Transactions of the American Mathematical Society*, 145:51–73.

[115] Pickands III, J. (1969). An iterated logarithm law for the maximum in a stationary Gaussian sequence. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 12(4):344–353.

[116] Piterbarg, V., Popivoda, G., and Stamatović, S. (2017). Extremes of Gaussian processes with a smooth random trend. *Filomat*, 31(8):2267–2279.

[117] Piterbarg, V. I. (1996). *Asymptotic methods in the theory of Gaussian processes and fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI.

[118] Piterbarg, V. I. (2001). Large deviations of a storage process with fractional Brownian motion as input. *Extremes*, 4(2):147–164.

[119] Piterbarg, V. I. (2015). *Twenty Lectures About Gaussian Processes*. Atlantic Financial Press, London, New York.

[120] Reich, E. (1958). On the Integrodifferential Equation of Takács. I. *The Annals of Mathematical Statistics*, pages 563–570.

[121] Resnick, S. I. (2007). *Heavy-tail phenomena*. Springer Series in Operations Research and Financial Engineering. Springer, New York. Probabilistic and statistical modeling.

[122] Ribeiro, C. and Webber, N. (2002). Valuing path dependent options in the variance-gamma model by Monte Carlo with a gamma bridge.

[123] Roy, R., Biswas, A., and Pal, A. (2024). Queues with resetting: a perspective. *Journal of Physics: Complexity*, 5(2):021001.

[124] Skorokhod, A. V. (1961). Stochastic equations for diffusion processes in a bounded region, part I. *Theory of Probability & Its Applications*, 6(3):264–274.

[125] Skorokhod, A. V. (1962). Stochastic equations for diffusion processes in a bounded region, part II. *Theory of Probability & Its Applications*, 7(1):3–23.

[126] Tang, Q. and Tsitsiashvili, G. (2003). Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stochastic Processes and their Applications*, 108(2):299–325.

[127] Taqqu, M. S., Willinger, W., and Sherman, R. (1997). Proof of a fundamental result in self-similar traffic modeling. *ACM SIGCOMM Computer Communication Review*, 27(2):5–23.

[128] Willinger, W., Taqqu, M. S., Leland, W. E., and Wilson, D. V. (1995). Self-similarity in high-speed packet traffic: analysis and modeling of Ethernet traffic measurements. *Statistical science*, pages 67–85.

[129] Wong, H. Y. and Kwok, Y. K. (2003). Multi-asset barrier options and occupation time derivatives. *Applied Mathematical Finance*, 10(3):245–266.

[130] Yang, H. and Sun, S. (2013). Subexponentiality of the product of dependent random variables. *Statistics & Probability Letters*, 83(9):2039–2044.

[131] Yang, Y. and Wang, Y. (2013). Tail behavior of the product of two dependent random variables with applications to risk theory. *Extremes*, 16(1):55–74.

[132] Yor, M. (2007). Some remarkable properties of gamma processes. *Advances in Mathematical Finance*, pages 37–47.

[133] Zhou, Y. and Xiao, Y. (2017). Tail asymptotics for the extremes of bivariate Gaussian random fields. *Bernoulli*, 23(3):1566–1598.

[134] Zubeldia, M. and Mandjes, M. (2021). Large deviations for acyclic networks of queues with correlated Gaussian inputs. *Queueing Systems*, pages 1–39.