University of Wrocław
Faculty of Mathematics and Computer Science
Mathematical Institute

Combinatorial Banach spaces

Sebastian Jachimek

doctoral dissertation supervised by

dr hab. Piotr Borodulin-Nadzieja, Professor at the University of Wrocław

Wrocław 2025

Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

Kombinatoryczne przestrzenie Banacha

Sebastian Jachimek

rozprawa doktorska napisana pod opieką
dr. hab. Piotra Borodulina-Nadziei, prof. UWr

Wrocław 2025

Abstract

This doctoral dissertation concerns combinatorial Banach spaces, that is, Banach spaces induced in a certain way by families $\mathcal{F}$ of finite subsets of $\mathbb{N}$ (or other infinite countable set). These spaces are denoted by $X_{\mathcal{F}}$. The thesis consists of four parts.

In the first part, we introduce the necessary notions, theorems, and facts that we use in the following chapters.

In the second part, we introduce various examples of combinatorial spaces. We investigate how combinatorial properties of families influence the structure of the spaces they induce. Particular attention is devoted to spaces associated with non-compact families, a subject for which the existing literature is rather sparse. In particular, we construct an example of an ℓ_1-saturated space failing the Schur property, and we provide a description of Pełczyński's universal space as a combinatorial space.

In the third part, we study the dual spaces of the combinatorial Banach spaces generated by compact families $\mathcal{F}$. Our aim is to obtain a convenient, equivalent description of the norm on the dual space. To do this, we introduce a quasi-Banach space $X^{\mathcal{F}}$ which, as it turns out, shares many properties with $X^*_{\mathcal{F}}$. In particular, we show that this quasi-Banach space provides yet another example of an ℓ_1-saturated space without the Schur property. Moreover, we prove that the Banach envelope of $X^{\mathcal{F}}$ is isometrically isomorphic to $X^*_{\mathcal{F}}$.

In the fourth part, we investigate the extreme points of the unit ball in combinatorial spaces and in related spaces. We provide characterizations of extreme points in several concrete cases. In addition, we adress the problem of describing extreme points in spaces induced by graphs.

Streszczenie

Niniejsza rozprawa doktorska dotyczy kombinatorycznych przestrzeni Banacha, tj. przestrzeni Banacha, indukowanych w określony sposób przez rodziny $\mathcal{F}$ skończonych podzbiorów $\mathbb{N}$ (lub innego zbioru przeliczalnego). Przestrzenie te są oznaczane symbolem $X_{\mathcal{F}}$. Praca składa się z czterech części.

W pierwszej części wprowadzamy niezbędne pojęcia oraz twierdzenia, z których korzystamy w dalszej części pracy.

W części drugiej wprowadzamy różne przykłady przestrzeni. Badamy, jak poszczególne kombinatoryczne własności rodzin wpływają na indukowane przez nie przestrzenie. Szczególną uwagę poświęcamy przestrzeniom związanymi z rodzinami niezwartymi, na których temat literatura jest raczej uboga. Podajemy m.in. przykład przestrzeni ℓ_1-nasyconej, która nie ma własności Schura, a także podajemy prezentację uniwersalnej przestrzeni Pełczyńskiego jako przestrzeni kombinatorycznej.

W części trzeciej zajmujemy się przestrzeniami dualnymi do przestrzeni kombinatorycznych, generowanych przez rodziny zwarte $\mathcal{F}$. Próbujemy znaleźć wygodny w użytku, równoważny opis normy na przestrzeni dualnej. W tym celu definiujemy przestrzeń quasi-Banacha $X^{\mathcal{F}}$, która, jak się okazuje, ma wiele wspólnych własności z $X_{\mathcal{F}}^*$. W szczególności pokazujemy, że ta przestrzeń quasi-Banacha jest kolejnym przykładem przestrzeni ℓ_1-nasyconej bez własności Schura. Ponadto, pokazujemy, że powłoka Banacha przestrzeni $X^{\mathcal{F}}$ jest izometrycznie izomorficzna z $X_{\mathcal{F}}^*$.

W części czwartej zajmujemy się tematyką punktów ekstremalnych kuli jednostkowej w przestrzeniach kombinatorycznych, a także w przestrzeniach z nimi związanymi. Podajemy charakteryzację punktów ekstremalnych w konkretnych przypadkach. Ponadto, podejmujemy tematykę punktów ekstremalnych w przestrzeniach indukowanych przez grafy.

Acknowledgments

First of all, I would like to express my gratitude to my supervisor, Piotr Borodulin-Nadzieja, for many things: for introducing me to the topic of combinatorial spaces, for his invaluable mathematical support, contributions to this doctoral thesis, great patience, and all the mathematical discussions that helped me grow mathematically. I am deeply grateful for this mathematical journey, which has continued since my undergraduate studies.

I would also like to thank my wife, Marta Kowalska-Jachimek, for believing in me, motivating me, and being by my side; for her patience and understanding of the difficult moments while writing my thesis, as well as for the support and all her words of encouragement in moments of doubt. I appreciate immensely having such a person by my side.

I would also like to thank all the people with whom I have had the pleasure of working and having mathematical discussions. In particular, I would like to thank Barnabás Farkas, Anna Pelczar-Barwacz, and Damian Sobota.

I am also very grateful to my parents, Ewa and Andrzej, and my sister, Milena, for their kindness, constant support, and for believing in me.

Contents

Introduction

The study of the structure of Banach spaces has been a central theme in modern analysis since the very beginning of their existence. Over the decades, an increasingly refined understanding of the geometry and other structural properties of Banach spaces has been developed. The tools for this development have often involved methods of combinatorics, set theory, and topology. As examples of deep results in Banach space theory in which these methods were used, we can point out:

- **James' space.** Construction of a separable, non-relexive space **J** without unconditional basis which is isometrically isomorphic to its double dual space ([34]).
- **Rosenthal's ℓ_1-theorem.** Every bounded sequence in an infinite-dimensional Banach space has either a weakly Cauchy subsequence or a subsequence that is equivalent to the standard basis of ℓ_1 ([48]).
- **Gowers-Maurey space.** There exists an infinite-dimensional Banach space such that its every infinite-dimensional subspace admits no unconditional Schauder basis ([32]),

and many other results.

This doctoral dissertation is concerned with a particular combinatorial method for defining Banach spaces. The method determines the name of the constructed space - *combinatorial Banach space*. It is defined as a completion of c_{00} with respect to the following norm

$$\|x\|_{\mathcal{F}} = \sup_{A \in \mathcal{F}} \sum_{n \in A} |x(n)|, \tag{1}$$

where $\mathcal{F}$ is a family of finite subsets of $\mathbb{N}$ (or any countable set) which is closed under taking subsets. The standard unit vectors (e_n) form an unconditional basis in this space.

The name *combinatorial Banach space* was coined by Gowers in 2009 (see [31]), however, investigations into this type of space date back much earlier.

Perhaps it started with the article ([49]) of J. Schreier. He showed that $C([0,1])$, the space of continuous functions on $[0,1]$, does not have the weak Banach-Saks property, thereby disproving the conjecture that this property might hold in all Banach spaces. His argument was based on a family of sets, nowadays known as Schreier sets, i.e., such sets A for which $|A| \leq \min(A)$. Later on, Baernstein used the notion of the Schreier set to construct a reflexive Banach space without the Banach-Saks property (see [8]), however, the norm of this space was slightly different than this of (1). In the late 1970s, Beauzamy used Schreier sets to define the space now called the Schreier space, exactly as described above - as the completion of c_{00} with respect to the norm (1) for the Schreier family of sets, which is usually denoted by $\mathcal{S}$. Interestingly, he used this space for the same purpose as Baernstein - to construct a counterexample of a reflexive space without the Banach-Saks property.

In the 1990s, Alspach and Argyros in [3] generalized the concept of Schreier sets using a certain inductive procedure. The families they obtained are known as the higher order Schreier families, and the Banach spaces induced by these families are accordingly called the higher order Schreier spaces.

One can say that combinatorial Banach spaces are the next step in the generalization of the Schreier space. There are no strictly imposed conditions on what assumptions should be made about the family $\mathcal{F}$; however, the minimal requirements are that the family contains all singletons and is closed under taking subsets. These were, for example, the only assumptions in Gowers' definition of (1). A common assumption is that the family $\mathcal{F}$ is regular, meaning that it is hereditary, compact, and spreading (for the definitions we refer the reader to Chapter 1, Subsection 1.3.1). Combinatorial Banach spaces understood in this way were studied extensively by many authors in various contexts (see e.g. [5], [12], [21], [6]).

The broad aim of this dissertation is to study various properties of combinatorial spaces (and other related Banach spaces), depending on the assumptions imposed on the family $\mathcal{F}$. In contrast to most authors, we do not assume that the family $\mathcal{F}$ is compact and spreading. Our standard requirements are that $\mathcal{F}$ is hereditary and covers $\mathbb{N}$ (or another countable set on which it is defined). Considering non-compact families in the context of combinatorial spaces is rather unusual, as reflected in the scarcity of literature on the subject.

Besides the case of the space ℓ_1, which isometrically isomorphic to $X_{[\mathbb{N}]^{<\infty}}$, non-compact families do not seem to be within the scope of interest of authors working on combinatorial spaces. Such a situation provides a wide field for exploration. For example, one can consider various families appearing in set theory or combinatorics (in particular, the theory of analytic P-ideals or graph theory) and generate Banach spaces out of them. It turned out that if the family is *interesting* or *generic*, then we may expect that the induced Banach space will also have interesting properties.

The worlds of combinatorial Banach spaces for compact and non-compact families are quite different. For example, it is known that $X_{\mathcal{F}}$ for any compact family $\mathcal{F}$ is c_0-

saturated. If $\mathcal{F}$ is non-compact, this is not the case. Since there is an infinite set A in the closure of $\mathcal{F}$, an associated space contains a copy of ℓ_1, which can be seen with the naked eye - it is spanned by vectors e_n for $n \in A$. It is not true, however, that combinatorial spaces induced by non-compact families are ℓ_1-saturated. We provide examples of spaces with both copies of c_0 and ℓ_1.

Of particular interest to us were the following properties of Banach spaces

(a) Schur property,

(b) ℓ_1-saturation,

(c) the lack of a copy of c_0.

It is known that for any Banach space the chain of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) holds. Our motivation was to explore whether these implications can be reversed in the realm of combinatorial Banach spaces. However, suitable counterexamples yield a negative answer for both implications. The space showing that (b) does not imply (a) thus provides an example of ℓ_1-saturated combinatorial space without the Schur property. For a long time, it was not sure if such spaces exist at all. The first example was given by J.Bourgain (see [20]), and then several other involved constructions of such spaces have been presented (see e.g. [7], [47]). The method used to construct the counterexample showing that (c) does not imply (a) also allows us to establish stronger results. As a consequence, we have obtained not only a space containing all the ℓ_p spaces for $1 \leq p < \infty$, but also all combinatorial spaces, and even all spaces with an unconditional basis. Hence, we obtained a combinatorial Banach space that is universal for the class of Banach spaces with an unconditional basis. This space is generated by a certain Fraïssé limit and the obtained space is isomorphic to the so-called Pełczyński space. This is one of the examples of the phenomenon mentioned above: a generic family of finite subsets induces an important example of a Banach space.

One of the possible reasons why the authors do not consider combinatorial spaces induced by non-compact families is that then the basis of $X_{\mathcal{F}}$ is not shrinking. Hence, the biorthogonal functionals do not form a basis in the dual space, which makes it more difficult and less convenient to study. In general, the dual spaces of combinatorial spaces seem to be rather mysterious objects in the theory of Banach spaces. Perhaps the reason lies in the lack of a nice description of the dual norm. Seeking such a description, we came up with the following formula

$$\|x\|^{\mathcal{F}} = \inf \left\{ \sum_{F \in \mathcal{P}} \sup_{i \in F} |x(i) : \mathcal{P} \subseteq \mathcal{F} \text{ is a partition of } \mathbb{N} \right\}. \tag{2}$$

Maybe it does not look nice, but in some sense this is dual to the combinatorial norm $\|\cdot\|_{\mathcal{F}}$ (see: Chapter 3). This formula, however, makes sense only for $\mathcal{F}$ which is compact. We have thus experienced firsthand that, at times, discarding compactness as an assumption on $\mathcal{F}$ leads to certain difficulties. We defined the space $X^{\mathcal{F}}$ as a completion of c_{00} with respect to the formula (2). For certain families $\mathcal{F}$, it is a Banach space that is isometrically isomorphic to $X_{\mathcal{F}}^*$. For instance, note that if $\mathcal{F}$ consists of singletons, then $X_{\mathcal{F}}$ is isometrically isomorphic to c_0, and $X^{\mathcal{F}}$ is isometrically isomorphic to ℓ_1.

In general, if a $\mathcal{F}$ is a hereditary, compact family of finite sets that covers $\mathbb{N}$, then $X^{\mathcal{F}}$ shares many properties with $X_{\mathcal{F}}^*$. It was difficult for us to find a property distinguishing those spaces. Hence, for quite a long time, we were convinced that these spaces must be isomorphic, but we could not prove that. Eventually, we understood why the previous attempts had failed: in general, $X^{\mathcal{F}}$ is not a Banach space! More precisely, $\|\cdot\|^{\mathcal{F}}$ does not satisfy the triangle inequality. Although this is not an encouraging observation, it turned out that the space $X^{\mathcal{F}}$ remains of interest. It belongs to the broader class of spaces, namely quasi-Banach spaces, which, under suitable assumptions, may be isomorphic to Banach spaces. More precisely, it is possible if the quasi-Banach space is 1-convex. For some particular cases, $X^{\mathcal{F}}$ satisfies this property and then it is isometrically isomorphic to $X_{\mathcal{F}}^*$; however, in general, this is not true.

Nevertheless, the connection between $X^{\mathcal{F}}$ and $X_{\mathcal{F}}^*$ is so strong that they share properties that are typically not invariant under isomorphism. For example, the unit balls in those spaces have 'the same' extreme points. Also, for every compact hereditary family $\mathcal{F}$, the spaces $X^{\mathcal{F}}$ and $X_{\mathcal{F}}^*$ have isometrically isomorphic duals. Besides, the quasi-norm $\|\cdot\|^{\mathcal{F}}$ is much easier to handle than the dual norm on $X_{\mathcal{F}}^*$.

In establishing our results, a notion we sometimes relied on was the set of extreme points of the unit ball. We used known facts concerning extreme points in the unit ball of $X_{\mathcal{F}}^*$, for compact, hereditary $\mathcal{F}$, to obtain that the Banach envelope of $X^{\mathcal{F}}$ (see: 3.4 in Chapter 3) is isometrically isomorphic to $X_{\mathcal{F}}^*$. The shape of the set of extreme points in combinatorial spaces (and some related ones) then began to be fascinating for its own sake.

Our motivation for pursuing this topic came from two factors: the known description of the extreme points in the unit ball of $X_{\mathcal{F}}^*$ for compact families $\mathcal{F}$, and the observation that, beyond the classical examples such as c_0 and ℓ_1, very little is known about the extreme points in $X_{\mathcal{F}}$. Even in the case of the Schreier family $\mathcal{S}$, the shape of the set of extreme points is not known.

If $\mathcal{F}$ is a compact family, then the extreme point in the dual unit ball has values in $\{-1, 0, 1\}$, and its support is a maximal set $F \in \mathcal{F}$. We generalized this fact to every hereditary family $\mathcal{F}$ covering $\mathbb{N}$. In accordance with the second motivation, we obtain a full characterization of extreme points for the specific families $\mathcal{F}$. We also presented combinatorial spaces defined by graphs and analyzed the extreme points in such spaces, indicating an interesting interplay between graph theory and convex analysis.

The thesis is organized as follows.

In Chapter 1, we introduce notions and facts which we use in the following part of this thesis.

In Chapter 2, we present a plethora of examples of combinatorial spaces. We investigate how the combinatorial properties of the family $\mathcal{F}$ influence the structure of the induced Banach space $X_{\mathcal{F}}$. In particular, a relatively simple example of ℓ_1-saturated Banach spaces without the Schur property is provided. Also, we give a new presentation of Pełczyński's universal space as a combinatorial space, and we also provide an answer to a question posed by Pełczyński in one of his papers, which appears to remain open.

In Chapter 3, we present quasi-Banach spaces which are closely related to the dual spaces of combinatorial Banach spaces and share many properties with them. More precisely, for a compact family $\mathcal{F}$, the Banach envelope of the defined quasi-Banach space is isometrically isomorphic to $X^*_{\mathcal{F}}$. We show that the quasi-Banach spaces induced by families from a certain class are ℓ_1-saturated and do not have the Schur property. In particular, it holds for the Schreier family $\mathcal{S}$, as it belongs to this class.

In Chapter 4 we study the extreme points in combinatorial spaces and their duals, as well as in the spaces $\mathrm{FIN}(\|\cdot\|_{\mathcal{F}})$. In addition, we provide a characterization of the extreme points in spaces defined by perfect graphs, together with partial results for non-perfect graphs, simultaneously pointing out the difference between these two cases.

The results presented in Chapters 2 and 3 are based on joint work [18] and [19], and are the outcome of collaboration with the co-authors. In all cases where the main idea does not originate from the author of this dissertation, this is explicitly indicated. The results from Chapter 4 were unpublished at the time of preparing this dissertation. Unless stated otherwise, they are due to the author of this dissertation.
In addition, Chapter 2 contains only the results from the article [18] to which the author of this dissertation contributed. However, both Chapters, 2 and 3, also contain new results obtained by the author of this dissertation, which do not appear in the publications cited above.

Chapter 1

Preliminaries

1.1 Basic notions

In this thesis, the set of natural numbers includes 0 and is denoted by $\mathbb{N}$. The symbol $\mathbb{N}_+$ is reserved for the set $\mathbb{N} \setminus \{0\}$. If $k \in \mathbb{N}$ and $M \subseteq \mathbb{N}$, then

$$[M]^{\leq k} = \{A \subseteq M : |A| \leq k\}.$$

Similarly, $[M]^k$ denotes the family of all subsets of M with exactly k elements, and $[M]^{<\infty}$ (respectively, $[M]^{\infty}$) denotes the family of all finite (respectively, infinite) subsets of M.

By a *partition* of a set C we mean a family $\mathcal{C}$ such that $\bigcup \mathcal{C} = C$ and any two distinct elements of $\mathcal{C}$ are disjoint. For technical reasons, which will be explained later, we also assume that $\emptyset \in \mathcal{C}$.

For any set A we denote by χ_A the *characteristic function* of A. The family of all subsets of $\mathbb{N}$ is denoted by $\mathcal{P}(\mathbb{N})$ and we identify it with the Cantor set $2^{\mathbb{N}}$ via the bijection

$$\mathcal{P}(\mathbb{N}) \ni A \mapsto \chi_A \in 2^{\mathbb{N}}.$$

Unless stated otherwise, we consider the standard product topology on the Cantor set. Thus, when we discuss topological properties of a family of sets $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ we mean the corresponding properties of its image in $2^{\mathbb{N}}$.

For any sets A, Ω, we denote by A^{Ω} the set of all functions from Ω to A. In most cases in this thesis $\Omega = \mathbb{N}$ (or, sometimes, another countably infinite set). Such a function is then called a *sequence*. For two sequences x, y $x + y$ and $x \cdot y$ denote their coordinate-wise addition and multiplication, i.e. $(x + y)(k) = x(k) + y(k), (x \cdot y)(k) = x(k)y(k)$ for every $k \in \Omega$.

If $f : \Omega \to A$, $\Gamma \subseteq \Omega$, and $B \subseteq A$, then an *image* of the set Γ under the function f is the set

$$f[\Gamma] = \{f(x) : x \in \Gamma\}.$$

Similarly, the *preimage* of the set B under the function f is the set

$$f^{-1}[B] = \{x \in \Omega : f(x) \in B\}.$$

For $k, l \in \mathbb{N}$ with $k < l$, we denote by $[k, l]$ the *interval* of natural numbers between k and l, namely

$$[k, l] = \{k, k+1, \cdots, l-1, l\}.$$

In a similar manner, we define half-open and half-closed intervals. We also write $n = [0, n)$, i.e., we identify each natural number with the set of all natural numbers less than n. For $A, B \subseteq \mathbb{N}$, by $A < B$ we mean that $\max(A) < \min(B)$.

1.2 Graphs

Let Ω be a countable set. A graph G is a pair (V, E), where $V \subseteq \Omega$ and $E \subseteq [V]^2$. An element of V is called a *vertex*, and an element of E is called an *edge*. If $\{v, w\} \in E$ for $v, w \in V$, then we say that vertices v and w are *adjacent*. In this thesis, graphs are always undirected and without loops, but they can be infinite (i.e. $|V| = \aleph_0$). By the complement of a graph $G = (V, E)$, we mean the graph $G^c = (V, [V]^2 \setminus E)$.
A *cycle* or a *hole* of size $n > 1$ (in short: n-hole) is a finite graph with vertices $\{v_0, ..., v_{n-1}\}$ such that $\{v_i, v_{i+1}\} \in E$ for every $i < n-1$, and $\{v_{n-1}, v_0\} \in E$. Such a graph is denoted by C_n. An *antihole* is a graph that is the complement of a hole.
We say that $C \subseteq V$ is a *clique* if every two distinct vertices from C are adjacent. A set $A \subseteq V$ is called *independent* or an *anticlique* if $[A]^2 \cap E = \emptyset$, i.e. no two vertices are adjacent.
We denote by $\omega(G)$ the size of a maximal clique in G, and call it a *clique number*. Similarly, the *anticlique number* $\alpha(G)$ is defined as the size of a maximal anticlique.
A *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color a graph G in such a way that each two adjacent vertices have different colors. Since the vertices of any clique need to have different colors, then $\chi(G) \geq \omega(G)$.
We say that a graph G is *perfect* if, for every induced finite subgraph H of G, we have $\chi(H) = \omega(H)$. The following two theorems give a characterization of perfect graphs.

Theorem 1.2.1 (Weak perfect graph theorem). *A graph G is perfect if and only if its complement is perfect.*

Theorem 1.2.2 (Strong perfect graph theorem). *A graph G is perfect if and only if it does not have either holes or antiholes of odd size at least* 5.

Both of these theorems were formulated for the first time as conjectures by C. Berge in [14]. The theorem 1.2.1 was proved by L. Lovász in 1972 (see [41]), and the theorem 1.2.2 - over thirty years later by M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas ([23]).

We say that graphs G and H are *isomorphic* if there exists a bijection f between sets of vertices $V(G)$ and $V(H)$ such that any two vertices v, v' are adjacent in G if and only if their images $f(v)$ and $f(v')$ are adjacent in H. Such f is called an *graph isomorphism* or an *edge-preserving bijection*.

1.3 Banach spaces

Every Banach space $(X, \|\cdot\|)$ is considered over $\mathbb{R}$. We omit the norm symbol when it is clear from the context. For a sequence (x_n) in a Banach space X by $[x_n]$ we denote its closed linear span, i.e. the closure (in the norm topology) of the set

$$\left\{ \sum_{n=1}^{N} a_n x_n : N \in \mathbb{N}, a_n \in \mathbb{R} \right\}.$$

Unless stated otherwise, by a *subspace* of a Banach space X we always mean a closed subspace.
For a Banach space X we denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively.
A linear map between Banach spaces X and Y is called an *operator*. It is a standard exercise to show that an operator is continuous if and only if it is bounded, i.e., there exists $C > 0$ such that for every $x \in X$

$$\|T(x)\|_Y \leq C\|x\|.$$

We say that T is an *isomorphism* if it is a bijective linear homeomorphism. Equivalently, a bijection T is an isomorphism if there exist $c, C > 0$ such that for all $x \in X$

$$c\|x\|_X \leq \|T(x)\|_Y \leq C\|x\|.$$

If $c = C = 1$ then we say that T is an isometric isomorphism. We write $X \simeq Y$ and $X \equiv Y$ to indicate that Banach spaces X and Y are isomorphic and isometrically isomorphic, respectively.
In the case $Y = \mathbb{R}$, the operator T is called a *functional*. The space of all continuous functionals defined on a space X is denoted by X^*.
We denote by e_n the sequence in $\mathbb{R}^{\mathbb{N}}$ whose only nonzero coordinate is 1 in the n-th position. We call it the *standard unit vector*.
We now recall the definitions of several classical Banach spaces.

- c_0 denotes the space of all sequences of real numbers convergent to 0, endowed with the supremum norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x(k)|$, for $x \in \mathbb{R}^{\mathbb{N}}$.

- For $0 < p < \infty$ $L_p(X,\mu)$ denotes the space of all μ-measurable functions f on X such that
$$\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}} < \infty.$$
This is a Banach space (with the norm $\|\cdot\|_p$) only for $p \geq 1$. We omit the underlying set and measure if they are clear from the context. In particular, L_p denotes the space $L_p([0,1],\lambda)$, where λ is the Lebesgue measure. On the other hand, ℓ_p denotes the space $L_p(\mathbb{N},\mu)$ with μ being the counting measure.

- $L_\infty(X,\mu)$ denotes the space of all essentialy bounded μ-measurable functions on X with the norm
$$\|f\|_\infty = \inf_{\mu(A)=0} \sup_{x\in X\setminus A} |f(x)|.$$
If $X=\mathbb{N}$ and μ is the counting measure, this space is denoted by ℓ_∞.

- $C(K)$ denotes the space of all continuous real-valued functions with a compact Hausdorff space K as a domain, with the norm
$$\|f\| = \sup_{t\in K} |f(t)|.$$

Let $(Y,\|\cdot\|_Y),(X_1,\|\cdot\|_{X_1}),(X_2,\|\cdot\|_{X_2}),...$ be Banach spaces. We consider the following set denoted by $\big(\bigoplus_{n=1}^\infty X_n\big)_Y$
$$\Big(\bigoplus_{n=1}^\infty X_n\Big)_Y = \Big\{(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty X_n : (\|x_n\|_{X_n})_n \in Y\}\Big\}.$$
Endowed with a coordinate-wise addition, scalar multiplication and the norm
$$\|(x_n)\| = \|(\|x_n\|_{X_n})\|_Y$$
it is a Banach space called Y*-direct sum* of the spaces (X_n).
In the theory of Banach spaces, finite-dimensional spaces play an important role too. Of particular interest are the spaces c_0^k and ℓ_p^k for $k\in\mathbb{N}_+$, that is the spaces $\mathbb{R}^k$ endowed with the supremum norm or the ℓ_p-norm, respectively.

Given Banach spaces X,Y, we say that X is Y*-saturated*, if every infinite-dimensional subspace of X contains an isomorphic copy of Y. In most cases, we are interested in spaces being c_0- or ℓ_1-saturated.
This notion can be generalized as follows. Let X be a Banach space and $\mathfrak{A}$ a family of Banach spaces. We say that X is $\mathfrak{A}$-saturated if for every infinite-dimensional subspace E of X there exists $Z\in\mathfrak{A}$ such that E contains a subspace isomorphic to Z.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is called *weakly Cauchy* (*weakly convergent*) if for every $x^* \in X^*$ the sequence $(x^*(x_n))_{n\in\mathbb{N}}$ is Cauchy (convergent). Every weakly convergent sequence is weakly Cauchy. Banach spaces in which every weakly Cauchy sequence is weakly convergent are called *weakly complete*. Reflexive spaces are the classical examples. A sequence (φ_n) in X^* is *weak-* convergent* to some $\varphi \in X^*$ if $\varphi_n(x) \xrightarrow{n\to\infty} \varphi(x)$ for every $x \in X$.

We say that a Banach space X has the *Schur property* if every weakly convergent sequence is also norm-convergent. The most well-known example of such a space is ℓ_1. The following classical result of H. Rosenthal implies that Banach spaces with the Schur property are ℓ_1-saturated.

Theorem 1.3.1 ([48]). *(ℓ_1-theorem) Let X be a Banach space and (x_n) be a bounded sequence in X. Then (x_n) has a subsequence (x_{n_k}) such that exactly one of the following holds:*

1. *(x_{n_k}) is weakly Cauchy;*
2. *(x_{n_k}) is equivalent to the standard basis of ℓ_1.*

Whether the converse holds - i.e., whether ℓ_1-saturated spaces necessarily have the Schur property - was an open question for a long time. The answer is negative: the first example of an ℓ_1-saturated Banach space without the Schur property was given by J. Bourgain (see [20]), and then several other involved constructions of such spaces have been presented (see e.g. [7], [47]). We discuss this phenomenon in subsequent chapters.

Let V be a real vector space. If $A \subseteq V$, then by $-A$ we denote the set $\{-a : a \in A\}$. We say that A is *symmetric* if $A = -A$.
We say that $K \subseteq V$ is *convex* if for every $a, b \in K$ and every $t \in [0, 1]$ we have $(1-t)a + tb \in K$. In other words, the set K is convex if every line segment between two points from K is contained in K.
For any $A \subseteq V$, the *convex hull* of A is the smallest convex set containing A. It is denoted by $\operatorname{conv} A$ and has the following equivalent definition

$$\operatorname{conv} A = \Big\{ \sum_{i\le n} \lambda_i v_i : n \in \mathbb{N}, \text{ for every } i \le n\ v_i \in A,\ \lambda_i > 0, \text{ and } \sum_{i\le n} \lambda_i = 1 \Big\}.$$

We say that $e \in K$ is an *extreme point* of K if there do not exist distinct $x, y \in K$ and $t \in (0, 1)$ such that $e = (1-t)x + ty$. We denote the set of all extreme points of K by $\operatorname{Ext}(K)$. There are many equivalent definitions of extreme points. We use the specific one that is the most convenient for our purposes.

Lemma 1.3.2. *Let K be a convex subset of a vector space V. Then e is an extreme point of K if and only if the only $v \in V$ such that $e + v \in K$ and $e - v \in K$ is $v = 0$.*

Proof. Suppose that there is a nonzero $v \in V$ such that $e \pm v \in K$. Then $e = \frac{1}{2}(e + v) + \frac{1}{2}(e - v)$ and thus, $e \notin \mathrm{Ext}(K)$.
Now, if $e \notin \mathrm{Ext}(K)$, then there are different $y, z \in K$ and $t \in (0,1)$ such that $e = (1-t)y + tz$. Note that we may assume that $t = \frac{1}{2}$, i.e., e is a midpoint of the line segment between points from K. Indeed, if $t < \frac{1}{2}$, put $y' = y$ and $z' = (1-2t)y + 2tz$. Then $y', z' \in K$ and

$$\frac{1}{2}(y' + z') = \frac{1}{2}(y + (1-2t)y + 2tz) = (1-t)y + tz = e.$$

Analogously, if $t > \frac{1}{2}$ then we get the same conclusion by taking $z' = z$ and $y' = (2-2t)y + (2t-1)z$.
Thus, if $e = \frac{1}{2}(y+z)$, then we have $y = e + \frac{1}{2}(y-z)$ and $z = e - \frac{1}{2}(y-z)$. Hence, the vector $v = \frac{1}{2}(y-z) \neq 0$ is such that $e \pm v \in K$. □

In Banach space theory, the study of extreme points usually focuses on the closed unit ball. Accordingly, by *extreme points in* X we always mean the extreme points of B_X, and we use the notation $\mathrm{Ext}(X)$ instead of $\mathrm{Ext}(B_X)$. It is easy to see that in any Banach space $\mathrm{Ext}(X) \subseteq S_X$.

Bases

Definition 1.3.3. A sequence $(x_n)_{n\in\mathbb{N}}$ of vectors in a Banach space X is called a *Schauder basis* if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n\in\mathbb{N}} a_n x_n.$$

In other words, the sequence $\left(\sum_{n=1}^{N} a_n x_n\right)_{N\in\mathbb{N}}$ converges to x in the norm topology of X.

Note that for finite-dimensional spaces the notions of a Schauder basis and a *Hamel basis* coincide. This is no longer true in the infinite-dimensional case, since a Hamel basis must then be uncountable.
Thus, in the context of Banach spaces, we use only the term *Schauder basis* and in the following part of this thesis, we will simply write *basis*.

Definition 1.3.4. Let (x_n) be a sequence in a Banach space X. Let (x_n^*) be a sequence in X^* such that for every $n, m \in \mathbb{N}$ $x_n^*(x_m) = \delta_{n,m}$ and for every $x \in X$ $x = \sum_{n\in\mathbb{N}} x_n^*(x_n)$. Such functionals (x_n^*) are called *biorthogonal functionals* associated with (x_n).

Having Definitions 1.3.3 and 1.3.4 one can deduce (see [2, Theorem 1.1.3]) that (x_n) is a basis for X if and only if for every $x \in X$ the expansion $\sum_{n\in\mathbb{N}} x_n^*(x)x_n$ is norm convergent to x. Since biorthogonal functionals are continuous, we have $x_n^*(x) = a_n$.

We say that a sequence (y_n) is a *basic sequence* if it is a basis for $[y_n]$.
If X has a basis (x_n), then $[x_n] = X$ and hence, it is separable (finite linear combinations of x_n with rational coefficients are dense in X). On the other hand, biorthogonal functionals associated with x_n form a basic sequence in X^*.

Example 1.3.5. Classical separable Banach spaces have Schauder bases.

(i) The standard unit vectors (e_n) form a basis for c_0 and ℓ_p for $1 \le p < \infty$. We will call it the *standard basis*.

(ii) The *Haar system* is a sequence (h_n) of functions defined on $[0, 1]$ as follows. Let $h_1 = 1$. For $k \in \mathbb{N}$ and $s \le 2^k$ let

$$h_{2^k+s}(x) = \begin{cases} 1, & \text{if } x \in [\frac{2s-2}{2^{k+1}}, \frac{2s-1}{2^{k+1}}] \\ -1, & \text{if } x \in [\frac{2s-1}{2^{k+1}}, \frac{2s}{2^{k+1}}] \\ 0 & \text{otherwise} \end{cases}$$

One can show that (h_n) is a basis of L_p for every $1 \le p < \infty$ (see: Proposition 6.1.3 in [2]).

(iii) $C([0, 1])$ admits a basis as well. It is the so-called *Schauder system* (f_n) and is defined as follows. Put $f_1 = 1$ and for $n > 1$ $f_n(t) = \int_0^t h_{n-1}(s)ds$ where h_n is the n-th Haar function from (ii). For the proof, see the note under Definition 1.a.4 in [39].

The question of whether every separable space admits a basis was posed by Stefan Banach in his book [9], and it was related to another problem, formulated by Stanisław Mazur in the *Scottish Book* (Problem 153). This question was answered negatively in 1973. Per Enflo ([26]) constructed a separable Banach space without the so-called *approximation property*, the lack of which also implies the lack of a Schauder basis.
Mazur, however, proved another result concerning bases.

Theorem 1.3.6. *Every infinite-dimensional Banach space contains a basic sequence.*

Remark 1.3.7. If (x_n) is a basis in a Banach spaces X, then for every $\varphi \in X^*$ we have $\varphi(y) = \sum_{n\in\mathbb{N}} \varphi(x_n)x_n^*(y)$ for every $y \in X$. Hence we may (and we will) identify φ with $(\varphi(x_n)) \in \mathbb{R}^{\mathbb{N}}$ and consider X^* as a subset of $\mathbb{R}^{\mathbb{N}}$. If $\alpha = (\varphi(x_n))$, then we will write $\langle \alpha, y\rangle$ for $\varphi(x) = \langle \varphi, x\rangle$.

We say that two bases (x_n) in X and (y_n) in Y are *equivalent* if the following holds

$$\sum_{n=1}^{\infty} a_n x_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} a_n y_n \text{ converges.}$$

An equivalence of two bases (or basic sequences) (x_n) and (y_n) we will denote by $(x_n) \sim (y_n)$.
From the closed graph theorem, we have the following result (see [2, Theorem 1.3.2 and Corollary 1.3.3])

Theorem 1.3.8. *For bases (x_n) and (y_n) in Banach spaces X and Y, respectively, the following conditions are equivalent*

(a) $(x_n) \sim (y_n)$.

(b) There is an isomorphism $T : X \to Y$ such that for every n $T(x_n) = y_n$.

(c) There exists $C > 0$ such that for every finitely nonzero sequence of real numbers (a_n) we have

$$\frac{1}{C}\Big\|\sum_{n=1}^{\infty} a_n y_n\Big\| \leq \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| \leq C\Big\|\sum_{n=1}^{\infty} a_n y_n\Big\|. \tag{1.1}$$

Thus, if we have a constant C as in the condition (c) in the Theorem 1.3.8 then we say that (x_n) and (y_n) are *C-equivalent*. Note that if (x_n) and (y_n) are C-equivalent, then they are also C'-equivalent for every $C' > C$. If $C = 1$, then (x_n) and (y_n) are said to be *isometrically equivalent*. Moreover, (x_n) and (y_n) are *permutatively equivalent* if there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $(x_{\pi(n)})$ is equivalent to (y_n).

Let (x_n) be a basic sequence in a Banach space X and let (p_n) be an increasing sequence of natural numbers. A sequence of vectors (z_k) in X of the form $z_k = \sum_{n=p_k+1}^{p_{k+1}} a_n x_n$ is called a block basic sequence of the (x_n) (here (a_n) is a sequence of scalars).
The notion of block basic sequence (or block basis) is very useful, which is seen in the result of Bessaga and Pełczyński (see [15]).

Proposition 1.3.9. *Let X be a Banach space with a Schauder basis and let Y be its infinite-dimensional subspace. Then there is a subspace Z of Y with a basis, which is equivalent to a block basis of (x_n).*

A basis (x_n) of X is called *unconditional* if for every permutation $\pi : \mathbb{N} \to \mathbb{N}$ $(x_{\pi(n)})$ is a basis of X. This is equivalent to say that for every $x \in X$ and for every choice of signs $\theta_n \in \{-1, 1\}^{\mathbb{N}}$ the series $\sum_{n\in\mathbb{N}} \theta_n x_n$ is convergent.

The standard bases of c_0 and ℓ_p are unconditional, but the two bases from Example 1.3.5 are not. Another simple example of a non-unconditional basis is also the so-called *summing basis* of c_0. This is the basis (x_n) defined as $x_n = \sum_{i=1}^{n} e_i$ for $n \in \mathbb{N}$.
We say that a basis (x_n) of a Banach space X is *shrinking* if the sequence of biorthogonal functionals (x_n^*) is a basis for X^*.
The classical example of a space with a shrinking basis is c_0. On the other hand, since $\ell_1^* = \ell_\infty$, then ℓ_1 is an example of a space without a shrinking basis.
For spaces with shrinking bases, there is a useful representation of their second dual spaces.

Proposition 1.3.10. *Let X be a Banach space with a shrinking basis (x_n). Then X^{**} can be identified with the space*

$$\mathrm{FIN} = \Big\{(a_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n \Big\| \sum_{i=1}^{n} a_i x_i \Big\| < \infty \Big\}$$

via the map $X^{**} \ni x^{**} \mapsto \big(x^{**}(x_i)\big) \in \mathrm{FIN}$.

A dual notion for shrinking basis is a *boundedly complete* basis. We say that a basis (x_n) of a Banach space X is boundedly complete, if for every sequence of scalars (a_n) such that $\sup_n \big\| \sum_{i=1}^{n} a_i x_i \big\| < \infty$, the series $\sum_{n=1}^{\infty} a_n x_n$ is convergent.
The standard unit vector basis is an example of a boundedly complete basis in ℓ_p for $p \geq 1$. However, this basis is not boundedly complete in c_0. Indeed, take $a_n = 1$ for every $n \in \mathbb{N}_+$. Then $\sup_n \big\| \sum_{i=1}^{n} a_i x_i \big\| = 1$ but a $\sum_{n=1}^{\infty} e_n$ is not convergent in c_0.

It is known that if (x_n) is a shrinking basis in X, then (x_n^*) is a boundedly complete basis in X^*. It explains why these two notions are considered dual.
One can ask whether for a boundedly complete basis, the opposite is true. Namely, is the Banach space X with a boundedly complete basis (x_n) isomorphic to some dual space? The answer to this question is affirmative (see [39, Proposition 1.b.4]).

Using the notions of shrinking and boundedly complete bases, R. C. James provided a convenient characterization of reflexivity for spaces with bases.

Theorem 1.3.11 ([35])**.** *Let X be a Banach space with a Schauder basis (x_n). Then X is reflexive if and only if (x_n) is both shrinking and boundedly complete.*

Quasi-Banach spaces

In this part of the preliminaries, we introduce a broader class of spaces than the class of Banach spaces. This notion will be used in one of the following chapters.
A *quasi-norm* in a vector space X is a function $\|\cdot\| \to \mathbb{R}$ satysfing

- $\|x\| = 0 \Leftrightarrow x = 0$,
- For every $\lambda \in \mathbb{R}$ $\|\lambda x\| = |\lambda|\|x\|$,
- There is $c \geq 1$ such that $\|x + y\| \leq c(\|x\| + \|y\|)$.

The minimal constant c working above is sometimes called the *modulus of concavity* of the quasi-norm. In particular, for $c = 1$ we get the definition of a norm.
In what follows, we will sometimes allow quasi-norms to take possibly infinite values. If $\|\cdot\|$ is a quasi-norm (taking only finite values) on a vector space X, then the pair $(X, \|\cdot\|)$ is called a *quasi-normed space*.
Note that a quasi-Banach space X that is not a Banach space cannot be locally convex. Therefore, results that hold in Banach spaces and rely on local convexity (e.g. Hahn-Banach extension property or Krein-Milman theorem), in general, are no longer valid in quasi-Banach spaces (see [36]). However, the standard results of Banach space theory such as the Open Mapping Theorem, Uniform Boundedness Principle and the Closed Graph Theorem can be applied in quasi-Banach spaces since they depend only on the completeness of the space.

1.3.1 Combinatorial spaces

In this subsection, we introduce the main notion of this thesis - a combinatorial Banach space.
We introduce a few definitions leading to the final notion.
For $A \subseteq \mathbb{N}$ we denote by $P_A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ the usual coordinate projection along the set, i.e.

$$P_A(x)(k) = \begin{cases} x(k), & \text{if } k \in A \\ 0, & \text{otherwise} \end{cases} .$$

For $x \in \mathbb{R}^{\mathbb{N}}$ we denote by $\operatorname{supp}(x)$ the *support* of x, i.e. the set of all $k \in \mathbb{N}$ such that $x(k) \neq 0$. By c_{00} we denote the set of all sequences with finite support.

Definition 1.3.12. We say that a function $\varphi : \mathbb{R}^{\mathbb{N}} \to [0, \infty]$ is a *nice extended (quasi)-norm* if it enjoys all conditions of being (quasi)-norm, possibly attains infinity, and, in addition, it satisfies the following conditions

(a) (*Non-degeneration*) $\varphi(x) < \infty$ for every $x \in c_{00}$,

(b) (*Monotonicity*) For $x, y \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$ $|x(n)| \leq |y(n)|$ implies $\varphi(x) \leq \varphi(y)$,

(c) (*Lower semicontinuity*) $\lim\limits_{n\to\infty} \varphi(P_n(x)) = \varphi(x)$ for every $x \in \mathbb{R}^{\mathbb{N}}$.

For an extended (quasi)-norm, we define the following sets

$$\mathrm{FIN}(\varphi) = \{x \in \mathbb{R}^{\mathbb{N}} : \varphi(x) < \infty\}, \tag{1.2}$$

$$\mathrm{EXH}(\varphi) = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} \varphi(P_{\mathbb{N}\setminus n}(x)) = 0\}. \tag{1.3}$$

Note that $\mathrm{EXH}(\varphi)$ has an equivalent definition: it is simply the completion of c_{00} with respect to φ.
The notion of FIN and EXH is inspired by the theory of ideals. In [17], the authors presented an interplay between the theory of ideals on $\mathbb{N}$ and Banach space theory. For example, they proved that $\mathrm{FIN}(\varphi)$ and $\mathrm{EXH}(\varphi)$, equipped with the nice extended norm φ, are Banach spaces and $\mathrm{EXH}(\varphi)$ has an unconditional basis consisting of standard unit vectors ([17], Proposition 5.1). The last part of this sentence can be reversed. Namely, every Banach space with an unconditional basis is isometrically isomorphic to $\mathrm{EXH}(\varphi)$ for some nice extended norm φ.
Note that $\mathrm{EXH}(\varphi) \subseteq \mathrm{FIN}(\varphi)$, not only as a subset but also as a (closed) subspace. The other inclusion holds if and only if $\mathrm{EXH}(\varphi)$ does not contain an isomorphic copy of c_0, i.e. when (e_n) is a boundedly complete basis in $\mathrm{EXH}(\varphi)$ (see [17, Theorem 5.4].)

Now we can finally present the most important definition. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be *hereditary* (i.e. closed under taking subsets) and *covering* $\mathbb{N}$ (i.e. $\bigcup \mathcal{F} = \mathbb{N}$). For $x \in \mathbb{R}^{\mathbb{N}}$ consider the following expression

$$\|x\|_{\mathcal{F}} = \sup_{F\in\mathcal{F}} \sum_{k\in F} |x(k)|. \tag{1.4}$$

It is easy to see that this is a nice extended norm and thus $\mathrm{EXH}(\|\cdot\|_{\mathcal{F}})$ is a Banach space with unconditional basis. This space is called *combinatorial Banach space* associated with the family $\mathcal{F}$ (sometimes we will also say that it is $\mathcal{F}$'s combinatorial space). It is convenient and also common in literature to denote this space by $X_{\mathcal{F}}$. In addition, to abbreviate the notation, we will denote the space $\mathrm{FIN}(\|\cdot\|_{\mathcal{F}})$ by $Z_{\mathcal{F}}$.

The name *combinatorial space* comes from the weblog of Gowers (see [31]), although such spaces were studied much earlier.
In 1930, Banach and Saks proved that every bounded sequence in L_p (for $p > 1$) has a subsequence with norm convergent arithmetic means (see [10]). Such property is nowadays called the *Banach-Saks* property. They asked whether $C([0,1])$ also satisfies this property. The negative answer to this question was given in the same year by Schreier. In [49], he constructed a sequence of continuous functions weakly convergent to 0 without a subsequence whose arithmetic means are convergent in a norm. So, in particular, he presented an example of a Banach space without the *weak Banach-Saks property*. In his construction, Schreier used a family of subsets of $\mathbb{N}$, which is now known as the *Schreier family*, and its elements are called the *Schreier sets*. We say that $F \in [\mathbb{N}_+]^{<\infty}$

is the Schreier set, if $F = \emptyset$ or $|F| \leq \min(F)$.
In 1979, Beauzamy in his paper ([13]) used a notion of the Schreier family $\mathcal{S}$ (he called its elements *admissible*) to construct a Banach space which is now known as the *Schreier space*. He used this space to construct another Banach space, being an example of a reflexive space without the Banach-Saks property (that all spaces with Banach-Saks properties are necessarily reflexive was proved by Nishiura and Waterman in [42]). The Schreier space was defined by Beauzamy as the completion of c_{00} with respect to the norm

$$\|x\|_{\mathcal{S}} = \sup \Big\{ \sum_{i \in A} |x(i)| : A \text{ is admissible} \Big\}$$

for $x \in c_{00}$. So, in the light of our definition, it is a combinatorial Banach space associated with the Schreier family.

As authors of [4] stated, there is some inconsistency in the term *combinatorial space*. The most common assumption is that family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ is

- hereditary,
- compact,
- *spreading*, meaning that for every $k \in \mathbb{N}$ and every $\{m_1, ..., m_k\} \in \mathcal{F}$, if $m_i \leq n_i$ for each $i \leq k$, then $\{n_1, ..., n_k\} \in \mathcal{F}$.

In this case, the family $\mathcal{F}$ is called *regular*. Combinatorial Banach spaces associated with regular families are quite well studied in the literature, as then they resemble the Schreier space (e.g., all such spaces have a shrinking basis consisting of standard unit vectors). However, as mentioned just before introducing the norm (1.4), we *only* assume that our families are hereditary and covering $\mathbb{N}$. Any additional assumption about $\mathcal{F}$ will be clearly indicated.

Chapter 2

The zoo of combinatorial Banach spaces

This chapter is entirely devoted to the consideration of various examples of combinatorial Banach spaces. We will present how combinatorial properties of a family $\mathcal{F}$ influence the properties of the induced Banach space.

2.1 c_0 and ℓ_1

We start with examples being the *classical* Banach spaces. Let $\mathcal{F}_1 = [\mathbb{N}]^{\leq 1}$. It is easy to verify that in these cases, the standard basis of $X_\mathcal{F}$ is isometrically equivalent to the standard basis of c_0 and thus these two spaces are isometrically isomorphic.
It is important that c_0 can also be seen as a combinatorial space related to other families, but only isomorphically. Namely, fix natural number $n > 1$ and let $\mathcal{F}_n = [\mathbb{N}]^{\leq n}$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_n$, then $\|\cdot\|_{\mathcal{F}_1} \leq \|\cdot\|_{\mathcal{F}_n}$. On the other hand, for every $x \in c_{00}$ and every $F \in \mathcal{F}_n$ we have

$$\sum_{i \in F} |x(i)| \leq n \max_{i \in F} |x(i)| \leq n \sup_{k \in \mathbb{N}} |x(k)| = n\|x\|_{\mathcal{F}_1} \tag{2.1}$$

Since c_{00} is dense in combinatorial spaces, it is enough to conclude that for each n $X_{\mathcal{F}_n}$ is isomorphic to c_0.

It is quite obvious, but worth mentioning that the above inequality cannot be improved to the isometric equivalence. One of the arguments is that in c_0 endowed with a standard sup norm (i.e. $X_{\mathcal{F}_1}$), the unit ball has no extreme points, whereas for each $n > 1$ the standard unit vectors e_i are extreme. Another reason for which the spaces $X_{\mathcal{F}_n}$ and $X_{\mathcal{F}_m}$ are not isometrically isomorphic for $n \neq m$ is given by a result from the paper of Brech, Ferenczi and Tcaciuc (see [21, Corollary 12]). The authors proved that two combinatorial spaces related to the regular families $\mathcal{F}$ and $\mathcal{G}$ are isometrically isomorphic if and only if there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{G} = \{\pi[F] : F \in$

$\mathcal{F}\}$. For each n, $\mathcal{F}_n$ is regular, but for $n \neq m$, there is no permutation for which $\mathcal{F}_n$ and $\mathcal{F}_m$ would be in such a relation.

The second classical Banach space from the title of this section can also be viewed as a combinatorial space. If $\mathcal{F} = [\mathbb{N}]^{<\infty}$, then it is straightforward that the standard basis of $X_\mathcal{F}$ is isometrically equivalent to the standard basis of ℓ_1. It is worth noting that if we allowed infinite sets in the definition of a combinatorial norm, there would be one more family that induces a space isometrically isomorphic to ℓ_1. Indeed, one can easily see that for $\mathcal{I} = \mathcal{P}(\mathbb{N})$, the combinatorial norm $\|\cdot\|_\mathcal{I}$ is isometrically equivalent to ℓ_1-norm. Note that $\overline{\mathcal{I}} = \mathcal{F}$. Such a phenomenon is a general fact concerning combinatorial spaces, namely for every family $\mathcal{G}$ $X_\mathcal{G} \equiv X_{\overline{\mathcal{G}}}$, i.e., the family and its topological closure give the same combinatorial space.
In further sections of this chapter, we will see many other examples of families related isomorphically to ℓ_1.

The heuristic and informal intuition about c_0 and ℓ_1 is that they are completely different in many ways. This section confirms that these spaces have a different combinatorial flavor, because they are induced by families coming from opposite ends of the spectrum. One is given by singletons, whereas the other is associated with a power set of $\mathbb{N}$. We can, however, obtain a combinatorial space which is, in some sense, a mix of these two spaces.
Let $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ be a partition of the set of natural numbers such that $|C_n| < \infty$ for every $n \in \mathbb{N}$. Let $\mathcal{F}$ be its hereditary closure, i.e. the smallest hereditary family containing $\mathcal{C}$. We show that the space $X_\mathcal{F}$ is isometrically isomorphic to c_0-direct sum of the spaces $\ell_1^{|C_n|}$.
For every n, let σ_n denote an increasing bijection between $|C_n|$ and C_n. Note that for every $m \in \mathbb{N}$ there is exactly one n_m and $j < |C_n|$ such that

$$m = \sigma_{n_m}(j) \tag{2.2}$$

So define $T : \left(\bigoplus_{n=1}^\infty \ell_1^{|C_n|}\right)_{c_0} \to X_\mathcal{F}$ by

$$T((x_n))(m) = x_{n_m}\big(\sigma_{n_m}^{-1}(m)\big),$$

where m and n_m are in the correspondence (2.2). By the assumption, $\|x_n\|_1 \xrightarrow{n\to\infty} 0$, thus T is well-defined. Also we have

$$\|T((x_n))\|_\mathcal{F} = \sup_{F\in\mathcal{C}} \sum_{k\in F} |T((x_n))(k)| = \sup_{n\in\mathbb{N}} \sum_{k\in C_n} |x_n(k)| = \sup_{n\in\mathbb{N}} \|x_n\|_1.$$

Hence, T is an isometry. T is also surjective, because for every $y \in \mathcal{F}$ using (2.2) we can build in a natural way an element $x \in \left(\bigoplus_{n=1}^\infty \ell_1^{|C_n|}\right)_{c_0}$ such that $T(x) = y$.

2.2 Schreier spaces and compact families

The notion of the Schreier family was mentioned in the previous chapter. Recall that it is defined as follows

$$\mathcal{S} = \{\emptyset\} \cup \{A \subseteq \mathbb{N}_+ : |A| \leq \min(A)\}. \tag{2.3}$$

The Schreier space is the best-known and most studied combinatorial space in the literature. For example, it was proved (see [49]) that this space has no weak Banach-Saks property. Also, in [22], the authors prove that the Schreier space is c_0-saturated. In fact, even more is true - E.Odell proved in [43] that every quotient of the Schreier space is c_0-saturated.
Alspach and Argyros in their paper [3] generalized Schreier families by the following inductive procedure. Let $\mathcal{S}_0 = [\mathbb{N}]^{\leq 1}$. If $\alpha < \omega_1$ is a successor ordinal and $\alpha = \beta + 1$ let

$$\mathcal{S}_\alpha = \Big\{ \bigcup_{i=1}^{k} F_i : k \leq F_1 < F_2 < ... < F_k \text{ and } F_i \in \mathcal{S}_\beta \text{ for every } i \leq k \Big\} \cup \big\{\emptyset\big\}.$$

For α being a limit ordinal, let α_n be an increasing sequence convergent to α. Then

$$\mathcal{S}_\alpha = \big\{ F \subseteq \mathbb{N}_+ : \text{ there is } n \geq 1 \text{ with } F \in \mathcal{S}_{\alpha_n} \text{ and } n \leq F \big\} \cup \big\{\emptyset\big\}.$$

The family $\mathcal{S}_\alpha$ is called *the Schreier family of order* α and the Banach space associated with it is called *the Schreier space of order* α. Note that, in particular, the *standard* Schreier family is a Schreier family of order 1. In most cases we denote it rather by $\mathcal{S}$ (like above) than $\mathcal{S}_1$.

The Schreier families are examples of regular families of subsets of $\mathbb{N}$, a notion of which was introduced in Preliminaries. Some of the results concerning the Schreier space can be upgraded to any combinatorial space associated with regular families. For instance, for any regular family $\mathcal{F}$, $X_\mathcal{F}$ is c_0-saturated. In fact, the assumption of $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ being spreading can be omitted and only its compactness and being hereditary are important (see [17, Theorem 6.3]). Thus, we obtain a convenient characterization expressed topologically: $X_\mathcal{F}$ is c_0-saturated if and only if $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ is compact in $\mathcal{P}(\mathbb{N})$.
There is also another equivalent condition for a combinatorial space to be c_0-saturated. Namely, $X_\mathcal{F}$ is c_0-saturated if and only if it does not contain an isomorphic copy of ℓ_1 (i.e. the standard basis of $X_\mathcal{F}$ is shrinking, see [16, Proposition 3.10]). Hence, it is natural to ask whether there is a similar characterization for ℓ_1-saturated spaces. Obviously, the lack of compactness of $\mathcal{F}$ implies the existence of a subspace isomorphic to ℓ_1; however, it can be given explicitly. Indeed, one can easily see that if $\mathcal{F}$ is not compact, then there is an infinite set A in $\overline{\mathcal{F}}$. As we mentioned in the previous section, $\mathcal{F}$

and $\overline{\mathcal{F}}$ *generate* the same combinatorial space. Since $(e_n)_{n\in A}$ is isometrically equivalent to the standard basis of ℓ_1, thus $[e_n]$ is a copy of ℓ_1 in $X_{\mathcal{F}}$.
Thus, a natural question arises: is it true that $\mathcal{F}$ is not compact if and only if it is ℓ_1-saturated? That would mean that also being ℓ_1-saturated and having no copy of c_0 is the same for combinatorial Banach spaces. There are, however, many examples of non-compact families for which the associated combinatorial Banach space is not ℓ_1-saturated. We will present them even in the next section.

2.3 Chains and Antichains

The following examples concern the families *living* on $2^{<\mathbb{N}}$, i.e., the set of all finite $0-1$ sequences. For any $s, t \in 2^{<\mathbb{N}}$ we say that t *extends* s if $s(k) = t(k)$ for every $k < |s|$ and $|s| \leq |t|$, where $|s|, |t|$ stands for the *length* of sequences s and t. This notion defines the natural order on $2^{<\mathbb{N}}$:

$$s \preceq t \Leftrightarrow t \text{ extends } s$$

Then consider the following families with respect to the order $\preceq$

$$\mathcal{A} = \{A \subseteq 2^{<\mathbb{N}} : A \text{ is a finite antichain}\}$$

and

$$\mathcal{C} = \{C \subseteq 2^{<\mathbb{N}} : C \text{ is a finite chain}\}$$

The spaces $X_{\mathcal{A}}$ and $X_{\mathcal{C}}$ were introduced by H. Rosenthal, and in literature they are usually denoted by S and B, respectively.
The space S is called the *(dyadic) stopping time space*. The name comes from the equivalent definition of this space expressed in the martingale language (see [11]). There is no unified name for the space B, however, we will call it the *chain space* in this thesis.

Unlike the Schreier families, $\mathcal{A}$ and $\mathcal{C}$ are far from being regular, and so S and B are not c_0-saturated. In fact, these spaces contain many copies both of ℓ_1 and c_0. Indeed, if D is an antichain (chain), then $(e_n)_{n\in D}$ is isometrically equivalent to the standard basis of ℓ_1 (c_0) in S and isometrically equivalent to the standard basis of c_0 (ℓ_1) in B.
One may deduce that, in general, combinatorial Banach spaces are $\{c_0, \ell_1\}$-saturated. Recall that it means that each infinite-dimensional subspace has copies of either c_0 or ℓ_1. However, S and B are counterexamples to that.
Namely, it was proved both by Schechtman and Rosenthal (both works were unpublished manuscripts) that S contains isomorphic copies of ℓ_p for each $p \in [1, \infty)$. The only proof of this fact seems to be presented by N. Dew in his PhD thesis (see [25, Section 7.6]). The proof involves machinery of probability theory and stochastic processes.

On the other hand, H. Bang and E. Odell proved that the space B is universal (for the definition see Section 2.6) for Banach spaces with unconditional basis (see [11, Theorem 2]).
Therefore, it shows that combinatorial Banach spaces can have way richer structure than one can expect, and it is not just a simple *amalgamation* of c_0 and ℓ_1.

The following fact from [11] presents an interesting relationship between stopping time space and the chain space.

Proposition 2.3.1 ([11])**.** *S^* is isometrically isomorphic to the space $Z_{\mathcal{C}}$, and B^* is isometrically isomorphic to $Z_{\mathcal{A}}$.*

So, in some sense, families $\mathcal{A}$ and $\mathcal{C}$ are *dual* to each other.

2.4 Farah spaces

In this section, we introduce a certain family of sets and its modifications that provide many examples of combinatorial spaces. This class of families is motivated by the definition of an analytic P-ideal due to Farah (see [28]).

For each $n \in \mathbb{N}$, let $I_n = [2^n, 2^{n+1})$. That is, $I_0 = \{1\}$, $I_1 = \{2, 3\}$, $I_2 = \{4, 5, 6, 7\}$ etc. *The Farah family* $\mathbf{F}$ is defined by

$$\mathbf{F} = \big\{A \in [\mathbb{N}]^{<\infty} : \forall n \in \mathbb{N}_+ \ \frac{|A \cap I_n|}{|I_n|} \leq \frac{1}{n}\big\}. \tag{2.4}$$

In other words, elements of $\mathbf{F}$ can *take* at most $\frac{1}{n}$ of interval I_n. This family can be slightly generalized in the following way. Fix function $g : \mathbb{N} \to [1, \infty)$. Then we consider *$g$-Farah family* $\mathbf{F}_g$ given by

$$\mathbf{F}_g = \big\{A \in [\mathbb{N}]^{<\infty} : \forall n \in \mathbb{N} \ |A \cap I_n| \leq g(n)\big\}.$$

In particular, the Farah family $\mathbf{F}$ is given by a function $g(n) = \frac{2^n}{n}$ (here the domain is $\mathbb{N}_+$, instead of $\mathbb{N}$). The class of spaces $X_{\mathbf{F}_g}$ is called the *Farah spaces*.
Note that for every function g, $\mathcal{F}_g$ is not compact and thus it contains a copy of ℓ_1. What is more, we can obtain an isometrically isomorphic copy of ℓ_1 as a g-Farah family. Namely, for a function given by $g(n) = 2^n$, we have $\mathbf{F}_g = [\mathbb{N}]^{<\infty}$, thus $X_{\mathbf{F}_g} \equiv \ell_1$.
In fact, we can show even a stronger result.

Theorem 2.4.1. *The Farah spaces have the Schur property.*

The proof for the *standard* Farah space $X_{\mathbf{F}}$ is presented in [17]. We present it here for any g-Farah space, with slight modifications.

Proof. Fix $g : \mathbb{N} \to [1, \infty)$. Suppose that $(x_n) \in X_{\mathbf{F}_g}$ is such that $\|x_n\|_{\mathbf{F}_g} \geq \varepsilon$ for some $\varepsilon > 0$ and for infinitely many (without loss of generality - for all) n's (i.e. (x_n) is not convergent to 0 in the norm). Without loss of generality we can assume that (x_n) is a block sequence, and thus $A_n := \operatorname{supp}(x_n)$ satisfy $|A_n| < \infty$ for every $n \in \mathbb{N}$, and $A_n < A_{n+1}$. There is a subsequence (x_{n_k}) such that for every $n \in \mathbb{N}$ there exists at most one $k \in \mathbb{N}$ such that $I_n \cap A_{n_k} \neq \emptyset$. Define a sequence of sets (B_n) as follows. If there is k such that $I_n \cap A_{n_k} \neq \emptyset$, then let $B_n (= B_n^k)$ be such subset of $I_n \cap A_{n_k}$ that $|B_n| \leq g(n)$ and $\|x_{n_k}\|_{\mathbf{F}_g} = \sum_{j \in B_n} |x_{n_k}(j)|$. If there is no such k, put $B_n = \emptyset$. Let $B = \bigcup_{n \in \mathbb{N}} B_n$ and note that $B \in \overline{\mathbf{F}_g}$ Define $\varphi : X_{\mathbf{F}_g} \to \mathbb{R}$ given by

$$\varphi(x) = \sum_{j \in B} \sum_{k=1}^{\infty} \operatorname{sgn}(x_{n_k}(j)) x(j)$$

Note that φ is linear and for every $x \in X_{\mathbf{F}_g}$ $|\varphi(x)| \leq \|x\|_{\mathbf{F}_g}$, hence $\varphi \in X^*_{\mathbf{F}_g}$. However, for every $l \in \mathbb{N}$ we have

$$\varphi(x_{n_l}) = \sum_{j \in B} \sum_{k=1}^{\infty} \operatorname{sgn}(x_{n_k}(j)) x_{n_l}(j) = \sum_{j \in B_n^l} |x_{n_l}(j)| = \|x_{n_l}\|_{\mathbf{F}_g} \geq \varepsilon.$$

Thus φ is not weakly null. □

In particular, we have an immediate corollary.

Corollary 2.4.2. *For every $g : \mathbb{N} \to [1, \infty)$, $X_{\mathbf{F}_g} = Z_{\mathbf{F}_g}$.*

Now fix functions $g, h : \mathbb{N} \to [1, \infty)$. We present some relations between values of g and h, and an isomorphic structure of their Farah spaces. However, before we show this result, we introduce briefly the notions that will be used in the proof.
Recall that $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is a function called a *floor function* and it is defined by

$$\lfloor r \rfloor = \max\{k \in \mathbb{Z} : k \leq r\}.$$

For every $x \in \mathbb{R}^{\mathbb{N}}$ consider such bijection $\sigma : \mathbb{N} \to \mathbb{N}$ that for every $n \in \mathbb{N}$ and every $k \in I_n$ $|x(\sigma(k))| \geq |x(\sigma(k+1))|$. Let y be a sequence defined by $y(k) = x(\sigma(k))$. Note that y restricted to every interval I_n is non-increasing and for every function $\psi : \mathbb{N} \to [1, \infty)$ $\|x\|_{\mathbf{F}_\psi} = \|y\|_{\mathbf{F}_\psi}$. For every $n \in \mathbb{N}$ and $0 \leq k < 2^n$ let

$$a_k^n = |y(2^n + k)|.$$

Then

$$\|y\|_{\mathbf{F}_\psi} = \sum_{n \in \mathbb{N}} \sum_{k=0}^{\lfloor \psi(n) \rfloor} a_k^n$$

Proposition 2.4.3. *Let $g, h : \mathbb{N} \to [1, \infty)$ be such that there exist positive constants $c_1 \leq c_2$ such that for every $n \in \mathbb{N}$*

$$c_1 \leq \frac{\lfloor g(n) \rfloor}{\lfloor h(n) \rfloor} \leq c_2$$

Then $X_{\mathbf{F}_g}$ and $X_{\mathbf{F}_h}$ are isomorphic.

Proof. We use notions introduced above. For convenience, we distinguish three cases.

(a) **Case 1.** If $c_2 \geq c_1 \geq 1$, then for every n $\lfloor h(n) \rfloor \leq \lfloor g(n) \rfloor$, and thus $\|y\|_{\mathbf{F}_h} \leq \|y\|_{\mathbf{F}_g}$. On the other hand, for every $n \in \mathbb{N}$ we have

$$a_1^n + ... + a_{\lfloor g(n) \rfloor}^n \leq c_2(a_1^n + ... + a_{\lfloor h(n) \rfloor}^n),$$

hence $\|y\|_{\mathbf{F}_g} \leq c_2 \|y\|_{\mathbf{F}_h}$.

(b) **Case 2.** $c_1 \leq 1$ and $c_2 \leq 1$. This case is symmetric to Case 1. Here we have $\|y\|_{\mathbf{F}_g} \leq \|y\|_{\mathbf{F}_h}$, because $\lfloor g(n) \rfloor \leq \lfloor h(n) \rfloor$. Also,

$$a_1^n + ... + a_{\lfloor h(n) \rfloor}^n \leq \frac{1}{c_1}(a_1^n + ... + a_{\lfloor g(n) \rfloor}^n),$$

and so $\|y\|_{\mathbf{F}_g} \geq c_1 \|y\|_{\mathbf{F}_h}$.

(c) **Case 3.** Let $c_1 \leq 1$ and $c_2 \geq 1$. Let $M_0 = \{n \in \mathbb{N} : \lfloor g(n) \rfloor \leq \lfloor h(n) \rfloor\}$ and $M_1 = \mathbb{N} \setminus M_0$. Then, for every $n \in M_0$ we repeat the step from Case 1, and for $n \in M_1$ - the step from Case 2.

These three cases combined yield

$$c_1 \|y\|_{\mathbf{F}_h} \leq \|y\|_{\mathbf{F}_g} \leq c_2 \|y\|_{\mathbf{F}_h}. \tag{2.5}$$

This implies that $X_{\mathbf{F}_g}$ and $X_{\mathbf{F}_h}$ are equal as sets and that the identity operator is an isomorphism. □

It is natural to ask whether the opposite theorem holds or, at least, whether there is any relation between g and h. We do not know the answer, hence we formulate the following problem.

Problem 2.4.4. Given functions g and h, suppose $X_{\mathbf{F}_g}$ and $X_{\mathbf{F}_h}$ are isomorphic. Is there any relation between the values of g and h? In particular, are there $0 < c_1 \leq c_2$ such that $c_1 \leq \frac{\lfloor g(n) \rfloor}{\lfloor h(n) \rfloor} \leq c_2$

Note that, if there is an isomorphism $T : X_{\mathbf{F}_h} \to X_{\mathbf{F}_g}$, then for every $x \in X_{\mathbf{F}_h}$

$$a\|x\|_{\mathbf{F}_h} \leq \|T(x)\|_{\mathbf{F}_g} \leq b\|x\|_{\mathbf{F}_h},$$

for some positive numbers $a \leq b$. Thus, a and b would be natural candidates for constants in the Problem 2.4.4.
Also, note that there are examples of Farah spaces which indicate a positive answer to the question. Indeed, consider $g = 1$ and h defined by $h(n) = 2^n$, i.e. $\frac{g(n)}{h(n)}$ can be arbitrarily small. As it was mentioned earlier, $X_{\mathbf{F}_h}$ is isometrically isomorphic to ℓ_1. If $X_{\mathbf{F}_g}$ were isomorphic to ℓ_1, then its basis would be equivalent to the standard unit vector basis of ℓ_1 (as ℓ_1 has a unique unconditional basis, see [38]). However, for every $N \in \mathbb{N}$ $\|\sum_{k<N} e_k\|_{\mathbf{F}_h} = N$, whereas $\|\sum_{k<N} e_k\|_{\mathbf{F}_g} = \max\{\lfloor \log_2 N \rfloor, 1\}$. Hence, these bases cannot be equivalent, and thus $X_{\mathbf{F}_g}$ and $X_{\mathbf{F}_h}$ are not isomorphic.

2.5 Modifications of Farah families

Theorem 2.4.1 provides a family of Banach spaces with the Schur property, and therefore with the ℓ_1-saturation property. Moreover, many of these spaces are not isomorphic to ℓ_1. More precisely, every g-Farah space with g satisfying $\lim_{n\to\infty} \frac{g(n)}{2^n} = 0$ serves as an appropriate exmaple. In particular, classical Farah space is not isomorphic to ℓ_1 (see [17]).
In this section, we present some modifications of Farah families whose combinatorial spaces differ from g-Farah spaces. As the choice of the function g is not crucial in these constructions, we fix in this section a function g associated with the classical Farah family **F** for simplicity (see the Definition 2.4).

2.5.1 F with intervals

Consider the following family

$$\mathbf{IF} = \{F \cup E : F \in \mathbf{F}, \text{ and } E \subseteq I_n \text{ for some } n \in \mathbb{N}\}.$$

So this family is created by adding some interval to the Farah. It may look like a cosmetic modification, but this extension changes the resulting combinatorial space quite fundamentally. Namely, $X_{\mathbf{IF}}$ contains an isomorphic copy of c_0.
To see this, consider sequence $x = \sum_{n\in\mathbb{N}} x_n$, where

$$x_n = \frac{1}{2^{2^n}} \chi_{I_{2^n}}$$

Then

$$\|x\|_{\mathbf{IF}} = \|\sum_{n\in\mathbb{N}} x_n\|_{\mathbf{IF}} \leq 1 + \sum_{n\in\mathbb{N}} \frac{1}{2^{2^n}} \cdot \frac{2^{2^n}}{2^n} = 1 + \sum_{n\in\mathbb{N}} \frac{1}{2^n} = 3 < \infty,$$

so $x \in Z_{\mathbf{IF}}$. However, since for every $n \in \mathbb{N}$ $I_n \in \mathbf{IF}$, then $\|P_{\mathbb{N}\setminus n}(x)\|_{\mathbf{IF}} \geq 1$, and so $x \notin X_{\mathbf{IF}}$. Thus $X_{\mathbf{IF}}$ contains a subspace isomorphic to c_0 (see [17, Theorem 5.4]).

2.5.2 The rapid Farah

Let $D \subseteq \mathbb{N}_+^{\mathbb{N}_+}$ consists of all strictly increasing sequences. For every $\sigma \in D$ define a function $s_\sigma : \mathbb{N}_+ \to [0, \infty)$ in the following way

$$s_\sigma(m) = \begin{cases} \dfrac{2^{\sigma(k)}}{k}, & \text{if there exists } k \in \mathbb{N} \text{ such that } m = \sigma(k) \\ 0, & \text{otherwise.} \end{cases}$$

Next, define the family $\mathcal{A}_\sigma$ given by

$$\mathcal{A}_\sigma = \big\{A \in [\mathbb{N}]^{<\infty} : \forall n \in \mathbb{N}_+ \ |A \cap I_n| \leq s_\sigma(n)\big\}.$$

Finally, we define a family $\mathbf{RF}$, called *the rapid Farah family*, in the following way

$$\mathbf{RF} = \big\{F \in [\mathbb{N}]^{<\infty} : \exists \sigma \in D \ \exists A \in \mathcal{A}_\sigma \ F \subseteq A\big\} \tag{2.6}$$

The formal definition of this family does not seem to be friendly; however, the intuition is clear. The finite set F is an element of $\mathbf{RF}$ if and only if for some increasing bijection of natural numbers σ, F can *take* the whole interval $I_{\sigma(1)}$, a half of the interval $I_{\sigma(2)}$ and so on. Note that for every $n \in \mathbb{N}_+$, I_n is an element of $\mathbf{RF}$ - in (2.6) it suffices to take any $\sigma \in D$ such that $\sigma(1) = n$. Also notice that $\mathbf{F} \subseteq \mathbf{RF}$ (consider $\sigma(n) = n$).

The rapid Farah space $X_{\mathbf{RF}}$ turns out to be another example of an ℓ_1-saturated Banach space without the Schur property.

Proposition 2.5.1. *$X_{\mathbf{RF}}$ does not have the Schur property.*

Proof. For every $n \in \mathbb{N}_+$ let $x_n = \dfrac{1}{2^n}\chi_{I_n}$. Since $I_n \in \mathbf{RF}$ for every n, then $\|x_n\|_{\mathbf{RF}} = 1$. However, we show that $(x_n)_{n\in\mathbb{N}_+}$ is weakly null.
Fix $x^* \in X^*_{\mathbf{RF}}$. Then $x^*(x) = \sum_{k\in\mathbb{N}_+} \alpha_k x(k)$. Suppose there exists $\varepsilon > 0$ and $A \in [\mathbb{N}_+]^\infty$ such that for every $n \in A$ $|x^*(x_n)| > \varepsilon$. It means that for each $n \in A$ we have

$$\big|\sum_{j\in I_n} \alpha_j\big| > 2^n\varepsilon.$$

Denote $A = \{n_1, n_2, n_3, ...\}$ and assume that $n_i < n_{i+1}$ for every i. For every $n \in \mathbb{N}$ let $M_n = \bigcup_{k=1}^{n} I_{n_k}$. Put $w = \sum_{k\in\mathbb{N}_+} \operatorname{sgn}(\alpha_k)x_k$ and $y_n = P_{M_n}(w)$. Then we have

$$\|y_n\|_{\mathbf{RF}} \leq 1 + \frac{1}{2} + ... + \frac{1}{n} \leq \log_2(n+1).$$

On the other hand,

$$|x^*(y_n)| = \Big|\sum_{j\in M_n} \alpha_j y_n(j)\Big| = \Big|\sum_{k=1}^{n}\sum_{j\in I_{n_k}} \alpha_j \frac{\operatorname{sgn}(\alpha_j)}{2^{n_k}}\Big| \geq \sum_{k=1}^{n} \frac{1}{2^{n_k}}\Big|\sum_{j\in I_{n_k}} \alpha_j\Big| > n\varepsilon$$

Hence $\dfrac{|x^*(y_n)|}{\|y_n\|_{\mathbf{RF}}} > \dfrac{n\varepsilon}{\log_2(n+1)}$, and so x^* is unbounded, which is a contradiction. ☐

To prove that $X_{\mathbf{RF}}$ is ℓ_1-saturated, we need to introduce some notation. For $1 \leq l \leq 2^n$ we define the formula $\|\cdot\|_{n,l}$ on $\mathbb{R}^{\mathbb{N}}$ by

$$\|x\|_{n,l} = \max\left\{\sum_{i\in F} |x(i)| : F \subseteq I_n \text{ and } |F| \leq \frac{|I_n|}{l}\right\}.$$

Of course, $\|\cdot\|_{n,l}$ is a norm on $\{x \in \mathbb{R}^{\mathbb{N}} : \operatorname{supp}(x) \subseteq I_n\} \simeq \mathbb{R}^{I_n}$. For example, $\|x\|_{\mathbf{F}} = \sum_{n=1}^{\infty} \|x\|_{n,n}$ and

$$\|x\|_{\mathbf{RF}} = \sup\left\{\sum_{k=1}^{\infty} \|x\|_{n_k,k} : (n_k) \in \mathbb{N}^{\mathbb{N}} \text{ is strictly increasing}\right\}.$$

We prove an easy observation basically saying that the sequence $\|x\|_{n,l}$ does not decrease too fast in l under a certain condition.

Lemma 2.5.2. *If $x \in \mathbb{R}^{\mathbb{N}}$, $1 \leq l \leq l'$, and $(l'+1)^2 \leq 2^n$, then*

$$\|x\|_{n,l'} \geq \frac{l}{l'+1}\|x\|_{n,l}. \tag{2.7}$$

Proof. It is easy to see that if $1 \leq K' \leq K$, $v \in \mathbb{R}^K$, and $v(1) \geq v(2) \geq \cdots \geq v(K) \geq 0$, then $(v(1) + \cdots + v(K'))/(v(1) + \cdots + v(K)) \geq K'/K$. It follows that

$$\begin{aligned}\frac{\|x\|_{n,l'}}{\|x\|_{n,l}} &\geq \frac{\lfloor 2^n/l' \rfloor}{\lfloor 2^n/l \rfloor} \geq \frac{2^n/l' - 1}{2^n/l} = \frac{l}{l'} - \frac{l}{2^n} \\ &\geq \frac{l}{l'} - \frac{l}{(l'+1)^2} > \frac{l \cdot l' \cdot (l'+1)}{l' \cdot (l'+1)^2} = \frac{l}{l'+1}.\end{aligned}$$

☐

Theorem 2.5.3. *The space $X_{\mathbf{RF}}$ is ℓ_1-saturated.*

Proof. Applying the Selection Principle (see e.g. [27, Theorem 4.26]), it is enough to find copies of ℓ_1 in subspaces of the form $[(x_m)]$ where (x_m) is a normalized block basic sequence. We can assume that the sets $D_m = \{n : \operatorname{supp}(x_m) \cap I_n \neq \emptyset\}$ are consecutive and fix

$$\{n_1^m < n_2^m < \cdots < n_{l_m}^m\} \subseteq D_m \text{ such that } 1 = \|x_m\|_{\mathbf{RF}} = \sum_{k=1}^{l_m} \|x_m\|_{n_k^m,k}.$$

The proof is based on the following technical statement:
Claim. Let $s \in \mathbb{N}$. Then there is a $y \in [(x_m)]$ such that the following holds:

(a) $\operatorname{supp}(y) \subseteq \mathbb{N} \setminus \bigcup_{n=1}^s I_n$ is finite and $\|y\|_{\mathbf{RF}} = 1$.

(b) If $z \in c_{00}$, $\operatorname{supp}(z) \subseteq \bigcup_{n=1}^s I_n$, and $\beta \in \mathbb{R}$, then

$$\|z + \beta y\|_{\mathbf{RF}} \geq \|z\|_{\mathbf{RF}} + |\beta|/2.$$

Let us first show that this implies the theorem. We can construct inductively a normalized block basic sequence $y_k \in [(x_m)]$ the following way: Let $y_1 = x_1$ and in general, let y_{k+1} be y from the claim above to an s satisfying $\operatorname{supp}(y_k) \subseteq \bigcup_{n=1}^s I_n$. To finish the argument, we show that (y_k) is equivalent to the standard basis of ℓ_1. If $K \in \mathbb{N}$ and $\theta \in \mathbb{R}^K$ then

$$\begin{aligned}\Big\|\sum_{k=1}^{K} \theta(k) y_k\Big\|_{\mathbf{RF}} &\geq \Big\|\sum_{k=1}^{K-1} \theta(k) y_k\Big\|_{\mathbf{RF}} + \frac{|\theta(K)|}{2} \\ &\geq \Big\|\sum_{k=1}^{K-2} \theta(k) y_k\Big\|_{\mathbf{RF}} + \frac{|\theta(K-1)|}{2} + \frac{|\theta(K)|}{2} \geq \dots \\ &\geq |\theta(1)| + \frac{|\theta(2)|}{2} + \cdots + + \frac{|\beta(K)|}{2} \geq \frac{1}{2}\sum_{k=1}^{K} |\theta(k)|.\end{aligned}$$

Regarding the claim, we distinguish two cases.
Case 1. $\max\{\|x_m\|_{n_i^m,1} : i = 1, \dots, l_m\} \xrightarrow{m\to\infty} 0$.
We show that $y = x_m$ is as required if m is large enough. Take an arbitrary $m \in \mathbb{N}$ such that $s \leq \min(D_m) - 4$. Then $s \leq n_1^m - 4$ and hence $s + i + 1 \leq n_1^m + (i-1) - 2 \leq n_i^m - 2$ for every $i \in [1, l_m]$. It follows that $(s + i + 1)^2 \leq 2^{n_i^m - 1}$ for every such i. The point is that, assuming $s \leq \min(D_m) - 4$ and $1 \leq i \leq l_m$,

(i) $\|x_m\|_{n_i^m, s+i}$ is defined, and

(ii) Lemma 2.5.2 applies with $x = x_m$, $l = i$, $l' = s + i$, and $n = n_i^m$.

By the definition of $\|\cdot\|_{\mathbf{RF}}$, we know that

$$\|z+\beta x_m\|_{\mathbf{RF}} \geq \|z\|_{\mathbf{RF}} + |\beta| \sum_{i=1}^{l_m} \|x_m\|_{n_i^m, s+i}.$$

Therefore, given any $r \in [1, l_m)$,

$$\begin{aligned} \|z+\beta x_m\|_{\mathbf{RF}} - \|z\|_{\mathbf{RF}} - \|\beta x_m\|_{\mathbf{RF}} &\geq |\beta| \left(\sum_{i=1}^{l_m} \|x_m\|_{n_i^m, s+i} - \sum_{i=1}^{l_m} \|x_m\|_{n_i^m, i} \right) \\ &\geq |\beta| \left(\sum_{i=r+1}^{l_m} \|x_m\|_{n_i^m, s+i} - \sum_{i=1}^{l_m} \|x_m\|_{n_i^m, i} \right). \quad (2.8) \end{aligned}$$

Now, we need to specify m a little further. Fix first r, then m from $\mathbb{N}$ such that

(r) $r/(s+r+1) \geq 3/4$;

(m) $s \leq \min(D_m) - 4$ and $\|x_m\|_{n_i^m, 1} \leq 1/4r$ for every $i \in [1, l_m]$.

Applying Lemma 2.5.2 as in (ii) above, for every $i \in (r, l_m]$ we have

$$\|x_m\|_{n_i^m, s+i} \geq \frac{i}{s+i+1} \|x_m\|_{n_i^m, i} \geq \frac{r}{s+r+1} \|x_m\|_{n_i^m, i} \geq \frac{3}{4} \|x_m\|_{n_i^m, i},$$

and hence the last difference of sums in (1) can be estimated as follows:

$$\begin{aligned} \sum_{i=r+1}^{l_m} \|x_m\|_{n_i^m, s+i} - \sum_{i=1}^{l_m} \|x_m\|_{n_i^m, i} &\geq -\frac{1}{4} \sum_{i=r+1}^{l_m} \|x_m\|_{n_i^m, i} - \sum_{i=1}^{r} \|x_m\|_{n_i^m, i} \\ &\geq -\frac{1}{4} \|x_m\|_{\mathbf{RF}} - r\|x_m\|_{n_i^m, 1} \geq -\frac{1}{4} - r\frac{1}{4r} = -\frac{1}{2}. \quad (2.9) \end{aligned}$$

Combining (2.8) and (2.9), $\|z+\beta x_m\|_{\mathbf{RF}} - \|z\|_{\mathbf{RF}} - |\beta| \geq -|\beta|/2$, hence $y = x_m$ is as desired.

Case 2. There are a $\delta > 0$, an $S \in [\mathbb{N}]^\infty$, and for every $m \in S$ an $i_m \in [1, l_m]$ such that $\|x_m\|_{n_{i_m}^m, 1} \geq \delta$.

Fix $J \in \mathbb{N}$ and $E = \{m_1 < m_2 < \cdots < m_J\} \subseteq S \setminus \{1, 2, 3\}$. Then, with $n_j = n_{i_{m_j}}^{m_j}$, we know that $1 \leq m_1 - 3 \leq n_1^{m_1} - 3 \leq n_1 - 3$, it follows that $j+1 \leq n_1 + (j-1) - 2 \leq n_j - 2$, and hence $(j+1)^2 \leq 2^{n_j - 1}$ and we can apply $(\star)$ with $l = 1$, $l' = j$, and $n = n_j$:

$$\Big\| \sum_{m \in E} x_m \Big\|_{\mathbf{RF}} \geq \sum_{j=1}^{J} \|x_{m_j}\|_{n_j, j} \geq \sum_{j=1}^{J} \frac{\|x_{m_j}\|_{n_j, 1}}{j+1} \geq \sum_{j=1}^{J} \frac{\delta}{j+1}.$$

Therefore, we can pick finite subsets $E_1 < E_2 < \dots$ of S such that $\| \sum_{m\in E_k} x_m\|_{\mathbf{RF}} \geq k$ for every k and define

$$\widetilde{x}_k = \frac{\sum_{m\in E_k} x_m}{\| \sum_{m\in E_k} x_m\|_{\mathbf{RF}}} \in [(x_m)],$$

a normalized block basic sequence. Instead of working with (x_m), we switch to $(\widetilde{x}_k)$ and define everything as above, $\widetilde{D}_k = \{n : \operatorname{supp}(\widetilde{x}_k) \cap I_n \neq \emptyset\}$, $\{\widetilde{n}_i^k : i = 1, \dots, \widetilde{l}_k\} \subseteq \widetilde{D}_k$ such that $1 = \|\widetilde{x}_k\|_{\mathbf{RF}} = \sum_{i=1}^{\widetilde{l}_k} \|\widetilde{x}_k\|_{\widetilde{n}_i^k, i}$, etc. Then

$$\max\left\{\|\widetilde{x}_k\|_{\widetilde{n}_i^k,1} : i \in [1, \widetilde{l}_k]\right\} \leq \frac{\max\left\{\|x_m\|_{\widetilde{n}_i^k,1} : m \in E_k, i \in [1, \widetilde{l}_k]\right\}}{k} \leq \frac{1}{k},$$

therefore, we can apply Case 1 to find the desired $y \in [(\widetilde{x}_k)] \subseteq [(x_m)]$. □

Remark 2.5.4. In fact, we obtained an even simpler example of an ℓ_1-saturated space without the Schur property. Consider $X = [(x_n)] \subseteq X_{\mathbf{RF}}$ where $x_n = \frac{1}{2^n}\chi_{I_n}$. Then (x_n) witnesses the failure of the Schur property, and, by the last theorem, X is ℓ_1-saturated. Considering $X \subseteq \mathbb{R}^{\mathbb{N}}$ along the 1-unconditional basis (x_n), the norm is of the following very simple form:

$$\|a\| = \sup\left\{\sum_{k=1}^{\infty} \frac{|a(n_k)|}{k} : (n_k) \in \mathbb{N}^{\mathbb{N}} \text{ is strictly increasing}\right\}.$$

In other words, X is the completion of c_{00} with respect to $\|\cdot\|$. Alternatively, $\|\cdot\|$ is an extended norm on $\mathbb{R}^{\mathbb{N}}$ and $a \in X$ if and only if $\|a\| < \infty$, if and only if $\|P_{\mathbb{N}\setminus n}(a)\| \to 0$, because $X_{\mathbf{RF}}$ does not contain copies of c_0, hence nor does X, therefore its basis is boundedly complete.
The space X is a special case of the so-called *Garling sequence space*. The Garling norm $\|\cdot\|_{w,p}$, where w is a decreasing sequence of positive numbers and $1 \leq p < \infty$, is defined by the formula

$$\|x\|_{w,p} = \sup_{\phi\in\mathcal{O}} \left(\sum_{n\in\mathbb{N}_+} |x(\phi(n))|^p w(n)\right)^{\frac{1}{p}},$$

where $\mathcal{O}$ is the set of all increasing sequences of natural numbers. The Garling space $\mathbf{g}(w,p)$ is defined as

$$\mathbf{g}(w,p) = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_{w,p} < \infty\}.$$

Hence, $X = \mathbf{g}(h,1)$, where $h(n) = \frac{1}{n}$ for $n \geq 1$.
For further details on Garling spaces, we refer the reader to [1].

2.5.3 Combinatorial spaces with prerequisite subspaces

Examples presented in the Subsection 2.3 witness that combinatorial spaces can contain any Banach space with an unconditional basis. However, it is rather difficult to track down and *really* see e.g., a copy of ℓ_2 in these examples. We will show that for every Banach space Y with an unconditional basis, there is a natural family $\mathcal{F}$ such that a complemented block basic sequence in $X_{\mathcal{F}}$ is equivalent to the basis of Y. The point is that we may encode a given "geometric" structure in the definition of $\mathcal{F}$. The idea is due to A. Pelczar-Barwacz and it is presented in the paper [18].

The family from this example is defined on the set $\Omega = \mathbb{N} \setminus \{0, 1\}$. Fix a Banach space Y with normalized 1-unconditional basis $(b_n)_{n\geq 2}$. We consider $Y \subseteq \mathbb{R}^\Omega$ along this basis (that is, $y = \sum_{n=2}^\infty y(n)b_n$), also, we consider $Y^* \subseteq \mathbb{R}^\Omega$ along (b_n^*). As (b_n) is 1-unconditional, if $\sigma \in Y^*$ then $\|\sigma\|_{Y^*} \leq \sum_{n=2}^\infty |\sigma(n)|$. Define

$$\mathcal{F}(Y) = \left\{ F \in [\Omega]^{<\infty} : \left(\frac{|F \cap I_n|}{|I_n|} \right) \in B_{Y^*} \right\} \tag{2.10}$$

and notice that it is a hereditary cover of Ω.

Theorem 2.5.5. *With Y and $\mathcal{F} = \mathcal{F}(Y)$ as above, the sequence $x_n = \frac{1}{|I_n|}\chi_{I_n}$ is a complemented normalized block basic sequence in $X_{\mathcal{F}}$ that is equivalent to (b_n).*

Proof. If $y \in c_{00}(\Omega)$, then

$$\begin{aligned} \Big\| \sum_{n=2}^\infty y(n)x_n \Big\|_{\mathcal{F}} &= \sup \left\{ \sum_{n=2}^\infty |F \cap I_n| \frac{|y(n)|}{|I_n|} : F \in \mathcal{F} \right\} \\ &= \sup \left\{ \left| \left\langle \left(\varepsilon_n \frac{|F \cap I_n|}{|I_n|} \right), y \right\rangle \right| : \varepsilon_n = \pm 1 \text{ and } F \in \mathcal{F} \right\} \\ &\leq \sup \left\{ |\langle \sigma, y \rangle| : \sigma \in B_{Y^*} \right\} = \|y\|_Y. \end{aligned}$$

Conversely, given $\sigma \in B_{Y^*}$, for each $n \geq 2$ we can fix an $F_n \subseteq I_n$ such that

$$\frac{|F_n|}{|I_n|} \leq |\sigma(n)| < \frac{|F_n| + 1}{|I_n|}.$$

Then $A_\sigma = \bigcup_{n=2}^\infty F_n \in \overline{\mathcal{F}}$ and

$$\left\| \sigma - \left(\operatorname{sgn}(\sigma(n)) \frac{|A_\sigma \cap I_n|}{|I_n|} \right) \right\|_{Y^*} \leq \sum_{n=2}^\infty \left| \sigma(n) - \operatorname{sgn}(\sigma(n)) \frac{|F_n|}{|I_n|} \right| < \sum_{n=2}^\infty \frac{1}{|I_n|} = \frac{1}{2}.$$

Therefore, if $y \in c_{00}(\Omega)$, then

$$\begin{aligned}\|y\|_Y &= \sup\left\{|\langle \sigma, y\rangle| : \sigma \in B(Y^*)\right\}\\ &\leq \sup\left\{\left|\left\langle \left(\varepsilon_n \frac{|F\cap I_n|}{|I_n|}\right), y\right\rangle\right| + \frac{\|y\|_Y}{2} : \varepsilon_n = \pm 1 \text{ and } F \in \mathcal{F}\right\}\\ &= \left\|\sum_{n=2}^{\infty} y(n)x_n\right\|_{\mathcal{F}} + \frac{\|y\|_Y}{2},\end{aligned}$$

and hence $\|y\|_Y \leq 2\|\sum_{n=2}^{\infty} y(n)x_n\|_{\mathcal{F}}$.
To show that $[(x_n)]$ is complemented in $X_{\mathcal{F}}$, define $T : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ as follows: For $x \in \mathbb{R}^\Omega$ and $k \in I_n$ let

$$T(x)(k) = \sum_{i\in I_n} \frac{x(i)}{2^n}.$$

In other words, $T(x)$ on I_n replaces the values of x with its arithmetic mean over I_n. Clearly, T is linear, $T\restriction_{[(x_n)]}$ is the identity, and $T^2 = T$. It remains to show that $T[X_{\mathcal{F}}] \subseteq X_{\mathcal{F}}$ (i.e. $T[X_{\mathcal{F}}] \subseteq [(x_n)]$) and that T is continuous.
Given $x \in X_{\mathcal{F}}$ and $F \in \mathcal{F}$, let $E \subseteq \Omega$ be such that

(a) $|E \cap I_n| = |F \cap I_n|$ for every n (hence $E \in \mathcal{F}$), and

(b) $\sum_{k\in E} |x(k)|$ is maximal with respect to (a).

It follows that $\sum_{k\in F} |T(x)(k)| \leq \sum_{k\in E} |x(k)| \leq \|x\|_{\mathcal{F}}$ holds for every $F \in \mathcal{F}$, hence $\|T(x)\|_{\mathcal{F}} \leq \|x\|_{\mathcal{F}}$. Applying this inequality, if $x \in X_{\mathcal{F}}$ and $n \geq 2$ then

$$\|P_{[2^n,\infty)}(T(x))\|_{\mathcal{F}} = \|T(P_{[2^n,\infty)}(x))\|_{\mathcal{F}} \leq \|P_{[2^n,\infty)}(x)\|_{\mathcal{F}},$$

therefore, $T(x) \in X_{\mathcal{F}}$, and so $T : X_{\mathcal{F}} \to X_{\mathcal{F}}$ is bounded. □

Example 2.5.6. Let $Y = \ell_2$ and let $\mathcal{F}$ be the associated family above. Then $X_{\mathcal{F}}$ contains an isomorphic copy of ℓ_2 and thus it is not ℓ_1-saturated. However, we will show that it also does not contain an isomorphic copy of c_0. Hence, this example provides an asymmetry - being c_0-saturated and having no copy of ℓ_1 is equivalent in the realm of combinatorial spaces, but if we replace c_0 and ℓ_1, then the statement is not true anymore.

We know that a normalized basic sequence (x_n) in a Banach space X is equivalent to the usual basis of c_0 if and only if

$$\exists\, K > 0\, \forall\, n\, \forall\, a \in \mathbb{R}^n \left\|\sum_{i=1}^{n} a(i)x_i\right\| \leq K \max_{i=1,\ldots,n} |a(i)|.$$

If X has an unconditional basis (b_n) and X contains a copy of c_0, then, assuming (b_n) is normalized, there is a normalized block basic (nbb) sequence (x_n) with respect to

(b_n) which is equivalent to the canonical basis of c_0 (see [2, Theorem 3.3.2]). Since nbb sequences in such a space is unconditional, it follows that such a sequence is equivalent to the basis of c_0 if and only if

$$\exists\, K > 0\, \forall\, n \left\| \sum_{i=1}^{n} x_i \right\| \leq K. \tag{2.11}$$

If $X = X_{\mathcal{F}}$, $b_n = e_n$, and, for a normalized block basic sequence (x_n), $s((x_n))$ stands for $\sum_{n=1}^{\infty} x_n$, then (2.11) is equivalent to $\|s((x_n))\|_{\mathcal{F}} < \infty$. Furthermore, in this case, we can always assume that such a normalized block basic sequence is *$\mathcal{F}$-supported*, that is, $\mathrm{supp}(x_n) \in \mathcal{F}$ for every n, because if $\|P_{F_n}(x_n)\|_{\mathcal{F}} = 1$ with some $F_n \in \mathcal{F}$ and $y_n = P_{F_n}(x_n)$, then (y_n) is an $\mathcal{F}$-supported normalized block basic sequence and $\|s((y_n))\|_{\mathcal{F}} \leq \|s((x_n))\|_{\mathcal{F}}$.
Of course, there are other natural ways to express $\|s((x_n))\|_{\mathcal{F}}$:

$$\|s((x_n))\|_{\mathcal{F}} = \sup_{H \in \mathcal{H}} \|P_H(s((x_n)))\|_{\mathcal{F}} = \sup_{H \in \mathcal{H}} \sum_{n=1}^{\infty} \|P_H(x_n)\|_{\mathcal{F}}$$

where $\mathcal{F} \subseteq \mathcal{H} \subseteq \overline{\mathcal{F}}$ and, in this case, $\|P_H(x)\|_{\mathcal{F}} = \|P_H(x)\|_1$ for every $x \in \mathbb{R}^{\mathbb{N}}$.
Reformulating the above, $X_{\mathcal{F}}$ does not contain a copy of c_0 if and only if the following holds:

$$\forall \mathcal{F}\text{-supported nbb sequence } (x_n) \text{ in } X_{\mathcal{F}} \sup_{A \in \overline{\mathcal{F}}} \sum_{n=1}^{\infty} \|P_A(x_n)\|_{\mathcal{F}} = \infty. \tag{2.12}$$

Hence, to show that $X_{\mathcal{F}}$ does not have a copy of c_0, we use the condition 2.12. The main idea of the proof and the above remarks are due to B. Farkas, and they come from the joint work [18].

Let (x_n) be an $\mathcal{F}$-supported normalized block basic sequence in $X_{\mathcal{F}}$, $\mathrm{supp}(x_n) = F_n \in \mathcal{F}$; by thinning our sequence, we can assume that the sets $D_n = \{k \geq 2 : F_n \cap I_k \neq \emptyset\}$ are consecutive and

$$\sum_{n=1}^{\infty} \frac{16}{2^{\min(D_n)}} < \frac{1}{4}. \tag{2.13}$$

For $k \in D_n$, let $F_{n,k} = F_n \cap I_k$ and pick an $E_{n,k} \subseteq F_{n,k}$ such that

$$|E_{n,k}| = \left\lceil \frac{|F_{n,k}|}{2n} \right\rceil \text{ and } \|P_{E_{n,k}}(x_n)\|_{\mathcal{F}} \geq \frac{\|P_{F_{n,k}}(x_n)\|_{\mathcal{F}}}{2n}. \tag{2.14}$$

We show that

$$A = \bigcup_{n=1}^{\infty} \bigcup_{k \in D_n} E_{n,k} \in \overline{\mathcal{F}}$$

and that $\sum_{n=1}^{\infty} \|P_A(x_n)\|_{\mathcal{F}} = \infty$ (hence (2.12) holds).
We have

$$\sum_{k=3}^{\infty} \frac{|A \cap I_k|^2}{|I_k|^2} = \sum_{n=1}^{\infty} \sum_{k \in D_n} \frac{|E_{n,k}|^2}{|I_k|^2} \leq \sum_{n=1}^{\infty} \sum_{k \in D_n} \left(\frac{|F_{n,k}|}{2n} + 1 \right)^2 \frac{1}{|I_k|^2}$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{(2n)^2} \sum_{k \in D_n} \frac{|F_{n,k}|^2}{|I_k|^2} + \frac{1}{n} \sum_{k \in D_n} \frac{|F_{n,k}|}{|I_k|^2} + \sum_{k \in D_n} \frac{1}{|I_k|^2} \right)$$

where we know the following:

$$\sum_{k \in D_n} \frac{|F_{n,k}|^2}{|I_k|^2} = \sum_{k=3}^{\infty} \frac{|F_n \cap I_k|^2}{|I_k|^2} \leq 1 \text{ because } F_n \in \mathcal{F}. \tag{1}$$

$$\frac{1}{n} \sum_{k \in D_n} \frac{|F_{n,k}|}{|I_k|^2} \leq \frac{1}{n} \sum_{k \in D_n} \frac{1}{2^{k-1}} \leq \frac{1}{n} \sum_{k=\min(D_n)}^{\infty} \frac{1}{2^{k-1}} = \frac{4}{n \cdot 2^{\min(D_n)}} < \frac{16}{2^{\min(D_n)}}. \tag{2}$$

$$\sum_{k \in D_n} \frac{1}{|I_k|^2} \leq \sum_{k=\min(D_n)}^{\infty} \frac{1}{2^{2k-2}} = \frac{16}{3 \cdot 2^{2\min(D_n)}} < \frac{16}{2^{\min(D_n)}}. \tag{3}$$

Now, substituting (1), (2), and (3) in the estimation above and applying (2.13):

$$\sum_{k=3}^{\infty} \frac{|A \cap I_k|^2}{|I_k|^2} < \sum_{n=1}^{\infty} \left(\frac{1}{(2n)^2} + \frac{16}{2^{\min(D_n)}} + \frac{16}{2^{\min(D_n)}} \right) < \frac{\pi^2}{24} + \frac{1}{4} + \frac{1}{4} < 1.$$

Thus $A \in \overline{\mathcal{F}}$. The second statement follows easily from (2.14):

$$\sum_{n=1}^{\infty} \|P_A(x_n)\|_{\mathcal{F}} = \sum_{n=1}^{\infty} \sum_{k \in D_n} \|P_{E_{n,k}}(x_n)\|_{\mathcal{F}} \geq \sum_{n=1}^{\infty} \sum_{k \in D_n} \frac{\|P_{F_{n,k}}(x_n)\|_{\mathcal{F}}}{2n}$$
$$= \sum_{n=1}^{\infty} \frac{\|P_{F_n}(x_n)\|_{\mathcal{F}}}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n} = \infty.$$

2.6 Universal spaces

We will finish this chapter with a construction which will provide another classical Banach space of the form $X_{\mathcal{F}}$.

Let $\mathfrak{A}$ be a family of Banach spaces. We say that a Banach space Z is *(complementably) universal* for the class $\mathfrak{A}$ if for every $X \in \mathfrak{A}$ there exists a (complemented) subspace of Z that is isomorphic to X. The classical example is $C([0,1])$ being a universal Banach

space for the class of all separable Banach spaces. On the other hand, W. Szlenk in his paper [51] proved that there is no separable reflexive space that is universal for all separable reflexive Banach spaces.
A. Pełczyński in [46] constructed a Banach space U, called *Pełczyński's space*, that is complementably universal for the class of Banach spaces with unconditional basis. We will show that there is a combinatorial Banach space $X_{\mathcal{P}}$ having this same property, and so being isomorphic to Pełczyński's space. An appropriate family $\mathcal{P}$ will be obtained as a result of Fraïssé type construction. This result is part of the joint work [18], but the proof presented here is slightly different.

Let $\mathbf{K}$ be a countable (up to isomorphism) class of finite structures. We say that $\mathbf{K}$ is a *Fraïssé class* if it

- is *hereditary*, that is for any $A \in \mathbf{K}$, if B is a substructure of A, then $B \in \mathbf{K}$,
- has *joint embedding property*, i.e. any $A, B \in \mathbf{K}$ can be embedded in some $C \in \mathbf{K}$.
- has *amalgamation* property, i.e. for any $A, B, C \in \mathbf{K}$ and embeddings $f : A \to B$ and $g : A \to C$, there are $D \in \mathbf{K}$ and embeddings $F : B \to D$, $G : C \to D$ such that $F \circ f = G \circ g$.

A classical Fraïssé's theorem says that there exists a unique (up to isomorphism) countable structure K containing structures from $\mathbf{K}$, and being *homogeneous*, which means that any isomorphism $\varphi : A \to B$ for $A, B \in \mathbf{K}$ can be extended to an automorphism $\Phi : K \to K$. It is called the *Fraïssé limit* of $\mathbf{K}$.
One can also see that homogeneity of Fraïssé limit implies so-called *extension property*, meaning that for any finite $A \subseteq K$, any $B \in \mathbf{K}$ such that A is a substructure of B, and any embedding $\varphi : A \to K$, there exists an embedding $\tilde{\varphi} : B \to K$ that extends φ.
To prove this, consider $B \in \mathbf{K}$. Then there is an embedding $j : B \to K$. Now, fix an embedding $\varphi : A \to K$. Then $\varphi : A \to \varphi[A]$ and $j|A : A \to j[A]$ are isomorphisms between finite substructures of K. Hence, the isomorphism $\psi = \varphi \circ (j|A)^{-1} : j[A] \to \varphi[A]$ extends to an automorphism Ψ on K. Then $\tilde{\varphi} = \Psi \circ j : B \to K$ is an embedding and for every $a \in A$

$$\tilde{\varphi}(a) = \Psi(j(a)) = \psi(j(a)) = \varphi(a).$$

Proposition 2.6.1. *The class $\mathbb{F}$ of all finite families of finite sets is a Fraïssé class.*

Proof. Clearly, $\mathbb{F}$ is hereditary and, up to isomorphism, countable. If $\mathcal{F}, \mathcal{G} \in \mathbb{F}$, then both can be embedded into $\mathcal{F} \cup \mathcal{G} \in \mathbb{F}$. To prove that $\mathbb{F}$ has amalgamation property, consider $\mathcal{F}_0, \mathcal{F}_1, \mathcal{G} \in \mathbb{F}$ and embeddings $f_i : \mathcal{G} \to \mathcal{F}_i$ for $i \in \{0, 1\}$. Since f_i are injective, consider bijection $\psi : f_1[\mathcal{G}] \to f_0[\mathcal{G}]$ given by $\psi(f_1(G)) = f_0(G)$ for $G \in \mathcal{G}$.

In such a way, we *identify* an image of $\mathcal{G}$ under f_0 with an image under f_1. Consider family $\mathcal{U} = \mathcal{F}_0 \cup \mathcal{F}_1$ and maps $F_i : \mathcal{F}_i \to \mathcal{U}$ given by $F_0(C) = C$ and

$$F_1(C) = \begin{cases} \psi(C), \text{ if } C \in f_1[\mathcal{G}] \\ C, \text{ otherwise.} \end{cases}$$

It is easy to see that F_i are embeddings and we have

$$(F_0 \circ f_0)(G) = f_0(G),$$

and

$$(F_1 \circ f_1)(G) = F_1(f_1(G)) = \psi(f_1(G)) = f_0(G).$$

□

Let $\mathcal{H}$ denote the Fraïssé limit of $\mathbb{F}$ and let $\mathcal{P}$ be its hereditary closure. One can easily deduce that $\mathcal{P}$ is a unique (up to isomorphism) homogeneous family containing all finite hereditary families of finite sets. However, an even stronger condition is satisfied.

Proposition 2.6.2. *Every infinite hereditary family $\mathcal{F}$ of finite sets embedds in $\mathcal{P}$, i.e. there is $M \subseteq \mathbb{N}$ and a bijection $b : (\mathbb{N}, \mathcal{F}) \to (M, \mathcal{P})$ such that $F \in \mathcal{F} \Leftrightarrow b[F] \in \mathcal{P}$. Then we say that $\mathcal{P}$ is universal.*

Proof. Fix an infinite hereditary family $\mathcal{F}$. We construct an embedding of $\mathcal{F}$ into $\mathcal{P}$ by finite-stage extension.
For every $n \in \mathbb{N}$, let $\mathcal{F}_n$ denotes a restriction of $\mathcal{F}$ to the initial segment $[0, n)$, that is

$$\mathcal{F}_n = \{A \in \mathcal{F} : A \subseteq n\}.$$

Note that for every n, $\mathcal{F}_n$ is hereditary. Let $b_0 : (\{0\}, \mathcal{F}_0) \to (M, \mathcal{P})$ be any embedding and suppose we have defined embeddings $b_n : ([0, n), \mathcal{F}_n) \to (M, \mathcal{P})$. Then, using extension property of Fraïssé limit, we define $b_{n+1} : ([0, n+1), \mathcal{F}_{n+1}) \to (M, \mathcal{P})$ as an extension of b_n. Then the final embedding of $\mathcal{F} \to \mathcal{P}$ is given by a bijection $b = \bigcup_{n \in \mathbb{N}} b_n$. □

What follows is that for every hereditary family $\mathcal{F}$, its combinatorial Banach space $X_\mathcal{F}$ is isomorphic to the complemented subspace of $X_\mathcal{P}$.

Theorem 2.6.3. *Let $\mathcal{P}$ be the hereditary closure of the Fraïssé limit of the class of all finite families of finite sets. Then $X_\mathcal{P}$ is complementably universal for the class of all Banach spaces with unconditional basis. Consequently, $X_\mathcal{P}$ is isomorphic to Pełczyński's space U.*

Proof. Let Y be a Banach space with an unconditional basis. Theorem 2.5.5 gives us a family $\mathcal{F}$ such that $X_{\mathcal{F}}$ contains a complemented copy of Y. The space $X_{\mathcal{P}}$ contains a complemented copy of $X_{\mathcal{F}}$, and hence of Y as well. By [46, Corollary 4] $X_{\mathcal{P}}$ is isomorphic to Pełczyński's universal space U. □

Remark 2.6.4. The above example provides a solution for one of Pełczyński's questions, [46, Problem 4], which seems to be still open. The canonical basis (e_n) of $X_{\mathcal{P}}$, where $\mathcal{P}$ is as in Theorem 2.6.3, is not permutatively equivalent to Pełczyński's universal unconditional basis (u_n) of his universal space (see [46, Problem 4]), i.e. there is no permutation π such that $(e_{\pi(n)})$ is equivalent to (u_n). Indeed, contrary to the case of (u_n), the basis of our space is not universal. E.g., no subsequence of (e_n) is equivalent to the canonical basis of ℓ_2. To see this, let $H \subseteq \mathbb{N}$ be infinite and denote by $\mathcal{P}_H$ the *restriction* of $\mathcal{P}$ to H, i.e.

$$\mathcal{P}_H = \{A \in \mathcal{P} : A \subseteq H\}.$$

Then, either $X_{\mathcal{P}_H} = [(e_n)_{n\in H}]$ contains a copy of ℓ_1 or $X_{\mathcal{P}_H}$ is c_0-saturated.

Chapter 3

On the dual to combinatorial Banach spaces

The main motivation of this chapter is an attempt to study Banach spaces dual to combinatorial Banach spaces. Even in the case of the Schreier space, not much seems to be known about its dual. Perhaps the reason lies in the lack of a nice description of the dual norm. Here we present the candidate for such a description.

3.1 Introduction

For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ we denote by $\mathbb{P}_{\mathcal{F}}$ the family of all such partitions of $\mathcal{P}$ of $\mathbb{N}$ that $\mathcal{P} \subseteq \mathcal{F}$. For $x \in c_{00}$ consider the following function

$$\|x\|^{\mathcal{F}} = \inf_{\mathcal{P} \in \mathbb{P}_{\mathcal{F}}} \sum_{F \in \mathcal{P}} \sup_{i \in F} |x(i)|. \tag{3.1}$$

Perhaps this formula does not look tempting at first glance, but in a 'combinatorial' sense it is dual to $\|\cdot\|_{\mathcal{F}}$. Indeed, we can think about evaluating $\|x\|_{\mathcal{F}}$ as partitioning $\mathbb{N}$ into pieces from $\mathcal{F}$, summing up $|x(i)|$ for i from one piece of the partition and then maximizing the result, for all partitions and all pieces. On the other hand, evaluating $\|x\|^{\mathcal{F}}$ comes down to partitioning $\mathbb{N}$ into pieces from $\mathcal{F}$, taking a maximum of $|x(i)|$ for i from one piece of the partition, summing up those maxima, and then minimizing the result for all possible partitions.

If $\mathcal{P}$ is a partition of $\mathbb{N}$, then for $x \in c_{00}$ by $\|x\|^{\mathcal{P}}$ we will denote $\|x\|^{\mathcal{F}}$, where $\mathcal{F}$ is a hereditary closure of $\mathcal{P}$. Note that in this case

$$\|x\|^{\mathcal{P}} = \sum_{P \in \mathcal{P}} \sup_{k \in P} |x(k)|.$$

So, for a family $\mathcal{P} \subseteq [\mathbb{N}]^{<\infty}$ we have

$$\|x\|^{\mathcal{F}} = \inf_{\mathcal{P} \in \mathbb{P}_{\mathcal{F}}} \|x\|^{\mathcal{P}}.$$

Note that in general, (3.1) does not define a norm.

Example 3.1.1. Let $\mathcal{S}$ be the Schreier family. Consider the finitely supported sequences $x = (0, 1, 1, 0, 0, 0, ...)$, $y = (0, 0, 1, 1, 1, 0, 0, 0, ...)$. We can easily check that $\|x\|^{\mathcal{S}} = \|y\|^{\mathcal{S}} = 1$, but $\|x + y\|^{\mathcal{S}} = 3$, so the triangle inequality is not satisfied.

The lack of triangle inequality is not something welcomed in the theory of Banach spaces. However, there are still good reasons to study $\|\cdot\|^{\mathcal{F}}$. It turns out that it is a quasi-norm, at least if $\mathcal{F}$ is a compact family of finite sets. Moreover, it is a nice quasi-norm (in the sense of definition 1.3.12).

Instead of showing that $\|\cdot\|^{\mathcal{F}}$ is a quasi-norm and then showing that it is nice, we will do the opposite: first, we will check that $\|\cdot\|^{\mathcal{F}}$ satisfies all the conditions of definition 1.3.12. The reason is that lower semicontinuity will allow us to focus on finitely supported sequences.
It is easy to check that if $\mathcal{F}$ is a family covering $\mathbb{N}$, then $\|\cdot\|^{\mathcal{F}}$ is monotone and non-degenerated (in the sense of Definition 1.3.12). However, it is not necessarily lower semicontinuous. Consider e.g. the family $\mathcal{F}$ of all finite subsets of $\mathbb{N}$ and x defined by $x(k) = 1$ for each k. Then $\|P_n(x)\|^{\mathcal{F}} = 1$ for each n but $\|x\|^{\mathcal{F}} = \infty$. We will show that if we additionally assume that $\mathcal{F}$ is compact, then $\|\cdot\|^{\mathcal{F}}$ is lower semicontinuous and so it is nice.

Theorem 3.1.2. *If $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is compact, hereditary and covering $\mathbb{N}$, then $\|\cdot\|^{\mathcal{F}}$ is a nice quasi-norm.*

Before we start the proof, we recall some definitions and facts about the Vietoris topology.
Fix a compact $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$. Every partition can be considered as a subset of $2^{\mathbb{N}}$ and thus we can treat the set $\mathbb{P}_{\mathcal{F}}$ as a subset of the power set of $2^{\mathbb{N}}$. We can endow this set with the Vietoris topology.

Definition 3.1.3. Let X be a compact topological space. By $\mathcal{K}(X)$, let denote the family of all closed subsets of X. The *Vietoris topology* is the one generated by sets of the form

$$\langle U_1, U_2, ..., U_n \rangle = \{K \in \mathcal{K}(X) : K \subseteq \bigcup_{i \leq n} U_i \wedge \forall i \leq n \; K \cap U_i \neq \emptyset\}, \tag{3.2}$$

where U_i are open subsets of X.

Note that if X is a compact space, then $K(X)$ endowed with the Vietoris topology is compact as well. Also, $K(X)$ is metrizable (by the Hausdorff metric).
In our case, the role of X is played by $\mathcal{F}$. According to the above, we would like to consider $\mathbb{P}_\mathcal{F}$ as a subspace of $\mathcal{K}(\mathcal{F})$. Notice that according to our definition of partition, it contains $\emptyset$ and so it is a closed subset of $\mathcal{F}$ (in fact, it forms a sequence converging to $\emptyset$).

Lemma 3.1.4. *$\mathbb{P}_\mathcal{F}$ is closed in $\mathcal{K}(\mathcal{F})$. Consequently, $\mathbb{P}_\mathcal{F}$ is a compact subspace of $\mathcal{K}(\mathcal{F})$.*

Proof. Let $\mathcal{G} \in \overline{\mathbb{P}_\mathcal{F}}$. We need to prove that $\mathcal{G}$ is a partition, i.e.

(i) $\emptyset \in \mathcal{G}$,

(ii) $\bigcup \mathcal{G} = \mathbb{N}$,

(iii) All elements of $\mathcal{G}$ are pairwise disjoint.

Of those (i) is straightforward.
Suppose now that there exists $n \in \mathbb{N}$ such that $n \notin \bigcup \mathcal{G}$. Put $U = \{x \in 2^\mathbb{N} : x(n) = 0\}$. The set U is an open subset of $2^\mathbb{N}$. Take the basic (in Vietoris topology) set $\mathcal{K}_U = \{K \in \mathcal{K}(\mathcal{F}) : K \subseteq U\}$, being an open neighborhood of $\mathcal{G}$. Then $\mathcal{K}_U \cap \mathbb{P}_\mathcal{F} = \emptyset$. Indeed, otherwise, there would be a partition $\mathcal{P}$ such that $\mathcal{P} \subseteq U$, which is impossible, because there is $A \in \mathcal{P}$ such that $n \in A$. The set $\mathcal{K}_U$, therefore, testifies that $\mathcal{G} \notin \overline{\mathbb{P}_\mathcal{F}}$, which is a contradiction. It proves (ii).

To prove (iii), suppose that there are $A, B \in \mathcal{G}$ such that $A \cap B \neq \emptyset$ and $A \setminus B \neq \emptyset$. Let $n \in A \cap B$ and $m \in A \setminus B$. Consider the following open subsets in $2^\mathbb{N}$

$$\begin{aligned} U_1 &= \{x \in 2^\mathbb{N} : x(n) = 1 \wedge x(m) = 1\}, \\ U_2 &= \{x \in 2^\mathbb{N} : x(n) = 1 \wedge x(m) = 0\}, \\ U_3 &= 2^\mathbb{N}, \end{aligned}$$

and the basic set $\langle U_1, U_2, U_3 \rangle$. Then $\mathcal{G} \cap U_1 \neq \emptyset$, because $A \in U_1$ and $\mathcal{G} \cap U_2 \neq \emptyset$, since $B \in U_2$. So the set $\langle U_1, U_2, U_3 \rangle$ is an open neighborhood of $\mathcal{G}$. If $\mathbb{P}_\mathcal{F} \cap \langle U_1, U_2, U_3 \rangle \neq \emptyset$, then there is a partition $\mathcal{P}$ and sets $K, L \in \mathcal{P}$ such that $n, m \in K$, $n \in L$, and $m \notin L$. But it is impossible, since elements of $\mathcal{P}$ are pairwise disjoint. It implies that $\mathbb{P}_\mathcal{F} \cap \langle U_1, U_2, U_3 \rangle = \emptyset$, which is a contradiction. □

Proof of Theorem 3.1.2. As we have mentioned, it is enough to show lower semicontinuity. Fix $x \in \mathbb{R}^\mathbb{N}$.
Assume that $\|x\|^\mathcal{F} = D$ (possibly $D = \infty$). Then for every partition $\mathcal{P}$ we have $\|x\|^\mathcal{P} \geq D$. For each $n \in \mathbb{N}$ put $x_n = P_n(x)$. Suppose that there exists $M < D$ such that $\|x_n\|^\mathcal{F} < M$ for each n. Then for every n there is a partition $\mathcal{P}_n$ such that $\|x_n\|^{\mathcal{P}_n} < M$. By compactness, we may assume (passing to a subsequence if needed) that $(\mathcal{P}_n)_{n\in\mathbb{N}}$ converges (in the Vietoris topology) to a partition $\mathcal{P}$. Since $\|x\|^\mathcal{P} \geq D$, there is $N \in \mathbb{N}$

such that $\|x_N\|^{\mathcal{P}} \geq D$. There are only finitely many elements $R_1, R_2, ..., R_j$ of $\mathcal{P}$ having non-empty intersection with $\{1, 2, ..., N\}$. For $k \leq j$ put

$$U_k = \{x \in 2^{\mathbb{N}} : \forall i \in R_k \cap [1, \dots, N]\ x(i) = 1\}$$

and consider the basic open set $\langle U_1, U_2, ..., U_j, U_{j+1} \rangle$, where $U_{j+1} = 2^{\mathbb{N}}$. Then

$$\mathcal{P} \in \langle U_1, U_2, ..., U_j, U_{j+1} \rangle.$$

Indeed, trivially $\mathcal{P} \cap U_{j+1} = \mathcal{P}$ and for $k \leq j$ we have $R_k \in \mathcal{P} \cap U_k \neq \emptyset$. Since $\mathcal{P}_n$ converges to $\mathcal{P}$, there is $k > N$ such that $\mathcal{P}_k \in \langle U_1, U_2, ..., U_j, U_{j+1} \rangle$. It means that

$$\{P \cap \{1, \dots, N\} \colon P \in \mathcal{P}_k\} = \{P \cap \{1, \dots, N\} \colon P \in \mathcal{P}\}.$$

So,

$$\|x_k\|^{\mathcal{P}_k} \geq \|x_N\|^{\mathcal{P}_k} = \|x_N\|^{\mathcal{P}} > M,$$

a contradiction. □

Now we can prove that $\|\cdot\|^{\mathcal{F}}$ is indeed a quasi-norm.

Theorem 3.1.5. *Let $\mathcal{F}$ be a compact hereditary family. Then for every $x, y \in \mathbb{R}^{\mathbb{N}}$*

(a) if x, y have disjoint supports, then $\|x + y\|^{\mathcal{F}} \leq \|x\|^{\mathcal{F}} + \|y\|^{\mathcal{F}}$,

(b) $\|x + y\|^{\mathcal{F}} \leq 2(\|x\|^{\mathcal{F}} + \|y\|^{\mathcal{F}})$, and so $\|\cdot\|^{\mathcal{F}}$ is a quasi-norm.

Proof. Of the above (a) is clear. We will check (b).
Let $x, y \in \mathbb{R}^{\mathbb{N}}$. By lower semicontinuity of $\|\cdot\|^{\mathcal{F}}$ it is enough to consider the case when x and y are finitely supported. Moreover, we will assume that $x(k), y(k) \geq 0$ for every k (since $\|x + y\| \leq \||x| + |y|\|$ and $\|x\| = \||x|\|$ for each $x, y \in X^{\mathcal{F}}$) (by $|x|$ we mean a sequence defined by $|x|(k) = |x(k)|$ for each $k \in \mathbb{N}$).
Now, let $\mathcal{P}$, $\mathcal{Q} \subseteq \mathcal{F}$ be partitions witnessing $\|x\|^{\mathcal{F}}$ and $\|y\|^{\mathcal{F}}$ respectively (here we take partitions of the supports of x and y). Enumerate $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ and $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_l\}$. Let $a_i = \max\{x(j) \colon j \in P_i\}$ for $i \leq k$, and $b_i = \max\{y(j) \colon j \in Q_i\}$ for $i \leq l$. Re-enumerating $\mathcal{P}$ and $\mathcal{Q}$, if needed, we may assume that (a_i) and (b_i) are non-increasing.
Now we will define a partition $\mathcal{R}$ of $\mathrm{supp}(x+y)$, intertwining $\mathcal{P}$ and $\mathcal{Q}$ in the following way:

$$R_{2n+1} = P_n \setminus \bigcup_{i \leq 2n} R_i \text{ for } 0 \leq n \leq k, \text{ and } R_{2n} = Q_n \setminus \bigcup_{i < 2n} R_i \text{ for } 1 \leq n \leq l.$$

Then $\max\{x(i) + y(i) \colon i \in R_1\} \leq a_1 + b_1$, $\max\{x(i) + y(i) \colon i \in R_2\} \leq a_2 + b_1$ and so on. Therefore, the partition $\mathcal{R}$ witnesses that

$$\begin{aligned} \|x + y\|^{\mathcal{F}} &\leq (a_1 + b_1) + (b_1 + a_2) + (a_2 + b_2) + (b_2 + a_3) + \dots \leq \\ &\leq a_1 + 2(a_2 + \dots + a_k) + 2(b_1 + \dots + b_l) \leq 2(\|x\|^{\mathcal{F}} + \|y\|^{\mathcal{F}}). \quad (3.3) \end{aligned}$$

□

Hence, for a compact family $\mathcal{F}$, we denote by $X^{\mathcal{F}}$ the completion of c_{00} with respect to the quasi-norm $\|\cdot\|^{\mathcal{F}}$. Moreover, analogously to combinatorial spaces, $\mathrm{FIN}(\|\cdot\|^{\mathcal{F}})$ is denoted by $Z^{\mathcal{F}}$.

Theorem 3.1.6. *If $\mathcal{F}$ is a compact, hereditary family and $\mathcal{F}$ covers $\mathbb{N}$, then $Z^{\mathcal{F}}$ and $X^{\mathcal{F}}$ are quasi-Banach spaces.*

Proof. We prove it by mimicking the proof from [17].

That $\|\cdot\|^{\mathcal{F}}$ is a nice quasi-norm follows directly from Theorem 3.1.5 and from Theorem 3.1.2.
We are going to show that $Z^{\mathcal{F}}$ is complete and then that $X^{\mathcal{F}}$ is its closed subspace.
For simplicity denote $\varphi = \|\cdot\|^{\mathcal{F}}$. We use in this proof the symbol $\mathrm{FIN}(\varphi)$ for $Z^{\mathcal{F}}$ and $\mathrm{EXH}(\varphi)$ for $X^{\mathcal{F}}$.

First, we will prove that $\mathrm{FIN}(\varphi)$ is complete. Let (x_n) be a Cauchy sequence in $\mathrm{FIN}(\varphi)$. Applying monotonicity, $\varphi(P_{\{k\}}(x_n - x_m)) \leq \varphi(x_n - x_m)$ for every k, n, m, and hence $(P_{\{k\}}(x_n))_{k\in\mathbb{N}}$ is a Cauchy sequence in the kth 1-dimensional coordinate space of $\mathbb{R}^{\mathbb{N}}$ (which is a quasi-Banach space, as φ is finite on c_{00}), $P_{\{k\}}(x_n) \xrightarrow{n\to\infty} y_k$ for some y_k. Put $y = (y_k)$. We will first show that $y \in \mathrm{FIN}(\varphi)$. The sequence $\{x_n\colon n \in \mathbb{N}\}$ is bounded, let say $\varphi(x_n) \leq B$ for every n. We show that $\varphi(y) \leq 4B$, i.e. (by the lower semicontinuity of φ) $\varphi(P_M(y)) \leq 4B$ for every $M \in \mathbb{N}$. Fix an $M > 0$. If n is large enough, say $n \geq n_0$, then $\varphi(P_{\{k\}}(y - x_n)) \leq \dfrac{B}{M}$ for every $k < M$ and hence

$$\varphi(P_M(y)) \leq 2(\varphi(P_M(y-x_n))+\varphi(P_M(x_n))) \leq 2(\sum_{k<M} \varphi(P_{\{k\}}(y-x_n))+\varphi(x_n)) \leq 4B.$$

The first inequality follows from Theorem 3.1.5(c) and the second from Theorem 3.1.5(a).
Now we will prove that $x_n \to y$. If not, then there are $\varepsilon > 0$ and $n_0 < n_1 < \cdots < n_j < \ldots$ such that $\varphi(x_{n_j} - y) > \varepsilon$, that is, $\varphi(P_{M_j}(x_{n_j} - y)) > \varepsilon$ for some $M_j \in \mathbb{N}\setminus\{0\}$ for every j. Pick j_0 such that $\varphi(x_{n_{j_0}} - x_n) < \dfrac{\varepsilon}{2}$ for every $n \geq n_{j_0}$ and then pick $j_1 > j_0$ such that $\varphi(P_{\{k\}}(x_{n_{j_1}} - y)) \leq \dfrac{\varepsilon}{2M_{j_0}}$ for every $k < M_{j_0}$. Then, using Theorem 3.1.5(a)

$$\varepsilon < \varphi(P_{M_{j_0}}(x_{n_{j_0}} - y)) \leq \varphi(P_{M_{j_0}}(x_{n_{j_0}} - x_{n_{j_1}})) + \sum_{k<M_{j_0}} \varphi(P_{\{k\}}(x_{n_{j_1}} - y)) < \varepsilon,$$

a contradiction.
Now we will show that $\mathrm{EXH}(\varphi) = \overline{c_{00}}$. The space c_{00} is dense in $\mathrm{EXH}(\varphi)$ because $\varphi(x - P_n(x)) = \varphi(P_{\mathbb{N}\setminus n}(x)) \xrightarrow{n\to\infty} 0$ for every $x \in \mathrm{EXH}(\varphi)$. We have to show

that $\mathrm{EXH}(\varphi)$ is closed. Let $x \in \mathrm{FIN}(\varphi)$ be an accumulation point of $\mathrm{EXH}(\varphi)$. For any $\varepsilon > 0$ we can find $y \in \mathrm{EXH}(\varphi)$ such that $\varphi(x - y) < \varepsilon$, and then n_0 such that $\varphi(P_{\mathbb{N}\setminus n}(y)) < \varepsilon$ for every $n \geq n_0$. If $n \geq n_0$ then $\varphi(P_{\mathbb{N}\setminus n}(x)) \leq 2(\varphi(P_{\mathbb{N}\setminus n}(x - y)) + \varphi(P_{\mathbb{N}\setminus n}(y))) < 4\varepsilon$. □

The main corollary of this section is the following reformulation of (a part of) Theorem 3.1.6:

Theorem 3.1.7. *If $\mathcal{F}$ is compact, hereditary and covering $\mathbb{N}$, then $X^{\mathcal{F}}$ is a quasi-Banach space.*

The following is a simple consequence of (a) of Theorem 3.1.5.

Corollary 3.1.8. *If a family $\mathcal{F}$ is a hereditary closure of a partition $\mathcal{P}$, then the formula (3.1) defines a norm.*

As we already know, in general, the formula (3.1) does not need to define a norm, but we can consider *the Banach envelope* of $X^{\mathcal{F}}$. Let $\mathcal{F}$ be a compact, hereditary family. Let

$$|||x|||^{\mathcal{F}} = \inf\left\{ \sum_{i=1}^{n} \|x_i\|^{\mathcal{F}} : n \in \mathbb{N}, x_1, ..., x_n \in X, x = \sum_{i=1}^{n} x_i \right\}. \tag{3.4}$$

Since $\|\cdot\|^{\mathcal{F}}$ is a quasi-norm, this formula defines a norm. The space $\widehat{X}^{\mathcal{F}} = \mathrm{EXH}(|||\cdot|||^{\mathcal{F}})$ is caled *the Banach envelope* of $X^{\mathcal{F}}$ (see [37]).

Remark 3.1.9. Clearly, for every compact, hereditary family $\mathcal{F}$ and for $x \in c_{00}$ we have

$$|||x|||^{\mathcal{F}} \leq \|x\|^{\mathcal{F}}.$$

If there is $C > 0$ such that for each sequence (x_i) of vectors in c_{00}

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^{\mathcal{F}} \leq C \sum_{i=1}^{n} \|x_i\|^{\mathcal{F}}, \tag{3.5}$$

then

$$C\,|||x|||^{\mathcal{F}} \geq \|x\|^{\mathcal{F}}.$$

Property (3.5) is called *1-convexity* and it is equivalent to the normability of a quasi-Banach space (by $|||\cdot|||^{\mathcal{F}}$). In the next section, we will show that in general $\|\cdot\|^{\mathcal{F}}$ does not have to be 1-convex.
Also, we will prove that the Banach space induced by $|||\cdot|||^{\mathcal{F}}$ is isomorphic to $X^{*}_{\mathcal{F}}$.

The following theorem shows that the quasi-Banach spaces $X^{\mathcal{F}}$ and $Z^{\mathcal{F}}$ are identical.

Proposition 3.1.10. *If $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a compact hereditary family covering $\mathbb{N}$, then*

$$X^{\mathcal{F}} = Z^{\mathcal{F}}$$

Proof. For each $x \in \mathbb{R}^{\mathbb{N}}$ and for fixed n we can write $x = x_n + x_n'$, where $x_n = P_n(x)$ and $x_n' = P_{\mathbb{N}\setminus n}(x)$. Thus if $x \in X^{\mathcal{F}}$ then $\|x_n'\|^{\mathcal{F}} \to 0$ and

$$\|x\|^{\mathcal{F}} \leq 2(\|x_n\|^{\mathcal{F}} + \|x_n'\|^{\mathcal{F}}) < \infty,$$

because x_n is finitely supported. It shows that $X^{\mathcal{F}} \subseteq Z^{\mathcal{F}}$. On the other hand, if $x \in Z^{\mathcal{F}}$, then there is a partition $\mathcal{G} = \{G_k : k \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $\sum_{k\in\mathbb{N}} \sup_{j\in G_k} |x(j)| < \infty$. It implies that

$$\sum_{k\geq n} \sup_{j\in G_k} |x(j)| \xrightarrow{n\to\infty} 0.$$

Let $\varepsilon > 0$ and fix m such that $\sum_{k\geq m} \sup_{j\in G_k} |x(j)| < \varepsilon$. Let $n > \max(\bigcup_{i<m} G_i)$. Then

$$\|x_n'\|^{\mathcal{F}} \leq \|x_n'\|^{\mathcal{G}} \leq \sum_{k\geq m} \sup_{j\in G_k} |x(j)| < \varepsilon.$$

It finishes the proof. □

3.2 $X^{\mathcal{F}}$ and the dual of $X_{\mathcal{F}}$

In this section, we will examine how close $X^{\mathcal{F}}$ is to $X_{\mathcal{F}}^*$, the space dual to $X_{\mathcal{F}}$.
In case $\mathcal{F}$ is simple enough (i.e., it is generated by a partition), it is not hard to see that $X^{\mathcal{F}}$ is isometrically isomorphic to $X_{\mathcal{F}}^*$ (Proposition 3.2.1). In general, this is not true. However, $X_{\mathcal{F}}^*$ is always the Banach envelope of $X^{\mathcal{F}}$.
We start with the aforementioned result for spaces generated by partitions.

Proposition 3.2.1. *Suppose $\mathcal{P}$ is a partition of $\mathbb{N}$ (into finite sets) and $\mathcal{F}$ is its hereditary closure. Then $X_{\mathcal{F}}^*$ is isometrically isomorphic to $X^{\mathcal{F}}$.*

Proof. Enumerate $\mathcal{P} = \{F_1, F_2, \dots\}$. It is known that for $\mathcal{F}$ being generated by partition, $X_{\mathcal{F}}$ is isometrically isomorphic to $\bigoplus_{c_0} \ell_1^{|F_n|}$ and so its dual space is isometrically isomorphic to $\bigoplus_{\ell_1} c_0^{|F_n|}$.
Let $y \in X^{\mathcal{F}}$. Then $\|y\|^{\mathcal{F}} = \sum_{n\in\mathbb{N}} \max_{k\in F_n} |y(k)|$. Taking $y_n = P_{F_n}(y)$ for each n, we can see y_n as element of $\mathbb{R}^{|F_n|}$. Thus $\|y\|^{\mathcal{F}} = \sum_{n\in\mathbb{N}} \|y_n\|_\infty$, which gives us the norm on $\bigoplus_{\ell_1} c_0^{|F_n|}$. □

Let $\mathcal{F}$ be a compact, hereditary family covering $\mathbb{N}$. Define $T\colon c_{00} \to X_{\mathcal{F}}^*$, a linear operator given by

$$T(y)(x) = \sum_{k\in\mathbb{N}} x(k)y(k) \tag{3.6}$$

for $x \in X_{\mathcal{F}}$. It is plain to check that T is injective. Also, let $T_0\colon c_{00}(\|\cdot\|^{\mathcal{F}}) \to X_{\mathcal{F}}^*$ and $T_1\colon c_{00}(\|\!|\cdot|\!\|^{\mathcal{F}}) \to X_{\mathcal{F}}^*$ denote the operators given by the same formula as T.

Proposition 3.2.2. *T_0 and T_1 are continuous with the norm 1.*

Proof. To prove that T_0 is continuous, take finitely supported y and let $\mathcal{P}$ be such that $\|y\|^{\mathcal{F}} = \sum_{F\in\mathcal{P}} \max_{k\in F} |y(k)|$. Then for every $x \in X_{\mathcal{F}}$ with $\|x\|_{\mathcal{F}} \le 1$ we have

$$\Big|\sum_{k\in\mathbb{N}} x(k)y(k)\Big| = \Big|\sum_{F\in\mathcal{P}}\sum_{k\in F} x(k)y(k)\Big| \le \sum_{F\in\mathcal{P}} \max_{k\in F}|y(k)| \sum_{k\in F}|x(k)| \le \|y\|^{\mathcal{F}}.$$

Thus

$$\|T(y)\|_{\mathcal{F}}^* \le \|y\|^{\mathcal{F}}, \tag{3.7}$$

and so T_0 is continuous.
To show that T_1 is continuous, we use (3.7). Notice that for $y = \sum_{i\le n} y_i$ we have

$$\|T(y)\|_{\mathcal{F}}^* \le \sum_{i\le n}\|T(y_i)\|_{\mathcal{F}}^* \le \sum_{i\le n}\|y_i\|^{\mathcal{F}}.$$

It implies that $\|T(y)\|_{\mathcal{F}}^* \le \|\!|y|\!\|^{\mathcal{F}}$, hence T_1 is continuous. ☐

Note that by above proposition, as $X_{\mathcal{F}}^*$ is complete, we can extend the operator T_0 to a continuous injective linear operator $X^{\mathcal{F}} \to X_{\mathcal{F}}^*$, denoted also by T_0. The same holds true for T_1 and $\widehat{X}^{\mathcal{F}}$.

We now state the main theorem of this section.

Theorem 3.2.3. *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ be a compact, hereditary family covering $\mathbb{N}$. Then $\widehat{X}^{\mathcal{F}}$ is isometrically isomorphic to $X_{\mathcal{F}}^*$.*

In the proof, we will use some general facts about the spaces of the form $X_{\mathcal{F}}^*$ and the extreme points of the unit ball in $X_{\mathcal{F}}^*$ and $X^{\mathcal{F}}$.

Recall that in a quasi-Banach space X which is not a Banach space, the unit ball is not convex. Although the notion of an extreme point is usually considered in the context of convex sets, the definition itself does not require convexity *a priori*. Thus, we can also consider extreme points of non-convex sets.

Now we introduce the notion which is very useful in the proof of Theorem 3.2.3.
We say that a quasi-Banach space X has *convex series representation property (CSRP)* if for every $x \in B_X$ there exists a sequence (λ_n) of positive real numbers with $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and a sequence (u_n) of extreme points of B_X such that

$$x = \sum_{n\in\mathbb{N}} \lambda_n u_n. \tag{3.8}$$

The combinatorial spaces and their duals were studied geometrically in the context of extreme points. Note that if $\mathcal{F}$ is a compact, hereditary family of finite sets, then all the extreme points of the unit ball of $X^*_{\mathcal{F}}$ are finitely supported and there are only finitely many extreme points with a given support (see [5], [21]). It is known (see [5]) that $X^*_{\mathcal{F}}$ has CSRP, for $\mathcal{F}$ as above. We will also show that the same holds in $X^{\mathcal{F}}$.

In [5], the authors provide proof of Gowers's theorem regarding the characterization of extreme points of the unit ball in $X^*_{\mathcal{F}}$. In his blog [31], Gowers states (without proof) that the set of extreme points is of the form

$$\Big\{ \sum_{i\in F} \varepsilon_i e_i^* : F \in \mathcal{F}^{\text{MAX}}, \varepsilon_i \in \{-1,1\} \Big\} \tag{3.9}$$

where

- (e_i^*) are biorthogonal functionals for the canonical Schauder basis (e_i),
- $\mathcal{F}^{\text{MAX}}$ is a family of *maximal* sets from $\mathcal{F}$, i.e. these sets F for which $F \cup \{k\} \notin \mathcal{F}$ for every $k \in \mathbb{N}$.

Actually, the fact that $\text{Ext}(X^*_{\mathcal{F}})$ is given by (3.9) was proven only for the Schreier space and for *higher order Schreier spaces*. However, that result also holds for a general compact, hereditary family $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ (see [5, Remark 4.4] and [21, Proposition 5]). We will show that $X^{\mathcal{F}}$ has basically *the same* extreme points, that is $T_0(\text{Ext}(X^{\mathcal{F}})) = \text{Ext}(X^*_{\mathcal{F}})$.

Proposition 3.2.4. *Assume that $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a compact, hereditary family covering $\mathbb{N}$. A vector $y \in X^{\mathcal{F}}$ is an extreme point of the unit ball of $X^{\mathcal{F}}$ if and only if it is of the form*

$$y(i) = \begin{cases} \varepsilon_i, & \text{if } i \in F \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

for some $F \in \mathcal{F}^{\text{MAX}}$ and $\varepsilon_i \in \{-1,1\}$.

Proof. First, assume that $|y(k)| = 1$ for each $k \in F$ for some $F \in \mathcal{F}^{\text{MAX}}$.
Suppose y is not extreme and $y = (1-t)x + tz$ for some $0 < t < 1$ and $x, z \in B_{X^{\mathcal{F}}}$. In particular, absolute values of x and z do not exceed 1. Suppose that e.g. $|x(k)| < 1$ for

some $k \in F$. Then $1 = |y(k)| \leq (1-t)|x(k)| + t|z(k)| < 1$, which is a contradiction. So $|x(k)| = |z(k)| = 1$ for each $k \in F$. On the other hand, if $x(k) \neq 0$ for $k \notin F$, then by maximality of F it follows that $\|x\|^{\mathcal{F}} > 1$. It implies that $x(k) = 0$ for every $k \notin F$. Hence $x = y$. This is a contradiction, and so y must be an extreme point.
Now suppose that $y \in \mathrm{Ext}(X^{\mathcal{F}})$. Then $\|y\|^{\mathcal{F}} = 1$. Let $\mathcal{P} \in \mathbb{P}_{\mathcal{F}}$ for which $\|y\|^{\mathcal{F}} = \|y\|^{\mathcal{P}}$. For each $P \in \mathcal{P}$ we have $|y(i)| = |y(j)|$ for every $i, j \in P$. Indeed, suppose otherwise. Then there is $P \in \mathcal{P}$ and $i, j \in P$ such that $|y(j)| < |y(i)|$ and so for $\eta < |y(i)| - |y(j)|$ we would have $\|y \pm \eta e_j\|^{\mathcal{F}} \leq \|y \pm \eta e_j\|^{\mathcal{P}} \leq 1$, hence y would not be an extreme point. It follows, that if $\mathrm{supp}(y) \in \mathcal{F}$, then y needs to be of the promised form (in particular $\mathrm{supp}(y)$ is a maximal set in $\mathcal{F}$, otherwise $\|y \pm e_i\| = 1$ for $i \notin \mathrm{supp}(y)$ with $\mathrm{supp}(y) \cup \{i\} \in \mathcal{F}$). If $\mathrm{supp}(y) \notin \mathcal{F}$, then we may find distinct $P_0, P_1 \in \mathcal{P}$ and $a_0, a_1 \neq 0$ such that $y(i) = a_j$ for $i \in P_j$, $j \in \{0, 1\}$. Since $\|y\|^{\mathcal{F}} = 1$, $|a_0|, |a_1| < 1$. But then for sufficiently small $\eta > 0$ and for $u \in B_{X^{\mathcal{F}}}$ defined by

$$u(i) = \begin{cases} \eta, & \text{if } i \in P_0 \\ -\eta, & \text{if } i \in P_1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}$$

we would have

$$\|y \pm u\|^{\mathcal{F}} \leq \|y \pm u\|^{\mathcal{P}} = \|y\|^{\mathcal{P}} = 1.$$

So, y has to be of the form (3.10). □

Proof of Theorem 3.2.3. We shall use the natural identification of $X^*_{\mathcal{F}}$ with a subspace of $\mathbb{R}^{\mathbb{N}}$ by the map $X^*_{\mathcal{F}} \ni f \mapsto (f(e_n))_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ (see Remark 1.3.7). In this setting the extended mapping T_1 of Proposition 3.2.2 (see the remark after Proposition 3.2.2) becomes the formal inclusion $\widehat{X}^{\mathcal{F}} \hookrightarrow X^*_{\mathcal{F}}$, and, by the fact that (e_i^*) is a basis for $X^*_{\mathcal{F}}$ (as $X_{\mathcal{F}}$ has a shrinking basis), $X^*_{\mathcal{F}}$ is the completion of $(c_{00}, \|\cdot\|^*_{\mathcal{F}})$.

By Proposition 3.2.2 we have $\|\cdot\|^*_{\mathcal{F}} \leq |||\cdot|||^{\mathcal{F}}$ on $\widehat{X}^{\mathcal{F}}$. We will prove that $|||\cdot|||^{\mathcal{F}} \leq \|\cdot\|^*_{\mathcal{F}}$ on c_{00}, which implies equality of $|||\cdot|||^{\mathcal{F}}$ and $\|\cdot\|^*_{\mathcal{F}}$ on c_{00}. Then the definition of $\widehat{X}^{\mathcal{F}}$ yield $\widehat{X}^{\mathcal{F}} = X^*_{\mathcal{F}}$ and equality of $|||\cdot|||^{\mathcal{F}}$ and $\|\cdot\|^*_{\mathcal{F}}$ on $\widehat{X}^{\mathcal{F}} = X^*_{\mathcal{F}}$.
We will prove that $|||\cdot|||^{\mathcal{F}} \leq \|\cdot\|^*_{\mathcal{F}}$ on c_{00} by showing that $B_{X^*_{\mathcal{F}}} \cap c_{00} \subseteq B_{\widehat{X}^{\mathcal{F}}}$.
Fix finitely supported $x \in B_{X^*_{\mathcal{F}}}$ and let $A = \mathrm{supp}(x)$. Since $X^*_{\mathcal{F}}$ has CSRP, we have $x = \sum_{k \in \mathbb{N}} \lambda_k u_k$ for $u_k \in \mathrm{Ext}(X^*_{\mathcal{F}})$ and λ_k such that $\sum_{k \in \mathbb{N}} \lambda_k = 1$.
By continuity of P_A we have $x = \sum_{k \in \mathbb{N}} \lambda_k P_A(u_k)$. By the form of extreme points (see (3.9)) the set $\{P_A(u_k) \colon k \in \mathbb{N}\}$ is finite and so we may enumerate it as $\{v_i \colon i \leq n\}$ for some $n \in \mathbb{N}$. Also, there are $\alpha_i > 0$, $i \leq n$, such that $\sum_{i=1}^n \alpha_i = 1$ and $x = \sum_{i=1}^n \alpha_i v_i$. It means that $x \in \mathrm{conv}(P_A[\mathrm{Ext}(B_{X^*_{\mathcal{F}}})])$, where $\mathrm{conv}(K)$ denotes the convex hull of a set K. Since each u_i is an extreme point, we have that $\varepsilon_0 P_A(u_i) + \varepsilon_1 P_{\mathbb{N} \setminus A}(u_i)$ is an extreme point for $\varepsilon_0, \varepsilon_1 \in \{-1, 1\}$. In particular, as $v_i = \dfrac{1}{2}\Big(u_i + (P_A(u_i) - P_{\mathbb{N} \setminus A}(u_i)\Big)$, we have $x \in \mathrm{conv}(\mathrm{Ext}(X^*_{\mathcal{F}}))$ and thus $B_{X^*_{\mathcal{F}}} \cap c_{00} = \mathrm{conv}(\mathrm{Ext}(X^*_{\mathcal{F}}))$.

On the other hand, by Proposition 3.2.4 we know that $\mathrm{Ext}(X^*_{\mathcal{F}}) \subseteq B_{X^{\mathcal{F}}} \subseteq B_{\widehat{X}^{\mathcal{F}}}$ and, since $B_{\widehat{X}^{\mathcal{F}}}$ is convex, we obtain that $B_{X^*_{\mathcal{F}}} \cap c_{00} \subseteq B_{\widehat{X}^{\mathcal{F}}}$. □

Now we present the aforementioned result that $X^{\mathcal{F}}$, same as $X^*_{\mathcal{F}}$, has a CSRP.

Theorem 3.2.5. *For any compact, hereditary family $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ covering $\mathbb{N}$, the space $X^{\mathcal{F}}$ has CSRP.*

Proof. We have to show that for every $x \in B_X$ there exists an appropriate sequence of extreme points and coefficients. First, we will prove it assuming that $\mathrm{supp}(x) \in \mathcal{F}$. Then we will generalize it for the case $x \in c_{00}$ and at the end we will show the final result.

1) $\mathrm{supp}(x) \in \mathcal{F}$

Assume $\mathrm{supp}(x) \subseteq F_0$ for some $F_0 \in \mathcal{F}^{\mathrm{MAX}}$. Put $\alpha = \min\{|x(k)|\colon k \in \mathrm{supp}(x)\}$. Define $\lambda_0 = \min\{\alpha, 1-\alpha\}$ and $\lambda_n = \dfrac{1-\lambda_0}{2^n}$ for $n \geq 1$. Let u_0 be an extreme point defined by

$$u_0(k) = \begin{cases} \mathrm{sgn}(x(k)), & \text{if } k \in \mathrm{supp}(x) \\ 1, & \text{if } k \in F_0 \setminus \mathrm{supp}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Put $v_0 = \lambda_0 u_0$ and define $S_0 = \{k \in \mathbb{N} : x(k) = v_0(k)\}$. Note that, a priori, it is possible for S_0 to be empty. If not, let $G_0 \subseteq \mathbb{N}$ be such that $F_0 < G_0$ and $F_1 := (F_0 \setminus S_0) \cup G_0 \in \mathcal{F}^{\mathrm{MAX}}$ (for $S_0 = \emptyset$ we take $G_0 = \emptyset$ as well).
We iterate this construction for $n \geq 1$, i.e., we put

$$u_n(k) = \begin{cases} \mathrm{sgn}(x(k) - \sum\limits_{j<n} v_j(k)), & \text{if } k \in F_{n-1} \\ 1, & \text{if } k \in F_n \setminus F_{n-1} \\ 0, & \text{otherwise,} \end{cases} \tag{3.12}$$

let $v_n = \lambda_n u_n$, $S_n = \{k \in \mathbb{N}\colon x(k) - \sum_{j<n} v_j(k) = v_n(k)\}$ and let G_n be such a set that $F_n < G_n$ and $F_{n+1} := (F_n \setminus S_n) \cup G_n$ is maximal (again, if $S_n = \emptyset$, then $G_n = \emptyset$ as well). Note that on each step of the construction, the sequence $x - \sum\limits_{j \leq n} v_j$ is supported on a subset of the maximal set F_n.
Now we show that the sequences (λ_n) and (u_n) are as desired by the definition of CSRP. It is clear that $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and for every n the vector u_n is an extreme point in $X^{\mathcal{F}}$ (by Lemma 3.2.4). It remains to check that the series $\sum_{n\in\mathbb{N}} \lambda_n u_n$ is convergent to x in the quasi-norm $\|\cdot\|^{\mathcal{F}}$.

Claim. For every $k \in \mathbb{N}$ $r_n(k) := \left| x(k) - \sum_{j\leq n} v_j(k) \right| \leq \dfrac{1-\lambda_0}{2^n}$.

Proof of claim. We prove it by induction with respect to n. If $n = 0$ and $k \in \operatorname{supp}(x)$ then we have $-\lambda_0 < x(k) - \lambda_0 \leq 1 - \lambda_0$ for $x(k) > 0$. Since $\lambda_0 \leq 1 - \lambda_0$ by the definition, the inequality holds for $x(k) > 0$. The definition of λ_0 implies also immediately that $r_0(k) \leq 1 - \lambda_0$ for $k \notin \operatorname{supp}(x)$. Finally, if $x(k) < 0$, then $v_0(k) = -\lambda_0$ and then $-1 + \lambda_0 \leq x(k) - v_0(k) < \lambda_0 \leq 1 - \lambda_0$.

Now suppose that $r_n(k) \leq \dfrac{1-\lambda_0}{2^n}$ for some n. If $x(k) - \sum_{j\leq n} v_j(k) > 0$, then $v_{n+1}(k) = \lambda_{n+1}$, if $k \in F_n$, and thus

$$-\frac{1-\lambda_0}{2^{n+1}} = -\lambda_{n+1} \leq x(k) - \sum_{j\leq n+1} v_j(k) \leq \frac{1-\lambda_0}{2^n} - \frac{1-\lambda_0}{2^{n+1}} = \frac{1-\lambda_0}{2^{n+1}}$$

If $x(k) - \sum_{j\leq n} v_j(k) < 0$ then $v_{n+1}(k) = -\lambda_{n+1}$ and the case is symmetric. Thus for $k \in F_n$, $r_{n+1}(k) \leq \dfrac{1-\lambda_0}{2^{n+1}}$. If $k \in F_{n+1} \setminus F_n$ then $x(k) - \sum_{j\leq n+1} v_j(k) = -v_{n+1}(k) = -\lambda_{n+1}$ which finishes the proof of the claim. □

Note that the above claim implies that $\sum_{n\in\mathbb{N}} \lambda_n u_n$ is convergent to x since

$$\|x - \sum_{j\leq n} \lambda_n u_n\|^{\mathcal{F}} = \max_{k\in F_n} \Big| x(k) - \sum_{j\leq n} v_n(k) \Big| \leq \frac{1-\lambda_0}{2^n} \xrightarrow{n\to\infty} 0$$

It finishes the proof for x having a support in $\mathcal{F}$.

2) $x \in c_{00}$**.**

If x is a finitely supported sequence, then $x = \sum_{i=1}^{m} x_i$ for some $m \in \mathbb{N}$, where x_i are sequences with supports contained in some $F_i \in \mathcal{F}^{\mathrm{MAX}}$. Then, we make a similar construction as in the previous case for each x_i separately. Put $M = \|x\|^{\mathcal{F}}$, and for each $1 \leq i \leq m$ let $\alpha_i = \min_{k\in F_i} |x(k)|$, $\beta_i = \max_{k\in F_i} |x(k)|$. Next, define a sequence $(\lambda_n^i)_n$ by taking

$$\lambda_0^i = \min\{\alpha_i, \frac{\beta_i}{M} - \alpha_i\} \text{ and } \lambda_n^i = \frac{\frac{\beta_i}{M} - \lambda_0^i}{2^n} \text{ for } n \geq 1.$$

For each i, $\sum_{n\in\mathbb{N}} \lambda_n^i = \dfrac{\beta_i}{M}$ and thus $\sum_{i=1}^m \sum_{n\in\mathbb{N}} \lambda_n^i = 1$.

The sequence of extreme points $(u_n^i)_n$ is defined as in the first case. Then, repeating arguments from the previous case for every i, we obtain

$$\|x_i - \sum_{j\leq n} \lambda_j^i u_j^i\|^{\mathcal{F}} \leq \frac{\frac{\beta_i}{M} - \lambda_0^i}{2^n}.$$

Thus, for each n we get

$$\|x - \sum_{i=1}^{m} \sum_{j \le n} \lambda_j^i u_j^i\|^{\mathcal{F}} \le 2^m \sum_{i=1}^{m} \|x_i - \sum_{j \le n} \lambda_j^i u_j^i\|^{\mathcal{F}} \le 2^m \sum_{i=1}^{m} \frac{\frac{\beta_i}{M} - \lambda_0^i}{2^n} \tag{3.13}$$

and the last expression tends to 0 when $n \to \infty$. Hence, every finitely supported sequence can be expressed as a convex series of extreme points.

3) The general case.

For $x \in X^{\mathcal{F}}$, the result follows from the lower semicontinuity of $\|\cdot\|^{\mathcal{F}}$ and the previous cases. Indeed, for any $\varepsilon > 0$ find $N \in \mathbb{N}$ such that $\|x - P_N(x)\|^{\mathcal{F}} < \frac{\varepsilon}{2}$. For that N, we can find a convex combination as in the second case, converging to $P_N(x)$. Namely, for sufficiently big $n \in \mathbb{N}$

$$\|P_N(x) - \sum_{i=1}^{m} \sum_{j \le n} \lambda_j^i u_j^i\|^{\mathcal{F}} < \frac{\varepsilon}{2}$$

Thus, using *quasi-triangle inequality*, we have

$$\|x - \sum_{i=1}^{m} \sum_{j \le n} \lambda_j^i u_j^i\|^{\mathcal{F}} \le 2\Big(\|x - P_N(x)\|^{\mathcal{F}} + \|P_N(x) - \sum_{i=1}^{m} \sum_{j \le n} \lambda_j^i u_j^i\|^{\mathcal{F}}\Big) < \varepsilon$$

It finishes the proof. □

Now we will show a result which indicates that the connection between $X^{\mathcal{F}}$ and $X^*_{\mathcal{F}}$ is quite strong. For each compact family $\mathcal{F}$ the space $X^{\mathcal{F}}$ is a (quasi-Banach) pre-dual of $(X_{\mathcal{F}})^{**}$. In other words, $X^{\mathcal{F}}$ and $X^*_{\mathcal{F}}$ have isometrically isomorphic dual spaces. In fact, this is a direct corollary of Theorem 3.2.3 and [37, Chapter 2.4]. We enclose a detailed proof.

Theorem 3.2.6. *If $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a compact hereditary family covering $\mathbb{N}$, then $(X^{\mathcal{F}})^*$ is isometrically isomorphic to $(X_{\mathcal{F}})^{**}$.*

Proof. By Proposition 1.3.10 the space $(X_{\mathcal{F}})^{**}$ is isometrically isomorphic to $Z_{\mathcal{F}}$. As in the proof of Theorem 3.2.3 we use the natural identification of $(X^{\mathcal{F}})^*$ with a subspace of $\mathbb{R}^{\mathbb{N}}$ via the map $(X^{\mathcal{F}})^* \ni f \mapsto (f(e_n))_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. We need to prove that $\|y\|_*^{\mathcal{F}} = \|y\|_{\mathcal{F}}$ for any $y \in \mathbb{R}^{\mathbb{N}}$, where $\|\cdot\|_*^{\mathcal{F}}$ denotes the functional norm on $X^{\mathcal{F}}$.
Take any $y \in \mathbb{R}^{\mathbb{N}}$. For any set $F_0 \in \mathcal{F}$ consider $x_0 \in X^{\mathcal{F}}$ given by

$$x_0(n) = \begin{cases} \text{sgn}(y(n)), & \text{if } n \in F_0, \\ 0, & \text{otherwise.} \end{cases}$$

This is a vector of norm at most 1 in $X^{\mathcal{F}}$ and thus

$$\|y\|_*^{\mathcal{F}} \ge |\sum_{n \in F_0} x_0(n) y(n)| = \sum_{n \in F_0} |y(n)|$$

As $F_0 \in \mathcal{F}$ was arbitrary, we obtain $\|y\|_*^{\mathcal{F}} \geq \|y\|_{\mathcal{F}}$. To prove the reverse inequality, take any $x \in c_{00}$ such that $\|x\|^{\mathcal{F}} = 1$. There exists a partition $\mathcal{P} = \{F_1, F_2, ..., F_j\}$ of the support of x at which the infimum in the definition of the quasi-norm is attained, namely

$$\|x\|^{\mathcal{F}} = \sum_{i=1}^{j} \sup_{k \in F_i} |x(k)|.$$

Let x' be defined by $x'(j) = a_i \cdot \mathrm{sgn}(y(j))$ if $j \in F_i$, where $a_i = \sup_{k \in F_i} |x(k)|$ (if $j \notin \bigcup_i F_i$, then let $x'(j) = 0$). Then $\|x'\|^{\mathcal{F}} = \|x\|^{\mathcal{F}} = 1$ and

$$\Big|\sum_{n \in \mathbb{N}} x(n)y(n)\Big| \leq \sum_{n \in \mathbb{N}} |x(n)y(n)| \leq \sum_{n \in \mathbb{N}} |x'(n)y(n)|.$$

Moreover

$$\sum_{n \in \mathbb{N}} |x'(n)y(n)| = \sum_{i=1}^{j} \sum_{n \in F_i} |x'(n)y(n)| = \sum_{i=1}^{j} a_i \sum_{n \in F_i} |y(n)| \leq \sum_{i=1}^{j} a_i \|y\|_{\mathcal{F}} = \|y\|_{\mathcal{F}}$$

which, as c_{00} is dense in $X^{\mathcal{F}}$, implies that $\|y\|_*^{\mathcal{F}} \leq \|y\|_{\mathcal{F}}$ and finishes the proof. □

Unfortunately, one cannot deduce from Theorem 3.2.3 that $X^{\mathcal{F}}$ and $X_{\mathcal{F}}^*$ are isomorphic. Below, we present an attempt to prove that these spaces are indeed isomorphic.

Fix a compact, hereditary family $\mathcal{F}$ and let $\varphi \in X_{\mathcal{F}}^*$. Then

$$\varphi(x) = \sum_{k \in \mathbb{N}} x(k)y(k)$$

where $y(k) = \varphi(e_k)$. It is straightforward that

$$\|\varphi\| \leq \|y\|^{\mathcal{F}}. \tag{3.14}$$

Indeed,

$$\begin{aligned} \|\varphi\| &= \sup_{x \in X_{\mathcal{F}} \setminus \{0\}} \frac{|\sum_{k \in \mathbb{N}} x(k)y(k)|}{\|x\|_{\mathcal{F}}} = \sup_{x \in X_{\mathcal{F}} \setminus \{0\}} \frac{|\sum_{k \in \mathbb{N}} x(k)y(k)|}{\sup_{\mathcal{P} \in \mathbb{P}_{\mathcal{F}}} \|x\|_{\mathcal{P}}} = \\ &= \sup_{x \in X_{\mathcal{F}} \setminus \{0\}} \inf_{\mathcal{P} \in \mathbb{P}_{\mathcal{F}}} \frac{|\sum_{k \in \mathbb{N}} x(k)y(k)|}{\|x\|_{\mathcal{P}}} \leq \inf_{\mathcal{P} \in \mathbb{P}_{\mathcal{F}}} \sup_{x \in X_{\mathcal{F}} \setminus \{0\}} \frac{|\sum_{k \in \mathbb{N}} x(k)y(k)|}{\|x\|_{\mathcal{P}}} = \|y\|^{\mathcal{F}}, \end{aligned} \tag{3.15}$$

where the last equality is a consequence of Proposition 3.2.1 and the only inequality above comes from the fact that $\sup_{x \in A} \inf_{y \in B} \theta(x, y) \leq \inf_{y \in B} \sup_{x \in A} \theta(x, y)$, whatever A, B and θ are.

The question is whether the other inequality in 3.14 holds or, at least, with some constant $c \in (0,1)$). In a very particular case, there is equality in (3.14). Namely, consider $\mathcal{F} = \mathcal{S}$, the Schreier family and for fixed $n \in \mathbb{N}_+$ let $y \in X^{\mathcal{S}}$ be given by $y = \chi_{A_n}$, where $A_n = \bigcup_{0 \le k \le n} I_k$ and, recall, $I_k = [2^k, 2^{k+1})$. Then

$$\|y\|^{\mathcal{S}} = \sum_{m \in \mathbb{N}} \max_{k \in I_m} |y(k)| = \sum_{m \le n} \max_{k \in I_m} |y(k)| = n$$

Now define x as follows. Let $P_{I_m}(x) = 2^{n-m}\chi_{I_m}$ for $0 \le m \le n$, and for every $k \in \bigcup_{m>n} I_m$ let $x(k) = 0$. Then $x \in X_{\mathcal{S}}$, $\sum_{k \in \mathbb{N}} x(k)y(k) = n2^n$, and one can easily see that $\|x\|_{\mathcal{S}} = \sup_{m \in \mathbb{N}} \sum_{k \in I_m} |x(k)| = 2^n$. Hence, the functional $\varphi \in X^*_{\mathcal{S}}$ related to the sequence y satisfies

$$\|\varphi\| \ge \frac{n2^n}{2^n} = n = \|y\|^{\mathcal{S}}$$

For an arbitrary non-increasing sequence, we do not have equality in (3.14). However, the other inequality is satisfied with a constant equal to $\frac{1}{2}$.

Proposition 3.2.7. *For the Schreier family $\mathcal{S}$ and a non-increasing sequence y with positive coordinates, the following holds*

$$\|y\|^{\mathcal{S}} \ge \|y\|^*_{\mathcal{S}} \ge \frac{1}{2}\|y\|^{\mathcal{S}}.$$

Proof. Fix $y \in c_{00}$ satisfying the assumptions. Since y is non-increasing, we observe that in the definition of the norm $\|\cdot\|^{\mathcal{S}}$, the partition for which the infimum is attained is

$$\mathcal{G} = \{I_n : n \in \mathbb{N}\}.$$

Then $\|y\|^{\mathcal{S}} = \sum_{k \le m} y(2^k)$ for such $m \in \mathbb{N}$ that $2^m \le \max(\text{supp}(y))$. Without loss of generality, we can assume that $\max(\text{supp}(y)) = 2^{m+1} - 1$. Distinct two cases:

Case 1. $y(1) < \sum_{k=1}^{m} y(2^k)$.

Then define a sequence $x \in X_{\mathcal{S}}$ in such a way that for $k \le m$ and $j \in I_k$ put $x(j) = \frac{1}{|I_k|} = \frac{1}{2^k}$, and for $j \notin \bigcup_{k \le m} I_k$ let $x(j) = 0$. We have $\|x\|_{\mathcal{S}} = 1$ and also

$$\sum_{k \in \mathbb{N}} x(k)y(k) = \sum_{k=0}^{m} \sum_{i=2^k}^{2^{k+1}-1} \frac{y(i)}{2^k}.$$

Since y is non-increasing, for every $k \le m$ and every $i \in I_k$ we have $y(i) \ge y(2^{k+1})$. Thus,

$$\sum_{k\in\mathbb{N}} x(k)y(k) \geq \sum_{k=0}^{m-1} \frac{2^k y(2^{k+1})}{2^k} = \sum_{k=1}^{m} y(2^k) = \frac{1}{2}\Big(\sum_{k=1}^{m} y(2^k) + \sum_{k=1}^{m} y(2^k)\Big) >$$

$$\frac{1}{2}\Big(\sum_{k=1}^{m} y(2^k) + y(1)\Big) = \frac{\|y\|^{\mathcal{S}}}{2}$$

Case 2. $y(1) \geq \sum_{k=1}^{m} y(2^k)$.

Take $x(1) = 1$ and $x(k) = 0$ for $k \neq 1$. Then $\|x\|_{\mathcal{S}} = 1$ and

$$\sum_{k\in\mathbb{N}} x(k)y(k) = y(1) \geq \frac{1}{2}\Big(y(1) + \sum_{k=1}^{m} y(2^k)\Big) = \frac{\|y\|^{\mathcal{S}}}{2}$$

Hence we obtain that $\|y\|^*_{\mathcal{S}} \geq \frac{1}{2}\|y\|^{\mathcal{S}}$. □

Unfortunately, a hope given by the result for non-increasing sequences was dashed, since we found an example of a family for which $X^{\mathcal{F}}$ and $X^*_{\mathcal{F}}$ are not isomorphic.

Example 3.2.8. In this example we will consider finite dyadic trees, i.e. the sets $T_N = \{0,1\}^{\leq N}$ of 0-1 sequences of length at most N. Notice that, identifying elements of T_N with natural numbers, using some fixed enumeration of T_N, we may think of T_N as a subset of integers. For $s \in \{0,1\}^N$ let $F_s = \{s_{\restriction k} \colon k \leq N\}$ and let $\mathcal{F}_N$ be the hereditary closure of the family $\{F_s \colon s \in \{0,1\}^N\}$. So, $\mathcal{F}_N$ is the family of chains in T_N and each F_s is a maximal chain (a branch). For each $s \in \{0,1\}^N$ let x_s be the vector in $\mathbb{R}^{T_N}$ given by $x_s = \chi_{F_s}$. Let $x = \sum_{s\in\{0,1\}^N} x_s$. Notice that $x(t) = |\{s \colon t \subseteq s\}|$. It can be checked by a simple induction (on N) that

$$\|x\|^{\mathcal{F}_N} = 2^N + 1\cdot 2^{N-1} + 2\cdot 2^{N-2} + 4\cdot 2^{N-3} + \cdots + 2^{N-1}\cdot 1$$

and so

$$\|x\|^{\mathcal{F}_N} = 2^N + N2^{N-1} = 2^N(1 + N/2).$$

Let $C > 0$. Take N so that $(1 + N/2) > C$. Then

$$\Big\|\sum_{s\in\{0,1\}^N} x_s\Big\|^{\mathcal{F}_N} > C\cdot 2^N = C\sum_{s\in\{0,1\}^N} \|x_s\|^{\mathcal{F}_N}.$$

So, at this point, for every $C > 0$ we are able to find an example which violates the inequality from Remark 3.1.9 for the chosen C. Now, we will amalgamate all those $\mathcal{F}_N$'s into one example.

For each N fix an injection $k_N\colon T_N \to \mathbb{N}$ in such a way that the images $(k_N[T_N])_N$ is a partition of $\mathbb{N}$. Let

$$\mathcal{F} = \bigcup_N \{k_N[F]\colon F \in \mathcal{F}_N\}.$$

Then $\mathcal{F}$ is a compact hereditary family of subsets of $\mathbb{N}$ covering $\mathbb{N}$. But there is no $C > 0$ such that

$$\Big\| \sum_{i\in A} x_i \Big\|^{\mathcal{F}} \leq C \sum_{i\in A} \|x_i\|^{\mathcal{F}}$$

for every $A \subseteq \mathbb{N}$ and so, according to Remark 3.1.9, $X^{\mathcal{F}}$ is not isomorphic to $X^*_{\mathcal{F}}$.

Now, we will show that for the Schreier family the same phenomenon occurs, that is there is no isomorphism between $X^*_{\mathcal{S}}$ and $X^{\mathcal{S}}$. The argument is more complicated and it was presented by A.Pelczar-Barwacz in [19]. It indicates that if $\mathcal{F}$ is complicated enough, $X^{\mathcal{F}}$ is not isomorphic to $X^*_{\mathcal{F}}$.

For any finite $A \subseteq \mathbb{N}$ let $\phi(A)$ be the minimal number of consecutive Schreier sets in A covering A.

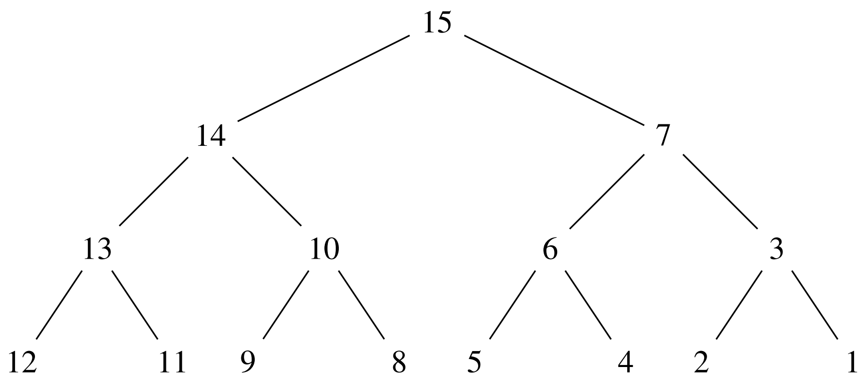
Lemma 3.2.9. *For any $N \in \mathbb{N}$ there are sets $F_1, \dots, F_{2^N} \in \mathcal{S}$ so that for $x = \sum_{j=1}^{2^N} \chi_{F_j}$ we have the following*

1. $x(i) \in \{2^r\colon r = 0, \dots, N\}$ *for any* $i \in \operatorname{supp}(x)$,
2. $\phi(A_r) \geq 2^{N-r}$, *where* $A_r = \{i \in \mathbb{N} : x(i) = 2^r\}$, *for any* $r = 0, \dots, N$.

Proof. Fix $N \in \mathbb{N}$. We shall again use the dyadic tree $T_N = \{0,1\}^{\leq N}$. This time we will assign to each element of T_N. First, we will linearize the inclusion ordering on T_N: define $\preceq$ on T_N by

$$s \preceq t \quad \text{if} \quad \big(t \subseteq s \text{ or } (t \text{ is incomparable with } s \text{ and } (s\cap t)^\frown 1 \subseteq s)\big).$$

Notice that $s \cap t$ is the longest element of T_N which is extended by both s and t. Below we enclose a drawing of T_3 with the nodes enumerated according to $\preceq$.



By t_0 we will denote the smallest element of T_N, i.e. the sequence constantly equal 1. For $s \in T_N$, $s \neq t_0$ denote by s^- be the immediate predecessor of s and let s' be the smallest, with respect to $\preceq$, descendant of s. Note that s' is always a terminal node. For $r \leq N$ let L_r be the r's level of T_N, i.e. the set of elements T_N of length r.
For every $s \in T_N$ we will define an interval I_s inductively, with respect to $\preceq$. Let $I_{t_0} = \{N+1\}$. If $s \neq t_0$ is a terminal node, then let I_s be an interval of length $2\max|I_{s^-}|$ and such that $\min I_s \geq (2N+1)\max I_{s^-}$. For a non-terminal node s let I_s be an interval of length $|I_{s'}|$ and such that $\min I_s > \max I_{s^-}$. In this way we will get a sequence of intervals (I_s) such that $s \preceq t$ iff $I_s < I_t$.
Each (maximal) branch $\mathcal{B}$ of T_N induces a set $F_{\mathcal{B}} = \bigcup_{s\in\mathcal{B}} I_s$; the sets obtained in this way form the family $(F_j)_{j=1}^{2^N}$ defining the vector x promised in the statement. First, notice that if $s \in L_{N-r}$, then s belongs to 2^r many branches. So, x satisfies the condition (1) of the statement. By the same reason we have $A_r = \bigcup_{s\in L_{N-r}} I_s$ for each r.
We will check that the family $\{F_{\mathcal{B}} \colon \mathcal{B} \text{ is a branch}\}$ is as desired by proving two claims.

Claim 1. $F_{\mathcal{B}} \in \mathcal{S}$ for every branch $\mathcal{B}$.

Consider first the branch $\mathcal{B}_0$ containing t_0. By definition $|I_s| = 1$ for each $s \in \mathcal{B}_0$, thus $|F_{\mathcal{B}_0}| \leq N+1$ (and, clearly, $\min F_{\mathcal{B}_0} = \min F_{t_0} = N+1$, so $F_{\mathcal{B}_0} \in \mathcal{S}$). Pick now a branch $\mathcal{B}$ containing any other terminal node $s \in T_N$. By definition, $\min F_{\mathcal{B}} = \min I_s \geq (2N+1)\max I_{s^-}$. On the other hand, for every $t \in \mathcal{B}$, $t' \preceq s$ and so $|I_t| = |I_{t'}| \leq |I_s| = 2\max I_{s^-}$. It follows that

$$|F_{\mathcal{B}}| \leq (2N+1)\max I_{s^-} \leq \min F_{\mathcal{B}},$$

and so $F_{\mathcal{B}} \in \mathcal{S}$.

Claim 2. $\phi(A_r) \geq |L_{N-r}| = 2^{N-r}$ for each r.

Fix $r < N$ and an interval $I \in \mathcal{S}$ in $A_r = \bigcup_{s\in L_{N-r}} I_s$ (i.e. $I = J \cap A_r$, where J is an interval). If for some $s_1 \preceq s_2 \in L_{N-r}$ we have $I \cap I_{s_1} \neq \emptyset$, then $\max I < \max I_{s_2}$. Indeed, notice that $|I| \leq \max I_{s_1}$. As $s_1 \preceq s_2$, and s_1, s_2 belong to the same level, $s_1 \preceq s_2'$, hence $2\max I_{s_1} \leq |I_{s_2'}| = |I_{s_2}|$ and so $\max I < \max I_{s_2}$. But this means that $\phi(A_r) \geq |L_{N-r}| = 2^{N-r}$ □

Theorem 3.2.10. *The space $X^{\mathcal{S}}$ is not isomorphic to $\widehat{X}^{\mathcal{S}}$ (and, thus, it is not isomorphic to $X^*_{\mathcal{S}}$).*

Proof. For every $N \in \mathbb{N}$ let $x_N = \sum_{j=1}^{2^N} \chi_{F_j^N}$ be as in Lemma 3.2.9. Then, we have $\|\chi_{F_j^N}\|^{\mathcal{S}} = 1$ for any $j = 1, \dots, 2^N$ and $N \in \mathbb{N}$, as each F_j^N is a Schreier set. Consequently,

$$\sum_j \|\chi_{F_j^N}\| = 2^N.$$

On the other hand, we will show that $2^{-N}\|x_N\|^{\mathcal{S}} \to \infty$ as $N \to \infty$. Therefore, for each $C > 0$ there is N such that

$$\|\sum_j \chi_{F_j^N}\| > C \cdot 2^N$$

and so, by Remark 3.1.9, we will be done.
Suppose, towards contradiction, that there is $M \in \mathbb{N}$ with $\|x_N\|^{\mathcal{S}} \leq 2^{M+N}$. For a fixed $N \in \mathbb{N}$ pick Schreier sets $(B_l)_l$ witnessing $\|x\|^{\mathcal{S}}$, i.e. such that $\|x_N\|^{\mathcal{S}} = \sum_l \max\{x_N(i)\colon i \in B_l\}$. Let

$$l_r = |\{l : \max\{x_N(i)\colon i \in B_l\} = 2^r\}|$$

for $r = 0, \dots, N$. Then

$$2^{N+M} \geq \sum_{r=0}^{N} l_r 2^r. \tag{3.16}$$

On the other hand, as $A^r \subseteq B^r := \bigcup\{B_l : \max\{x_N(i)\colon i \in B_l\} \geq 2^r\}$,

$$2^{N-r} \leq \phi(A^r) \leq \phi(B^r) \leq l_r + l_{r+1} + \dots + l_N \text{ for every } r = 0, \dots, N-1. \tag{3.17}$$

The first inequality follows from Lemma 3.2.9, whereas the last one from the definition of ϕ: the partition of a set into consecutive Schreier sets is the smallest in the sense of cardinality of all the partitions into Schreier sets.
In order to simplify the notation, we write $m_r = l_{N-r}$, $r = 0, \dots, N$. Then

$$\sum_{r=0}^{N} m_r 2^{-r} \leq 2^M, \tag{3.18}$$

thus

$$m_r \leq 2^{M+r} \text{ for every } r = 0, \dots, N.$$

So, for $r > M+2$ we have

$$m_0 + \dots + m_{r-M-2} \leq 2^M + \dots + 2^{M+r-M-2} = 2^M + \dots + 2^{r-2} \leq 2^{r-1}. \tag{3.19}$$

On the other hand $2^{N-r} \leq m_{N-r} + \dots + m_0$ for each $r = 0, \dots, N$, that is

$$2^r \leq m_r + \dots + m_0 \text{ for any } r = 0, \dots, N \tag{3.20}$$

Therefore, for $r > M+2$ we have, by (3.19) and (3.20),

$$2^r \leq m_r + \dots + m_{r-M-2} + \dots + m_0 \leq m_r + \dots + m_{r-M-1} + 2^{r-1}. \tag{3.21}$$

Hence, for every $r > M+2$ there is $i \in \{r-M-1, \dots, r\}$ with

$$m_i \geq 2^{r-1}(M+2)^{-1}$$

and so

$$m_i 2^{-i} \geq \frac{1}{2(M+2)}.$$

By diving (sufficiently big) N into intervals of size $M+2$ and subsequently using the above fact we see that

$$\sum_{r=0}^{N} m_r 2^{-r} \geq \frac{N}{M+2} \cdot \frac{1}{2(M+2)} = \frac{N}{2(M+2)^2} \tag{3.22}$$

which yields a contradiction with (3.18) for sufficiently big N. □

Remark 3.2.11. There is also another approach to producing a norm on the dual space of combinatorial-like spaces, different from the one considered above. In [44] D. Ojeda-Aristizabal proposed a formula for the norm of the original space constructed by Tsirelson, which can also be adapted to the case of mixed Tsirelson spaces. The case of the original Tsirelson space is somewhat similar to the case of duals to combinatorial spaces; its (pre)dual norm can be derived from a precise formula, whereas the norm of the very space does not possess an analogous expression. The formula proposed in [44] is based on a dualization of the Figiel-Johnson norm (similar to our case), but yields a norm, instead of just a quasi-norm, via including in the definition of $\|x\|$ the expression $\inf\{\|y\| + \|z\| : y + z = x\}$, which forces the triangle inequality. As it is noted in [44], this definition does not permit calculating the norm of a vector in a way similar to the case of Figiel-Johnson norm on the dual of the original Tsirelson space (see [29]). In contrast, not including the above expression in our definition allows us to work with the quasi-norm on duals of combinatorial spaces, as shown in the next section, at the price of the lack of the triangle inequality.

3.3 On ℓ_1-saturation of $X^{\mathcal{F}}$'s

In this section, we consider spaces $X^{\mathcal{F}}$ for a family $\mathcal{F}$ satisfying an additional condition (see definition 3.3.1).

In Preliminaries, we give references to the constructions of ℓ_1-saturated Banach spaces which do not have the Schur property. The result of Galego, González, and Pello in [30] says that you do not have to *construct* such space: the dual to the Schreier space is already a good example. Although the lack of Schur property in $X_{\mathcal{S}}^*$ is quite straightforward, the proof that it is ℓ_1-saturated is rather difficult.

We will show that the same holds for $X^{\mathcal{F}}$, for any compact family $\mathcal{F}$. We think that our proof is considerably easier and it indicates that studying $X^{\mathcal{F}}$ is easier than $X_{\mathcal{F}}^*$. This is why $X^{\mathcal{F}}$ may be helpful.

We will prove the lack of Schur property using Theorem 3.2.6 for families satisfying a certain property introduced by J.Lopez-Abad and S.Todorcevic in [40].

Definition 3.3.1. A family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called *large* when $\mathcal{F} \cap [M]^n \neq \emptyset$ for every infinite subset M of $\mathbb{N}$ and for every $n \in \mathbb{N}$.

Proposition 3.3.2. *Suppose that $\mathcal{F}$ is a large family of finite subsets of $\mathbb{N}$. Then $X^{\mathcal{F}}$ does not have the Schur property.*

Proof. Consider the sequence (e_n), the standard unit vector basis. We claim that this sequence is weakly null, but it is not convergent to zero in the quasi-norm.
Indeed, fix $\varphi \in (X^{\mathcal{F}})^*$ and denote $y(n) = \varphi(e_n)$. By Theorem 3.2.6 we have $y = (y(n)) \in Z_{\mathcal{F}}$, so $\|y\|_{\mathcal{F}} < \infty$. If $\lim_{n\to\infty} y(n) \neq 0$, then there is an infinite $M \subseteq \mathbb{N}$ and $c > 0$ such that $|y(m)| \geq c$ for each $m \in M$. By the assumption, for each $k \in \mathbb{N}$ there is $F \in \mathcal{F}$ such that $F \subseteq M$, $|F| = k$ and so $\sum_{i\in F} |y(i)| = c \cdot k$. Hence, $\|y\|_{\mathcal{F}} = \sup_{F\in\mathcal{F}} \sum_{i\in F} |y(i)| = \infty$, a contradiction.
On the other hand, for every $n \in \mathbb{N}$ we have $\|e_n\|^{\mathcal{F}} = 1$, so (e_n) is not convergent to zero in the quasi-norm. □

One can easily see that the Schreier family $\mathcal{S}$ is large in the sense of the Definition 3.3.1. Hence, in particular, $X^{\mathcal{S}}$ does not have the Schur property.

Definition 3.3.3. We say that the vector $x \in X^{\mathcal{F}}$ is *k-stable*, $k \in \mathbb{N}$, if $\|x\|_\infty \leq \frac{1}{2k}\|x\|^{\mathcal{F}}$.

Note that for any k-stable $x \in X^{\mathcal{F}}$ and any $F \in \mathcal{F}$ we have $\|P_F(x)\|^{\mathcal{F}} \leq \frac{1}{2k}\|x\|^{\mathcal{F}}$.

Lemma 3.3.4. *Let $x_1, x_2 \in c_{00}$ be such that $\operatorname{supp}(x_1) < \operatorname{supp}(x_2)$. If $\mathcal{P}$ is a partition such that for every $P \in \mathcal{P}$ we have $P \cap \operatorname{supp}(x_1) = \emptyset$ or $P \cap \operatorname{supp}(x_2) = \emptyset$, then*

$$\|x_1 + x_2\|^{\mathcal{P}} = \|x_1\|^{\mathcal{P}} + \|x_2\|^{\mathcal{P}}.$$

Proof. Let $P \in \mathcal{P}$. Then, either x_1 or x_2 vanishes on P, so for each $k \in P$, $|x_1(k) + x_2(k)| = |x_1(k)| + |x_2(k)|$. It implies immediately that $\|x_1 + x_2\|^{\mathcal{P}} = \|x_1\|^{\mathcal{P}} + \|x_2\|^{\mathcal{P}}$. □

Proposition 3.3.5. *Let $x, y \in c_{00}$ be such that*

(i) $\operatorname{supp}(x) < \operatorname{supp}(y)$,

(ii) y is k_0-stable, where $k_0 = \max \operatorname{supp}(x)$.

Then for each scalar λ

$$\|x + \lambda y\|^{\mathcal{F}} \geq \|x\|^{\mathcal{F}} + \frac{\lambda}{2}\|y\|^{\mathcal{F}}.$$

Proof. First, notice that if y is k-stable, then λy is k-stable and so we may assume that $\lambda = 1$.
Since $x+y$ is finitely supported, there exists partition $\mathcal{P}$ such that $\|x+y\|^{\mathcal{F}} = \|x+y\|^{\mathcal{P}}$.
Let $A = \{n \in \operatorname{supp}(y) : \forall P \in \mathcal{P}\ (n \in P \Rightarrow P \cap \operatorname{supp}(x) \neq \emptyset)\}$. Note that the sequences x, $P_{\mathbb{N}\setminus A}(y)$ and the partition $\mathcal{P}$ satisfy the assumption of Lemma 3.3.4. In addition, there are pairwise disjoint sets $F_1, ..., F_{k_0} \in \mathcal{F}$ such that $A \subseteq \bigcup_{i \leq k_0} F_i$. So, using the assumption of k_0-stability (see the remark after Definition 3.3.3) and Lemma 3.1.5 (a) we obtain

$$\|P_A(y)\|^{\mathcal{F}} \leq \sum_{i \leq k_0} \|P_{F_i}(y)\|^{\mathcal{F}} \leq \frac{1}{2}\|y\|^{\mathcal{F}}$$

and thus

$$\begin{aligned}\|x+y\|^{\mathcal{F}} = \|x+y\|^{\mathcal{P}} \geq \|x + P_{\mathbb{N}\setminus A}(y)\|^{\mathcal{P}} = \|x\|^{\mathcal{P}} + \|P_{\mathbb{N}\setminus A}(y)\|^{\mathcal{P}} \\ \geq \|x\|^{\mathcal{F}} + \|P_{\mathbb{N}\setminus A}(y)\|^{\mathcal{F}} \geq \|x\|^{\mathcal{F}} + \frac{\|y\|^{\mathcal{F}}}{2}. \quad (3.23)\end{aligned}$$

□

Theorem 3.3.6. *$X^{\mathcal{F}}$ is ℓ_1-saturated.*

Proof. Let (x_n) be a sequence in $X^{\mathcal{F}}$ and let E be a subspace of $X^{\mathcal{F}}$. We are going to show that E contains an isomorphic copy of ℓ_1. By the standard arguments (see e.g. [15]) we may assume that $E = [x_n]$, where for each $n \in \mathbb{N}$ we have $x_n \in c_{00}$, $\|x_n\|^{\mathcal{F}} = 1$ and $\operatorname{supp}(x_n)$ is finite. Additionally, we assume that $\operatorname{supp}(x_n) < \operatorname{supp}(x_{n+1})$ for each $n \in \mathbb{N}$.
It is enough to construct a sequence (y_n) of unit vectors in E which will be equivalent to the standard ℓ_1-basis, i.e. such that for each sequence $(\lambda_i)_{i \leq n}$ of scalars

$$\Big\| \sum_{i=1}^{n} \lambda_i y_i \Big\|^{\mathcal{F}} \geq \frac{1}{2} \sum_{i=1}^{n} |\lambda_i|.$$

The sequence (y_n) will be of the form of a block sequence of (x_n) We define by induction sequences of natural numbers (k_n), (l_n) and the sequence of vectors (y_n). Let $l_1 = 1$, $y_1 = x_1$ and $k_1 = \max \operatorname{supp}(x_1)$. Next, let

(1) $l_{n+1} \in \mathbb{N}$ be such that $\sum_{i=l_n+1}^{l_{n+1}} x_i$ is k_n-stable,

(2) $y_{n+1} = \dfrac{\sum_{i=l_n+1}^{l_{n+1}} x_i}{\Big\| \sum_{i=l_n+1}^{l_{n+1}} x_i \Big\|^{\mathcal{F}}}$, and

(3) $k_{n+1} = \max \operatorname{supp}(y_{n+1})$.

We will show that we are able to perform such a construction. Only condition (1) needs some explanation.

Claim. For each $l \in \mathbb{N}$ and each $k \in \mathbb{N}$ there is $L > l$ such that $\sum_{i=l}^{L} x_i$ is k-stable.

Suppose it is not true. Then there are $k, l \in \mathbb{N}$ such that for each $L > l$

$$\|\sum_{i=l}^{L} x_i\|_\infty > \frac{1}{2k} \|\sum_{i=l}^{L} x_i\|^{\mathcal{F}}.$$

Denote $x = \sum_{i=l}^{\infty} x_i$. Then, for every L we have $\|P_L(x)\|^{\mathcal{F}} < 2k\|x\|_\infty \leq 2k$, and so, by lower semicontinuity, $\|x\|^{\mathcal{F}} \leq 2k$. Thus, $x \in Z^{\mathcal{F}}$. On the other hand, for every $i \in \mathbb{N}$ we have $\|x_i\|^{\mathcal{F}} = 1$, so $\|P_{\mathbb{N}\setminus L}(x)\|^{\mathcal{F}} \geq 1$ for every L, which means that $x \notin X^{\mathcal{F}}$. This is a contradiction with Proposition 3.1.10.

Having this construction, fix $n \in \mathbb{N}$ and a sequence $(\lambda_i)_{i \leq n}$. Of course, $\|y_i\|^{\mathcal{F}} = 1$ for each i and subsequently using Proposition 3.3.5 we have

$$\Big\| \sum_{i=1}^{n} \lambda_i y_i \Big\|^{\mathcal{F}} = \Big\| \sum_{i=1}^{n-1} \lambda_i y_i + \lambda_n y_n \Big\|^{\mathcal{F}} \geq \Big\| \sum_{i=1}^{n-1} \lambda_i y_i \Big\|^{\mathcal{F}} + \frac{|\lambda_n|}{2} \geq ... \geq |\lambda_1| + \frac{1}{2}\sum_{i=2}^{n} |\lambda_i| \geq \frac{1}{2}\sum_{i=1}^{n} |\lambda_i|$$

and so (y_n) is equivalent to the standard ℓ_1-basis. □

Chapter 4

On the extreme points

This chapter is concerned with a geometric property of combinatorial spaces and those related to them: the extreme points of the unit ball. The general question of this chapter is as follows:

Problem 4.0.1. Given a hereditary family $\mathcal{F}$ covering $\mathbb{N}$ (or another countable set), what is the shape of the set $\mathrm{Ext}(X_{\mathcal{F}})$?

It is a classical result that $\mathrm{Ext}(c_0) = \emptyset$ and $\mathrm{Ext}(\ell_1) = \{\pm e_n : n \in \mathbb{N}\}$. But ideally, one would like to find a universal description of the set $\mathrm{Ext}(X_{\mathcal{F}})$ for any family $\mathcal{F}$. This seems to be difficult, as the literature on this topic is rather sparse. For example, not much is known about the set of extreme points in the Schreier space $X_{\mathcal{S}}$. It was proved, for instance, that $\mathrm{Ext}(X_{\mathcal{S}}) \subseteq c_{00}$ and that its cardinality is $\aleph_0$ (see [50]). This result was later generalized to any combinatorial space associated with regular family (see [12]). Nevertheless, there is still no full characterization of $\mathrm{Ext}(X_{\mathcal{S}})$. In [50], the authors present some examples of extreme points, but they do not appear to indicate any clear pattern.
For non-compact families even less is known about this topic. For example, we do not know whether each extreme point in related combinatorial spaces is necessarily finitely supported. The argument presented in [12] for regular families does not seem to work; however, we did not find any combinatorial space $X_{\mathcal{F}}$ and its extreme point with an infinite support.

Remark 4.0.2. Fix a family $\mathcal{F}$. For every $\theta \in \{-1, 1\}^{\mathbb{N}}$ consider the linear map $T_\theta : X_{\mathcal{F}} \to X_{\mathcal{F}}$ given by

$$T_\theta(x) = \theta \cdot x \tag{4.1}$$

It is easy to check that T_θ is an isometric isomorphism, hence it maps extreme points onto extreme points. It follows that if $x \in \mathrm{Ext}(X_{\mathcal{F}})$, then for every $\theta \in \{-1, 1\}^{\mathbb{N}}$, $\theta \cdot x \in \mathrm{Ext}(X_{\mathcal{F}})$. Thus, if we want to show that some $x \in X_{\mathcal{F}}$ is or is not an extreme point, we may assume, without loss of generality, that each coordinate of x is nonnegative (we will denote this fact by $x \geq 0$). We make such an assumption unless stated otherwise.

Now we present two simple lemmas which will be useful in the following part of this chapter.

Lemma 4.0.3. *Let $n \in \mathbb{N}$ and let $(\lambda_i)_{i\leq n}$ be a sequence of positive numbers with $\sum_{i\leq n} \lambda_i = 1$. Let $(\alpha_i)_{i\leq n}$ be a sequence of real numbers such that*

$$\sum_{i\leq n} |\lambda_i \pm \alpha_i| \leq 1.$$

Then $|\alpha_i| \leq \lambda_i$ for every $i \leq n$.

Proof. To simplify the notation, for a set $K \subseteq \mathbb{N}$ and a sequence β, we denote $\sum_{k\in K} \beta(k)$ by $S(\beta, K)$. Let $A = \{k \leq n : |\alpha_k| > \lambda_k\}$, $B = [0, n] \setminus A$. Suppose $A \neq \emptyset$. Let $A_0 = \{k \in A : \alpha_k > \lambda_k\}$ and $A_1 = \{k \in A : \alpha_k < -\lambda_k\}$. Then

$$S(\lambda, A_0) < S(\alpha, A_0) \text{ and } S(\lambda, A_1) < -S(\lambda, A_1) \tag{4.2}$$

We have

$$\begin{aligned} 1 \geq \sum_{i\leq n} |\lambda_i + \alpha_i| = S(|\lambda + \alpha|, [0, n]) = \\ = S(\lambda, A_0) + S(\alpha, A_0) - S(\lambda, A_1) - S(\alpha, A_1) + S(\lambda, B) + S(\alpha, B) = \\ = 1 - 2S(\lambda, A_1) + S(\alpha, A_0) + S(\alpha, B) - S(\alpha, A_1), \end{aligned}$$

and thus

$$2S(\alpha, A_1) \geq S(\alpha, A_0) + S(\alpha, B) - S(\alpha, A_1). \tag{4.3}$$

Furthermore, expanding $S(|\lambda - \alpha|, [0, n])$ in a similar manner, we obtain

$$2S(\lambda, A_0) \geq S(\alpha, A_0) - S(\alpha, A_1) - S(\alpha, B). \tag{4.4}$$

Hence, summing up both sides of (4.3) and (4.4) we obtain

$$2S(\lambda, A_0) + 2S(\alpha, A_1) \geq 2S(\alpha, A_0) - 2S(\alpha, A_1)$$

which is a contradiction with (4.2). □

Lemma 4.0.4. *Let $n \in \mathbb{N}_+$ and let $k < n$. Suppose $(\lambda_i)_{1\leq i\leq n}$ is a sequence of real numbers with the following property: for every $A \in [n]^k$ $\sum_{j\in A} \lambda_j = 0$. Then $\lambda_i = 0$ for every $1 \leq i \leq n$.*

Proof. Note that each i appears in exactly $\binom{n-1}{k-1}$ subsets of size k. Hence, we have

$$0 = \sum_{A \in [n]^k} \sum_{i \in A} \lambda_i = \binom{n-1}{k-1} \sum_{i \le n} \lambda_i,$$

and so $\sum_{i \le n} \lambda_i = 0$. Furthermore, for a fixed i and every $S \in [n \setminus \{i\}]^k$ (which exists, because $k < n$) we have $\sum_{j \in S} \lambda_j = 0$. Hence, the sequence $\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_n$ also sums up to 0, which implies that $\lambda_i = 0$ and since i was arbitrary, this completes the proof. □

4.1 Extreme points in dual spaces

Although not much is known about extreme points in combinatorial spaces, it turns out that in their dual spaces, there is a nice characterization. For compact families $\mathcal{F}$, the formula (3.9) presented in Chapter 3 describes an extreme point in $X_{\mathcal{F}}^*$. Below, we show that the same characterization works in the general setting.

Fix $\varphi \in X_{\mathcal{F}}^*$ and let α denotes its *sequence representation*, that is $\alpha(n) = \varphi(e_n)$ (see Remark 1.3.7). If $\alpha \in B_{X_{\mathcal{F}}^*}$, then $|\alpha(n)| \le 1$ for every n; conversely, if $\operatorname{supp}(\alpha) \in \overline{\mathcal{F}}$ and $|\alpha(n)| \le 1$ for every n, then $\alpha \in B_{X_{\mathcal{F}}^*}$. In particular, if $\beta \in \{-1, 0, 1\}^{\mathbb{N}}$ is such that $\operatorname{supp}(\beta) \in \overline{\mathcal{F}}$, then $\beta \in B_{X_{\mathcal{F}}^*}$ and, unless $\operatorname{supp}(\beta) = \emptyset$, $\|\beta\| = 1$. Now, for $\mathcal{H} \subseteq \overline{\mathcal{F}}$ define

$$W(\mathcal{H}) = \left\{ \beta \in \{-1, 0, 1\}^{\mathbb{N}} : \operatorname{supp}(\beta) \in \mathcal{H} \right\}$$

Notice that $W(\mathcal{H}) \subseteq B_{X_{\mathcal{F}}^*}$ and if $\mathcal{F} \subseteq \mathcal{H}$, then $W(\mathcal{H})$ is a *norming set*, that is,

$$\|x\|_{\mathcal{F}} = \sup \left\{ \left| \langle \beta, x \rangle \right| : \beta \in W(\mathcal{H}) \right\} \text{ for every } x \in X_{\mathcal{F}}.$$

It is easy to see that $W(\mathcal{H})$ is symmetric. Then, by [21, Lemma 4], we have

$$\overline{\operatorname{conv}}^{w^*}(W(\mathcal{H})) = B(X_{\mathcal{F}}^*).$$

Also, one can easily check that the weak* topology on $W(\mathcal{H})$ coincides with the inherited topology from the product $\{-1, 0, 1\}^{\mathbb{N}}$.

Proposition 4.1.1. *For every hereditary family $\mathcal{F}$ covering* $\mathbb{N}$ $\operatorname{Ext}(X_{\mathcal{F}}) = W(\overline{\mathcal{F}}^{\mathrm{MAX}})$.

Proof. By the Banach-Alaoglu theorem $B_{X_{\mathcal{F}}^*} = \overline{\operatorname{conv}}^{w^*}(W(\overline{\mathcal{F}}))$ is weak* compact. Since $X_{\mathcal{F}}^*$ with the weak* topology is locally convex, applying Milman's theorem (see [27, Theorem 3.66]) we obtain $\operatorname{Ext}(X_{\mathcal{F}}) \subseteq \overline{\operatorname{conv}}^{w^*}(W(\overline{\mathcal{F}})) = W(\overline{\mathcal{F}})$. Now suppose that $\beta \in W(\overline{\mathcal{F}})$ and that $\mathrm{supp}(\beta)$ is not maximal in $\overline{\mathcal{F}}$. Let $k \in \mathbb{N} \setminus \mathrm{supp}(\beta)$ be such that $\mathrm{supp}(\beta) \cup \{k\} \in \overline{\mathcal{F}}$. It follows that $\|\beta \pm e_k^*\| \leq 1$ and so β is not an extreme point. To see that $W(\overline{\mathcal{F}}^{\mathrm{MAX}}) \subseteq \mathrm{Ext}(X_{\mathcal{F}})$, consider $\sigma \in W(\overline{\mathcal{F}}^{\mathrm{MAX}})$ and suppose there are $\alpha, \beta \in B_{X_{\mathcal{F}}^*}$, different than σ, such that $\sigma = \frac{\alpha+\beta}{2}$. It follows that for every $n \in \mathrm{supp}(\sigma)$ $\alpha(n) = \beta(n) = \sigma(n)$, and hence there is $k \in \mathrm{supp}(\alpha) \setminus \mathrm{supp}(\sigma)$. As $\mathrm{supp}(\sigma) \cup \{k\} \notin \overline{\mathcal{F}}$, we can pick a finite non-empty $S \subseteq \mathrm{supp}(\sigma)$ such that $S \cup \{k\} \notin \mathcal{F}$. Define $x \in X_{\mathcal{F}}$ such that $\mathrm{supp}(x) = S \cup \{k\}$ and $x(n) = \frac{\mathrm{sgn}(\alpha(n))}{|S|}$ for every $n \in \mathrm{supp}(x)$. Then $\|x\|_{\mathcal{F}} = 1$, but $\langle \alpha, x\rangle = 1 + \frac{|\alpha(k)|}{|S|} > 1$, which is a contradiction. □

4.2 Extreme points in Farah spaces

In this section, we present a full characterization of extreme points in g-Farah spaces. The key fact turns out to be the following lemma.

Lemma 4.2.1. *Let* $g : \mathbb{N} \to [1, \infty)$ *and let* $x \in X_{\mathbf{F}_g}$. *If* $x \in \mathrm{Ext}(X_{\mathbf{F}_g})$, *then* $|\{n \in \mathbb{N} : \mathrm{supp}(x) \cap I_n \neq \emptyset\}| = 1$.

Proof. Following Remark 4.0.2, we assume that $x(j) \geq 0$ for each $j \in \mathbb{N}$. Suppose there are $n_0 < n_1$ such that $B_{n_i} = \mathrm{supp}(x) \cap I_{n_i} \neq \emptyset$ for $i \in \{0, 1\}$. Let $A \in \overline{\mathbf{F}_g}$ be such that $1 = \|x\|_{\mathbf{F}_g} = \sum_{j \in A} x(j)$. Let $A_{n_i} = A \cap I_{n_i}$. Then $|A_{n_i}| \leq \lfloor g(n_i) \rfloor$ and, without loss of generality, we can assume that $A_{n_i} \subseteq B_{n_i}$. For a sufficiently small $\varepsilon > 0$ define

$$z = \frac{\varepsilon}{|A_{n_0}|}\chi_{B_{n_0}} - \frac{\varepsilon}{|A_{n_1}|}\chi_{B_{n_1}}.$$

Then $\|z\|_{\mathbf{F}_g} \leq 2\varepsilon$ and, since z is piecewise constant, then $\|x \pm z\|_{\mathbf{F}_g}$ is attained on A. In particular, if $C = I_{n_0} \cup I_{n_1}$, then

$$\|P_C(x \pm z)\|_{\mathbf{F}_g} = \sum_{j \in A_{n_0}} (x \pm z)(j) + \sum_{j \in A_{n_1}} (x \pm z)(j) =$$
$$\sum_{j \in A_{n_0}} x(j) \pm \varepsilon + \sum_{j \in A_{n_1}} x(j) \mp \varepsilon = \sum_{j \in A_{n_0}} x(j) + \sum_{j \in A_{n_1}} x(j),$$

and so $\|x \pm z\|_{\mathbf{F}_g} = 1$, a contradiction.

□

Hence, if $x \in \mathrm{Ext}(X_{\mathbf{F}_g})$, then $\mathrm{supp}(x) \subseteq I_n$ for some n.

Lemma 4.2.2. *The vector* $\pm e_n$ *is an extreme point in* $X_{\mathbf{F}_g}$ *if and only if* $n \in I_m$, *where* m *is such that* $g(m) \geq 2$.

Proof. Suppose $e_n \in \mathrm{Ext}(X_{\mathbf{F}_g})$ and let m be such that $n \in I_m$. Then, if $g(m) < 2$, then for every $k \in I_m \setminus \{n\}$ $\|e_n \pm e_k\|_{\mathbf{F}_g} = 1$.
Now let $m \in \mathbb{N}$ be such that $g(m) \geq 2$ and $n \in I_m$. Suppose that e_n is not an extreme point and let $z \neq 0$ be such that $\|e_n \pm z\|_{\mathbf{F}_g} \leq 1$. One can see that $n \notin \mathrm{supp}(z)$ and $\mathrm{supp}(z) \cap I_m \neq \emptyset$, because otherwise $\|e_n \pm z\|_{\mathbf{F}_g} > 1$. So there is $j \in \mathrm{supp}(z) \cap I_m$ different from n. Since $g(m) \geq 2$, then $\|e_n \pm z\|_{\mathbf{F}_g} \geq 1 + |z(j)| > 1$, a contradiction. □

It is natural to ask whether there are any other extreme points in $X_{\mathbf{F}_g}$. If there is such x, then there is $m \in \mathbb{N}$ such that $|x(k)| < 1$ for every $k \in I_m$ and $x(k) \neq 0$ for some k.

Lemma 4.2.3. *Suppose $x \in \mathrm{Ext}(X_{\mathbf{F}_g})$ and $x \neq \pm e_n$ for every n. Then $\mathrm{supp}(x) = I_m$ for some $m \in \mathbb{N}$.*

Proof. Suppose that $\mathrm{supp}(x) \subsetneq I_m$. Then, if $|\mathrm{supp}(x)| < \lfloor g(m) \rfloor$, there exists $\varepsilon > 0$ such that $y = \varepsilon(e_k - e_l)$, where $k, l \in \mathrm{supp}(x)$ with $k \neq l$, yields $\|x \pm y\|_{\mathbf{F}_g} = 1$, which is a contradiction. On the other hand, if $|\mathrm{supp}(x)| \geq \lfloor g(m) \rfloor$, then choose $0 < \delta < \min\{x(k) : k \in \mathrm{supp}(x)\}$ and define $y = \delta e_j$ for some $j \in I_m \setminus \mathrm{supp}(x)$. Then clearly, $\|x \pm y\|_{\mathbf{F}_g} = \|x\|_{\mathbf{F}_g} = 1$, and thus $x \notin \mathrm{Ext}(X_{\mathbf{F}_g})$. □

Theorem 4.2.4.

$$\mathrm{Ext}(X_{\mathbf{F}_g}) = \big\{ \pm e_n : \exists m \in \mathbb{N}\big(g(m) \geq 2 \wedge n \in I_m\big)\big\}$$
$$\cup \big\{ \frac{1}{\lfloor g(n) \rfloor}\theta \cdot \chi_{I_n} : n \in g^{-1}\big[[1, 2^n)\big],\ \theta \in \{-1, 1\}^{\mathbb{N}} \big\}.$$

Proof. Let E_0 and E_1 denote, respectively, the first and second sets in the union on the right-hand side above. By Lemma 4.2.2 $E_0 \subseteq \mathrm{Ext}(X_{\mathbf{F}_g})$, so now we will show that the same holds for E_1.
Suppose that for a fixed m $x = \frac{1}{\lfloor g(m) \rfloor}\chi_{I_m}$ is not an extreme point. Then there is nonzero $y \in X_{\mathbf{F}_g}$ such that $\|x \pm y\|_{\mathbf{F}_g} \leq 1$. Let $A \in \overline{\mathbf{F}_g}$ be such that $|A \cap I_n| = \lfloor g(n) \rfloor$ for every n. Then

$$\sum_{j \in A \cap I_m} |x(j) \pm y(j)| + \sum_{n \neq m} \sum_{j \in A \cap I_n} |y(j)| \leq 1. \tag{4.5}$$

Note that by Lemma 4.0.3 $x(j) \pm y(j) \geq 0$. Hence, we can rewrite (4.5) as follows

$$1 \pm \sum_{j \in A \cap I_m} y(j) + \sum_{n \neq m} \sum_{j \in A \cap I_n} |y(j)| \leq 1. \tag{4.6}$$

Thus

$$\sum_{n \neq m} \sum_{j \in A \cap I_n} |y(j)| \leq 0,$$

hence for each $n \neq m$ and each $j \in A \cap I_n$ we have $y(j) = 0$, and what follows - $y(j) = 0$ for every $j \notin I_m$. Having this, from (4.6) we obtain that

$$\sum_{j \in A \cap I_m} y(j) = 0,$$

and by Lemma 4.0.4 (as $A \cap I_m$ is an arbitrary subset of I_m of size $\lfloor g(m) \rfloor < 2^m$) we conclude that $y(j) = 0$ for each $j \in I_m$. This is a contradiction with $y \neq 0$.
Now suppose that $x \in \mathrm{Ext}(X_{\mathbf{F}_g})$. Then, by Lemma 4.2.1, there is $n \in \mathbb{N}$ such that $\mathrm{supp}(x) \subseteq I_n$. If $|\mathrm{supp}(x)| = 1$, then $x = \pm e_k$, and by Lemma 4.2.2 we know that in this case $g(n) \geq 2$, i.e. $x \in E_0$. If $|\mathrm{supp}(x)| > 1$, then by Lemma 4.2.3 $\mathrm{supp}(x) = I_n$. By Remark 4.0.2 we can assume that $x(k) > 0$ for each $k \in I_n$. Suppose that $x \notin E_1$, that is, x is not constant on the I_n. Define $B_0 = \{k \in I_n : x(k) < \frac{1}{\lfloor g(n) \rfloor}\}$ and $B_1 = I_n \setminus B_0$. Notice that by the assumption these sets are non-empty and, in addition, $|B_1| < \lfloor g(n) \rfloor$. Let $A_0 \subseteq B_0$ be such that

$$\min_{k \in A_0} x(k) \geq \max_{k \in B_0 \setminus A_0} x(k) \tag{4.7}$$

and $|A_0| + |B_1| = \lfloor g(n) \rfloor$. Then we have

$$1 = \|x\|_{\mathbf{F}_g} = \sum_{k \in A_0} x(k) + \sum_{k \in B_1} x(k).$$

There exists $\varepsilon > 0$ such that for every $k \in B_1$ $x(k) - \varepsilon > x(j)$ for every $j \in B_0$ and, what follows, $x(j) + \varepsilon < \frac{1}{\lfloor g(n) \rfloor}$ for each $j \in B_0$. Define

$$y = \varepsilon e_k - \frac{\varepsilon}{|A_0|} \chi_{B_0}.$$

Then we have

$$\|x \pm y\|_{\mathbf{F}_g} = \sum_{k \in A_0} \big(x(k) \mp \frac{\varepsilon}{|A_0|}\big) + \sum_{k \in B_1} x(k) \pm \varepsilon = 1,$$

thus $x \notin \mathrm{Ext}(X_{\mathbf{F}_g})$, a contradiction. □

On the family determined by a fixed vector

Let x be a vector given by

$$x = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \chi_{I_n}. \tag{4.8}$$

In this short part of the chapter, we consider the family $\mathcal{F}_x$ determined by x in the following way

$$\mathcal{F}_x = \{A \in [\mathbb{N}]^{<\infty} : \sum_{j \in A} x(j) \leq 1\}. \tag{4.9}$$

Note that for every $n \in \mathbb{N}$, an interval I_n is an element of $\mathcal{F}_x$. In addition, $\mathcal{F}_x$ is not compact, as it contains infinite sets in its closure, e.g., $\{2^n : n \geq 1\} \in \overline{\mathcal{F}_x}$.
For a family defined in such a way, the Banach space $Z_{\mathcal{F}_x}$ has an interesting property. Namely, x is its extreme point.

Proposition 4.2.5. *The vector x given by (4.8) is the extreme point of $Z_{\mathcal{F}_x}$.*

Proof. Suppose x is not an extreme point and let z be, as usual, a nonzero vector such that $\|x \pm z\|_{\mathcal{F}_x} \leq 1$. Denote $z(j) = \varepsilon_j$ for every $j \geq 1$. In particular, for every maximal element A (i.e. such that $\sum_{j \in A} x(j) = 1$) we have $\sum_{j \in A} |x(j) \pm \varepsilon_j| \leq 1$. Hence, the assumptions of Lemma 4.0.3 are satisfied by $A \in \mathcal{F}_x^{\mathrm{MAX}}$ and numbers $x(j), \varepsilon_j$ for $j \in A$, and then we have

$$1 \pm \sum_{j \in A} \varepsilon_j \leq 1,$$

and thus $\sum_{j \in A} \varepsilon_j = 0$ for every maximal set A. In particular, for every $k \in \mathbb{N}$

$$\sum_{j \in I_k} \varepsilon_j = 0 \tag{4.10}$$

and for every $m \in I_k$

$$\sum_{j \in I_k \setminus \{m\}} \varepsilon_j + \varepsilon_{2^{k+1}} + \varepsilon_{2^{k+1}+1} = 0, \tag{4.11}$$

as $I_k \setminus \{m\} \cup \{l, l'\}$ is a maximal set in $\mathcal{F}_x$ for every pair $l, l' \in I_{k+1}$. Note that, combining (4.10) and (4.11), we obtain 2^k+1 equations with 2^k+2 variables $\varepsilon_{2^k}, \varepsilon_{2^k+1}, \ldots, \varepsilon_{2^{k+1}+1}$. The matrix M_k of size $(2^k+1) \times (2^k+2)$ that corresponds to this system of linear equations is, up to permutation of rows or columns, of the following form. Its first row consists of 2^k consequtive 1's and last two entries are equal to 0, and for $2 \leq j \leq 2^k+1$ the j-th row has only one 0 in $(j-1)$-th column, and the rest of the entries are equal to 1. For example, for $k = 2$ the matrix M_2 is equal to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

In particular, one can see that the last two columns are equal. The rank of this matrix is equal to $2^k + 1$. Indeed, one can see that submatrix B_k which is obtained from

M_k by removing its last column, is a square matrix of dimension $(2^k + 1)$ and, up to permuatation, its j-th row, for $1 \leq j \leq 2^k + 1$ is equal to $\mathbf{1} - e_j$, where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^{2^k+1}$ and e_j is j-th standard unit vector in $\mathbb{R}^{2^k+1}$. Hence, these rows are linearly independent vectors, thus $\det(B_k) \neq 0$, and so $\operatorname{rank}(M_k) = 2^k + 1$.
It is also easy to see that

$$u = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix} \in \ker(M_k).$$

Using the rank-nullity theorem, we obtain that $\dim(\ker(M_k)) = (2^k+2) - \operatorname{rank}(M_k) = 1$, so $\ker(M_k)$ is spanned by u. What follows, in particular, $\varepsilon_i = 0$ for every $i \in I_k$, and $\varepsilon_{2^{k+1}} = \varepsilon_{2^{k+1}+1}$. Since in (4.11) 2^{k+1} and $2^{k+1}+1$ can be exchanged for any $l, l' \in I_{k+1}$, we also obtain that $\varepsilon_i = 0$ for all $i \in I_{k+1}$. Since k was chosen arbitrarily, for every $j \in \mathbb{N}$ we have $\varepsilon_j = 0$, and hence $z = 0$. $\square$

Remark 4.2.6. Note that, by the definition of x,

$$A \in \mathcal{F}_x \Leftrightarrow \sum_{k\in\mathbb{N}} \sum_{j \in A\cap I_k} x(j) \leq 1 \Leftrightarrow \sum_{k\in\mathbb{N}} \frac{|A \cap I_k|}{2^k} \leq 1 \Leftrightarrow \left(\frac{|A\cap I_k|}{|I_k|}\right) \in B_{\ell_1}.$$

Hence, $\mathcal{F}_x$ is the family of the form (2.10), more precisely, $\mathcal{F}_x = \mathcal{F}(c_0)$.

4.3 Combinatorial spaces and graphs

In this section, we introduce a combinatorial norm $\|\cdot\|_G$ associated with a graph G. We present it in this chapter because we will be mostly interested in extreme points in a related combinatorial space. More precisely, we will study extreme points of the unit ball in Z_G. In many cases, G will be a finite graph, and thus Z_G and X_G will coincide (as finite-dimensional spaces).

Let Ω be a countable, possibly finite, set and let $G = (V, E)$ be a graph with $V \subseteq \Omega$. Let $\mathcal{C}_G$ and $\mathcal{A}_G$ denote the set of all cliques and the set of all anticliques of G, respectively. Then, a combinatorial Banach space associated with a graph G is defined as a completion of c_{00} with respect to the following norm

$$\|x\|_G = \sup_{C\in\mathcal{C}_G} \sum_{v\in C} |x(v)|.$$

Note that here we slightly abuse the notation compared to the previously defined combinatorial spaces. More precisely, in the subscript of the norm symbol, we should write $\mathcal{C}_G$, but we only write the symbol G. This should not confuse the reader.

It is worth mentioning that some of the families considered so far in this thesis can be viewed as graphs, such as $\mathcal{C}$ and $\mathcal{A}$ from Section 2.3. Indeed, $\mathcal{C} = (2^{<\mathbb{N}}, E_0)$, where E_0 consists of pairs of comparable sequences, and $\mathcal{A} = (2^{<\mathbb{N}}, E_1)$, where E_1 is a complement of E_0. These two graphs are perfect and *dual* in the sense that the cliques of $\mathcal{C}$ are the anticliques of $\mathcal{A}$ and vice versa.

It turns out that the extreme points in the case of perfect and non-perfect graphs differ significantly; hence, we divide the discussion into two subsections.

4.3.1 Extreme points and perfect graphs

The literature concerning perfect graphs is very rich. Recall that by Theorem 1.2.2 G is perfect if and only if it contains neither an odd hole nor an odd antihole. We will present an elegant characterization of the set of extreme points in Z_G. In more general settings, these results come from a manuscript by P. Borodulin-Nadzieja, B. Farkas, and J. Lopez-Abad, which had not yet been published at the time of writing this thesis.

Recall that by Theorem 1.2.1 of L. Lovász, a graph $G = (V, E)$ is perfect if and only if its complement is perfect. It turns out that the following theorem of V.Chvátal (see [24]) was an important part of Lovász's proof.

Theorem 4.3.1. *A graph $G = (V, E)$ is perfect if and only if*

$$\mathrm{conv}\{\chi_A : A \in \mathcal{A}_G\} = \left\{x \in [0,1]^V : \sum_{v \in C} x(v) \leq 1 \textit{ for every } C \in \mathcal{C}_G\right\}.$$

This theorem will also be an important part of our characterization of extreme points. Note that the equality in the theorem above can be written as follows

$$\mathrm{conv}\{x \in W(\mathcal{A}_G) : x \geq 0\} = \{x \in B_{X_G} : x \geq 0\},$$

where $W(\mathcal{H}) = \left\{\beta \in \{-1,0,1\}^{\mathbb{N}} : \mathrm{supp}(\beta) \in \mathcal{H}\right\}$ for a family $\mathcal{H}$ (see Section 4.1).

Theorem 4.3.2. *For a graph $G = (V, E)$ the following are equivalent*

(1) G is perfect.

(2) $W(\mathcal{A}_G) = B_{X_{\mathcal{C}_G}} \cap c_{00}$.

(3) For every finite induced subgraph H of G $B_{X_{\mathcal{C}_H}} = B_{X^*_{\mathcal{A}_H}}$.

(4) For every finite induced subgraph H of G $\mathrm{Ext}(X_{\mathcal{C}_H}) = W(\mathcal{A}_H^{\mathrm{MAX}})$.

Note that here we use the notation with $\mathcal{C}_H$ and $\mathcal{A}_H$ in a subscript to emphasize that we consider the spaces induced by cliques and anticliques, respectively.

Proof. We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2). Note that this implication follows immediately from Theorem 4.3.1, since both sets in (2) are invariant under changing signs in coordinates.
(2) $\Rightarrow$ (3). Note that (2) holds if and only if $W(\mathcal{A}_H) = B_{X_{c_H}}$ for every finite induced subgraph H. Then,

$$B_{X^*_{\mathcal{A}_H}} = \mathrm{conv}(\mathrm{Ext}(X^*_{\mathcal{A}_H})) = \mathrm{conv}(W(\mathcal{A}_H^{\mathrm{MAX}})) = \mathrm{conv}(W(\mathcal{A}_H)) = B_{X_{c_H}},$$

where the second equality follows from Proposition 4.1.1.
(3) $\Rightarrow$ (4). This is an immediate consequence of Proposition 4.1.1.
(4) $\Rightarrow$ (1). If G is not perfect, then it contains either an odd hole or an odd antihole. Both of these cases are considered in the next section. Then, by Proposition 4.3.7 and Proposition 4.3.9, we conclude that (4) is not satisfied. □

The following corollary follows straightforwardly from the previous theorem.

Corollary 4.3.3. *If G is a perfect graph, then* $\mathrm{Ext}(Z_G) = W(\mathcal{A}_G^{\mathrm{MAX}})$.

It is important to mention that Theorem 4.3.2 does not allow us to draw such conclusions about the shape of $\mathrm{Ext}(X_G)$ for an infinite graph G. Indeed, $\mathcal{F} = [\mathbb{N}]^{\leq 1}$ is a perfect graph, but $\mathrm{Ext}(X_{\mathcal{F}}) = \mathrm{Ext}(c_0) = \emptyset$. The same situation is with $\mathcal{A}$, family of all antichains on $2^{<\mathbb{N}}$. Indeed, if x was in $\mathrm{Ext}(X_{\mathcal{A}})$, then $\mathrm{supp}(x)$ would be some maximal chain C, and for each $\alpha \in C$ $|x(\alpha)| = 1$. But this is impossible, because the norm of tails of x should be convergent to 0.

Sierpiński coloring and combinatorial Banach spaces

This part of the chapter is an exception, which means we consider combinatorial Banach spaces related to some graphs, but we do not focus on their extreme points. We present a quite interesting family defined by a certain coloring. It is also considered in the manuscript of Nadzieja, Farkas, and Lopez-Abad that was mentioned before.
A function $c : [\Omega]^2 \to \{0, 1\}$ is called *a coloring*. Every set $A \subseteq \Omega$ such that $|c[[A]^2]| = 1$ is called *monochromatic*.
Having coloring c we can define a graph $G_c = (\Omega, E)$, where $E = \{\{a, b\} \in [\Omega]^2 : c(a, b) = 1\}$. Then the set of finite cliques (anticliques) consists of monochromatic sets of color 1 (0).
Fix a bijection $f : \mathbb{N} \to \mathbb{Q}$. The *Sierpiński's coloring* c_f associated with this bijection is a coloring defined on $[\mathbb{N}]^2$ in the following way: $c_f(m, n) = 1$ if and only if $(m < n \Leftrightarrow f(m) < f(n))$. Hence, the family of cliques consists of these sets $A = \{n_0 < n_1 < ... < n_k\}$, $k \in \mathbb{N}$, such that $f(n_0) < f(n_1) < ... < f(n_k)$. Analogously, anticliques are exactly these sets, on which f is decreasing. We denote by G_f a graph for a given coloring associated to the bijection f and, for simplicity, by X_f we denote the combinatorial Banach space related to G_f. We call it the *Sierpiński's space* associated with f.

Proposition 4.3.4. *For every bijections $f, g : \mathbb{N} \to \mathbb{Q}$, the graph G_f is isomorphic to an induced subgraph of G_g, and vice versa.*

The idea behind the proof is to *mimic the order* of f via the bijection g. For example, if f and g restricted to $[0, 3)$ are given by $f(0) = 0$, $f(1) = -1$, $f(2) = 3$, and $g(0) = -1$, $g(1) = -2$, $g(2) = 0$, then we can say that they mimic each other, because $f(i) < f(j) \Leftrightarrow g(i) < g(j)$ for every pair (i, j).

Proof. Without loss of generality, we can assume that $f(0) = g(0) = 0$. The isomorphism is a result of the following inductive procedure.
First note that there exists the smallest natural number $n_1 > 0$ such that $c_g(\{0, n_1\}) = c_f(\{0, 1\})$. At step k we find the smallest natural number $n_k > n_{k-1}$ such that $c_g(\{0, n_k\}) = c_f(\{0, k\})$ and $c_g(\{n_j, n_k\}) = c_f(\{j, k\})$ for every $j < k$. This is a formal expression of the fact that every relation between any two values of f, restricted to $[0, k]$ is exactly the same as the relation between any two values of g.
In this way, we find an infinite set $M = \{0 < n_1 < n_2 < ...\}$ such that the map $\mathbb{N} \ni j \mapsto n_j \in M$ is a graph isomorphism between G_f and a subgraph of G_g induced by M. □

As a consequence, we have the following corollary concerning combinatorial spaces induced by G_f and G_g.

Corollary 4.3.5. *For every bijections $f, g : \mathbb{N} \to \mathbb{Q}$, X_f is isometric to a complemented subspace of X_g and vice versa.*

Proof. We find infinite sets M, N such that G_f is isomorphic to a subgraph of G_g induced by M, and G_g is isomorphic to a subgraph of G_f induced by N. Then X_f is isometric to $[e_n : n \in M] \subseteq X_g$ and X_g is isometric to $[e_n : n \in N] \subseteq X_f$. Clearly, these isometric copies are complemented. □

However, the theorem of M. Wójtowicz (see [52]) states that if (x_n) and (y_n) are bases in Banach spaces, (x_n) is equivalent to a subbasis of (y_n) and (y_n) is equivalent to a subbasis of (x_n), then (x_n) and (y_n) are permutatively equivalent. Thus, we obtain the following result.

Theorem 4.3.6. *For any two bijections $f, g : \mathbb{N} \to \mathbb{Q}$, X_f and X_g are isomorphic.*

4.3.2 Extreme points and non-perfect graphs

In this subsection, we consider only non-perfect graphs. Recall that, by Theorem 1.2.2, such graphs must contain odd holes or odd antiholes of size at least 5. We begin our study of $\mathrm{Ext}(X_G)$ with the case where G is an odd hole or an odd antihole. Then we will extend it to some more complex non-perfect graphs.

Proposition 4.3.7. *If $G = (V, E)$ is an odd hole then*

$$\text{Ext}(X_G) = \left\{ \frac{1}{2}\theta \cdot \chi_V : \theta \in \{-1, 1\}^V \right\}$$

Proof. Denote $V = \{v_0, v_1, ..., v_{k-1}\}$, where $k > 3$ is an odd number. Suppose that a vector $x \in \mathbb{R}^V$ given by $x(v_j) = \frac{1}{2}$ for $0 \leq j < k$ is not an extreme point (note that $\|x\|_G = 1$, since all maximal cliques in G are of size 2). Take $z \neq 0$ such that $\|x \pm z\|_G \leq 1$. Denote $z(v_j) = \varepsilon_j$. Note that for every $0 \leq j < k$ we have $|\varepsilon_j| \leq \frac{1}{2}$, hence

$$\begin{cases} \frac{1}{2} + \varepsilon_j + \frac{1}{2} + \varepsilon_{j+1} \leq 1 \\ \frac{1}{2} - \varepsilon_j + \frac{1}{2} - \varepsilon_{j+1} \leq 1 \end{cases},$$

where an addition in the subscript of ε's is mod k. It implies that for every j we have $\varepsilon_j + \varepsilon_{j+1} = 0$. Hence $z = \varepsilon_0 \eta \cdot \chi_V$ where $\eta \in \{-1, 1\}^V$ is given by $\eta(v_j) = (-1)^j$ for $0 \leq j < k$. However, since k is odd, then $\eta(v_0) = \eta(v_{k-1})$, so $\varepsilon_0 = \varepsilon_{k-1} = 0$, and what follows, $z = 0$.

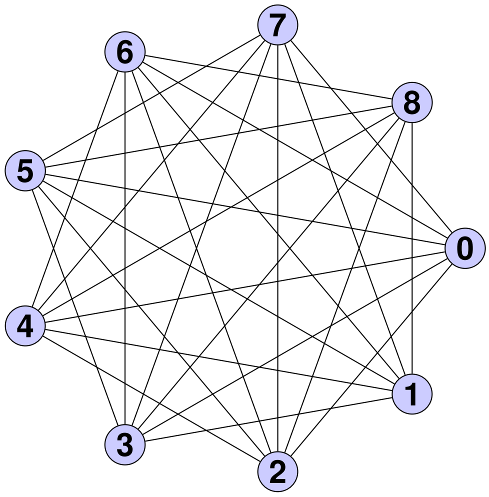
Now let $e \in \text{Ext}(X_G)$ and assume that all values of e are non-negative. Note that then $e(v_i) + e(v_{i+1}) = 1$ for every $0 \leq i < k$. Indeed, otherwise take a sufficiently small positive number δ and define $z \in \mathbb{R}^V$ as follows: if $e(v_0) + e(v_1) = 1$, then let $z(v_0) = \delta$ and $z(v_1) = -\delta$. Otherwise, put $z(v_0) = z(v_1) = \delta$. Next, for any $i \geq 1$, put $z(v_{i+1}) = -z(v_i)$, if $e(v_i) + e(v_{i+1}) = 1$ and $z(v_{i+1}) = z(v_i)$, otherwise. Then z, up to an absolute value, is equal δ and for every i $|e(v_i) + z(v_i)| + |e(v_{i+1}) + z(v_{i+1})|$ is either equal 1 or equal to $e(v_i) + e(v_{i+1}) + 2\delta$ and the δ is chosen such that it does not exceed 1. For $|e(v_i) - z(v_i)| + |e(v_{i+1}) - z(v_{i+1})|$ we have an analogous case, and hence $\|e \pm z\|_G \leq 1$. Note that we did not use an assumption that k is odd, so this is true for any hole.
Now suppose there is j such that $e(v_j) \neq \frac{1}{2}$. For simplicity, we can assume that $j = 0$. Since $e(v_i) + e(v_{i+1}) = 1$ for every i, then we have

$$e(v_i) = \begin{cases} e(v_0), & \text{if } i \text{ is even} \\ 1 - e(v_0), & \text{if } i \text{ is odd.} \end{cases}$$

However, since $k - 1$ is even we have $e(v_{k-1}) + e(v_0) = 2e(v_0) \neq 1$, which is a contradiction. □

Before we begin our analysis of extreme points in the antihole case, we need to introduce some notation.

Recall that an antihole is the complement of a hole, meaning that each vertex is connected to all the others except its two neighbors (in the original cycle). Below we present a graphical representation of a 9-antihole.



To describe the set of extreme points in the case of an odd antihole G, we need to present some facts about cliques in G.
We use the following notation. If $V = \{v_0, v_1, ..., v_{k-1}\}$ is the set of vertices of an antihole G and $\{v_{i_0}, v_{i_1}, ..., v_{i_m}\} \subseteq V$ is a clique, then we denote $C = \langle i_0, i_1, ..., i_m \rangle$ and we always assume that j-th entry of C is smaller than $(j+1)$-th.
It is easy to see that the size of a maximal clique in G is equal to $\lfloor \frac{k}{2} \rfloor$, since among the numbers $\{0, 1, \ldots, k-1\}$ there is at most $\frac{k}{2}$ pairs p, q such that $|p - q| \geq 2$ ($\mod (k-1))$. An interesting question is, how many of the maximal cliques are there in G?
This can be reformulated as the following combinatorial problem. Find the cardinality of $B \subseteq \{0,1\}^{2m+1}$ ($k = 2m+1$) of the sequences satisfying following properties

(1) Each element s of B has exactly m 1's,

(2) For every $s \in B$ there is no $i < 2m$ such that $s(i) = s(i+1) = 1$, i.e. the distance between 1's needs to be at least 2,

(3) $s(0)$ and $s(2m)$ cannot be both equal 1.

Lemma 4.3.8. $|B| = 2m + 1$

Proof. One can see that $s \in B$ if and only if s satisfies condition (2) and there is exactly one block of two zeros in s, i.e. there is exactly one $j \in \{0, 1, ..., 2m\}$ such that $s(j) = s(j+1) = 0$ (including the case that the first and last coordinates of s is a block as well). Thus, we have $2m + 1$ many choices of such j and this choice, combined with condition (2), determines the sequence s uniquely.

□

We also need to introduce a certain type of matrix. Let $n \in \mathbb{N}_+$. We say that an $n \times n$ matrix A is *circulant* (see [33]) if it is of the form

$$A = \begin{bmatrix} a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \dots & a_{n-3} & a_{n-2} \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ a_2 & a_3 & \dots & a_0 & a_1 \\ a_1 & a_2 & \dots & a_{n-1} & a_0 \end{bmatrix}. \tag{4.12}$$

In other words, A is determined by a single vector $(a_0, a_1, ..., a_{n-2}, a_{n-1})$, meaning that for every $1 < i \leq n$, the i-th row is a right shift of the first row by $(i-1)$ positions.
It is known (see [33]) that the eigenvalues λ_j of A, for $0 \leq j < n$ are given by the formula

$$\lambda_j = \sum_{k=0}^{n-1} a_k \zeta_n^{-kj}, \tag{4.13}$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$ is a primitive n-th root of unity. It is worth to mention that (4.13) is an $(j+1)$-th coordinate of a *discrete Fourier transform* of a sequence $(a_0, a_1, ..., a_{n-1})$.

Proposition 4.3.9. *Let $G = (V, E)$ be an odd antihole with $|V| = 2m+1$, where $m \geq 2$. Then*

$$\mathrm{Ext}(X_G) = \left\{ \frac{1}{m} \theta \cdot \chi_V : \theta \in \{-1, 1\}^V \right\}.$$

Proof. First, we show that x given by $x(v) = \frac{1}{m}$, for $v \in V$ is an extreme point.
Suppose otherwise, and let z be a nonzero vector for which $\|x \pm z\|_G \leq 1$. Then for every maximal clique C_i, where $0 \leq i < 2m+1$, we have $\sum_{v \in C_i} (x(v) \pm z(v)) \leq 1$ and, what follows, $\sum_{v \in C_i} z(v) = 0$. Fix a *clockwise* enumeration of V Then one can see that we obtained a system of linear equations $Az = 0$, where A is a $0-1$ square matrix of dimension $2m+1$ such that each row consists of m 1's, indicating a clique of G (e.g. for $m = 3$, i.e. when G is an 7-antihole, there is a row $(1, 0, 1, 0, 1, 0, 0)$ which corresponds to the clique $\langle 0, 2, 4 \rangle$). Such A is a circulant matrix determined by the vector a given by

$$a(k) = \begin{cases} 1, & \text{if } k \text{ is even and } k \leq 2(m-1) \\ 0, & \text{otherwise.} \end{cases}$$

Claim. $\det(A) = m$ and so A is invertible.

Proof of the Claim. By (4.13) we know that each eigenvalue λ_j of A is of the form

$$\lambda_j = \sum_{k=0}^{m-1} \zeta_n^{-2kj},$$

for $n = 2m+1$. Note that $\lambda_0 = m$ and for $1 \leq j \leq 2m$ we have

$$\lambda_j = \frac{1 - \zeta_n^{-2mj}}{1 - \zeta_n^{-2j}}.$$

Note that

$$\zeta_n^{-2mj} = (e^{\frac{4m\pi i}{2m+1}})^{-j} = (e^{\frac{2\pi i(2m+1)-2\pi i}{2m+1}})^{-j} = e^{2\pi i} \cdot (e^{\frac{-2\pi i}{n}})^{-j} = \zeta_n^j,$$

and thus

$$\det(A) = \prod_{j=0}^{2m} \lambda_j = m \frac{\prod_{j=1}^{2m}(1-\zeta_n^j)}{\prod_{j=1}^{2m}(1-\zeta_n^{-2j})} = m \frac{\prod_{j=1}^{2m}(1-\zeta_n^j)}{\prod_{j=1}^{2m}(1-\zeta_n^{-j})(1+\zeta_n^{-j})} = \frac{m}{\prod_{j=1}^{2m}(1+\zeta_n^j)},$$

where the last equality follows from the fact that the set of nth roots of unity forms a group and, what follows $\{\zeta_n^{-j} : 1 \leq j \leq 2m\} = \{\zeta_n^j : 1 \leq j \leq 2m\}$. It is known that the product of roots of a polynomial $P(z) = \sum_{k=0}^N a_k z^k$ is equal to $(-1)^N \frac{a_0}{a_N}$. Since $1 + \zeta_n^j$ is a root of $C(z) = (z-1)^{2m+1} - 1$ for every $0 \leq j \leq 2m$, their product is equal $(-1)^{2m+1} \cdot (-2) = 2$. Hence

$$\frac{1}{\prod_{j=1}^{2m}(1+\zeta_n^j)} = \frac{1+\zeta_n^0}{2} = 1,$$

and thus $\det(A) = m$. □

Thus z needs to be 0, which is a contradiction.
Now let $e \in \mathrm{Ext}(X_G)$. Then for every maximal clique C we have

$$\sum_{v \in C} e(v) = 1. \tag{4.14}$$

Indeed, if for some i $\sum_{v \in C_i} e(v) < 1$, then one can find a nonzero vector $z = A^{-1}u$, where u is such that $u(i) = 1 - \sum_{v \in C_i} e(v) \neq 0$. This z is a witness for $\|e \pm z|_G \leq 1$. Since x that is constantly equal to $\frac{1}{m}$ satisfies (4.14) and A defines a bijective linear operator, we conclude that this is the only vector satisfying

$$Ax = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

It finishes the proof. □

Remark 4.3.10. One can deduce from Propositions 4.3.7 and 4.3.9 that in both cases the set of extreme points can be described by a common formula. Namely, each coordinate of an extreme point $x \in X_G$ is, up to absolute value, equal to $\frac{1}{m}$, where m is the size of the maximal clique in G (equal to 2 when G is an odd hole, and equal to $\lfloor \frac{|V|}{2} \rfloor$ when G is an odd antihole). What is more, the proof of Proposition 4.3.7 can also be expressed in a similar manner as the one above. The appropriate circulant matrix, associated with an odd hole G, is determined by a vector $(1, 1, 0, 0, ..., 0)$.

Let us now consider the following situation. Suppose that to the set of vertices $V = \{v_0, v_1, ..., v_2m\}$ of an odd antihole ($m > 1$) we add an additional vertex $w \neq v_i$ for every $0 \leq i \leq 2m$. Then, for a fixed non-maximal clique $\langle i_0, i_1, ..., i_{j-1} \rangle \subseteq V$, we connect w to each of the vertices of this clique.

Proposition 4.3.11. *For a fixed m and $j < m$, let G be the graph defined as above. Then the vector x given by*

$$x(v_i) = \frac{1}{m} \text{ for all } 0 \leq i \leq 2m, \quad x(w) = \frac{m-j}{m}$$

is an extreme point of X_G.

Proof. If x was not in $\mathrm{Ext}(X_G)$, then find a vector $z \neq 0$ for which $\|x \pm z\|_G \leq 1$. Then we have one equation and one variable more than in the proof of Proposition 4.3.9. More precisely, we obtain $0-1$ square matrix B of dimension $2m+2$, which is of the form

$$\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline u & 1 \end{array}\right], \tag{4.15}$$

where A is the matrix from the proof of Proposition 4.3.9, $\mathbf{0}$ is a column vector of $2m+1$ zeros, and u is a row vector that corresponds to the clique $\langle i_0, i_1, ..., i_{j-1} \rangle$. It is easy to see that $\det(B) = \det(A) = m$. Thus, $z = 0$, which is a contradiction. □

Hence, the above proposition states that, given a $(2m+1)$-antihole $G = (V, E)$ and some $w \notin V$, the vector that is constantly equal to $\frac{1}{m}$ on V can be *extended* to an extreme point $\tilde{x} \in \mathbb{R}^{V \cup \{w\}}$ in such a way that $x(w)$ may take any value from $\{\frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}\}$, where the precise value depends on how w is connected to G.
Conversely, for any number $\frac{m}{n}$ with $0 < m < n$, consider a graph consisting of a $(2n+1)$-antihole and a vertex w adjacent to $n-m$ vertices of a fixed clique. Then the vector x, defined to be $\frac{1}{n}$ on the vertices of the antihole, and equal to $\frac{m}{n}$ on w, is an extreme point in the space induced by such a graph. Thus, we have the following corollary.

Corollary 4.3.12. *For every rational number $q \in (0, 1)$ there is a graph $G = (V, E)$ and $x \in \mathrm{Ext}(X_G)$ such that $q \in x[V]$.*

Extensions to certain non-perfect graphs

It is natural to ask, what the extreme points look like for spaces related to more complex, non-perfect graphs. In some trivial cases, the answers are straightforward. For example, if $G = (\{v_0, ..., v_{2k}\}, E)$ is an odd hole and we extend it to $\tilde{G} = (V \cup \{w\}, E \cup \{v_0, w\})$ for some vertex $w \neq v_i$ for every i and $v_0 \in V$, then it is easy to see that $x \in \mathbb{R}^{\tilde{G}}$ equal to $\frac{1}{2}$ is an extreme point in $Z_{\tilde{G}}$. It is also trivial that if we extend G by isolated vertices $w_0, ..., w_l$ for some $j \in \mathbb{N}$, then x defined by $x(v_i) = \frac{1}{2}$ and $x(w_j) = 1$ ($0 \leq i < 2m+1$, $0 \leq j \leq l$) is an extreme point in the space related to this extended graph.
These simple extensions led us to the interesting algorithm that allows us to produce extreme points in spaces Z_G generated by graphs containing an odd hole.
The procedure describing the graph G and defining an extreme point x is as follows.
Suppose that G contains an odd hole C_k for odd $k > 3$. Fix some enumeration $\{v_n : n \in M\}$ of the set of vertices, where M can be either equal to an initial segment of natural numbers, or equal to $\mathbb{N}$. The algorithm has two parts.

- First, let $D_0 = C_k$. For every vertex $v \in C_k$ put $x(v) = \frac{1}{2}$.
- Next, let

  $$k_0 = \min\{j \in M : \text{ there is } v \in D_0 \text{ such that } v_j \in G \setminus D_0 \text{ and } \{v_j, v\} \in E\}.$$

  If there is exactly one $v \in D_0$ such that $\{v_{k_0}, v\} \in E$, then put $x(v_{k_0}) = \frac{1}{2}$, otherwise let $x(v_{k_0}) = 0$. Put $D_1 = D_0 \cup \{v_{k_0}\}$.
- We proceed inductively, i.e., at step $n+1$ let

  $$k_n = \min\{j \in M : \text{ there is } v \in D_n \text{ such that } v_j \in G \setminus D_n \text{ and } \{v_j, v\} \in E\}.$$

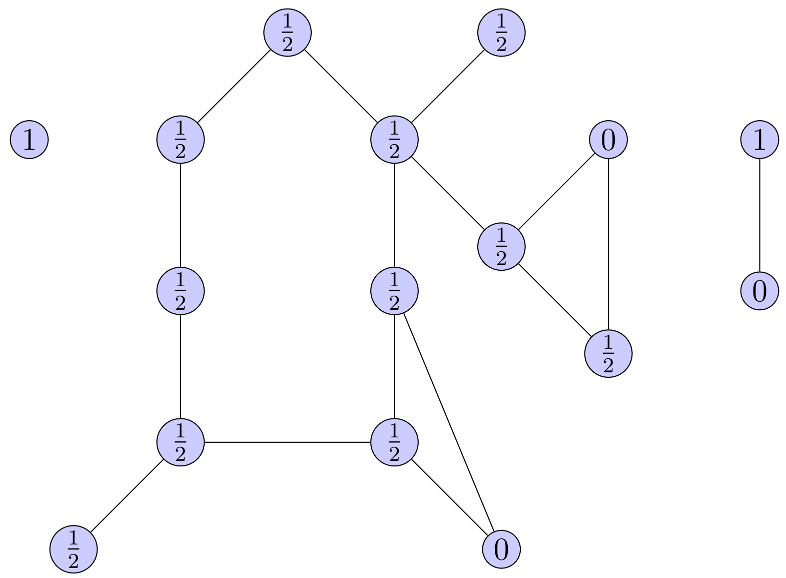
  We put $x(v_{k_n}) = \frac{1}{2}$, if there is exactly one $v \in D_n$ such that $\{v_{k_n}, v\} \in E$, otherwise we put $x(v_{k_n}) = 0$, and denote $D_{n+1} = D_n \cup \{v_{k_n}\}$.
- If there are no more such vertices, or we took infinitely many steps, we start the second part of the algorithm.
- Let $H_0 = \bigcup D_n$. Note that if $w \in G \setminus H_0$ then it is not adjacent to any $v \in V$ with $x(v) = \frac{1}{2}$. Let

  $$l_0 = \min\{j \in M : \text{ there is } v_j \in G \setminus H_0\}.$$

  We put $x(v_{l_0}) = 1$ and for every vertex u that is adjacent with v_{l_0} let $x(u) = 0$. Let $H_1 = H_0 \cup \{v_{l_0}\} \cup \{u \in V : \{v_{l_0}, u\} \in E\}$.

- We continue inductively, in a similar way as in the first part. At the step $n+1$, let l_n be the smallest natural number j for which $v_j \notin H_n$. We put $x(v_{l_n}) = 1$ and $x(u) = 0$ for every u with $\{v_{l_n}, u\} \in E$.

Below we can see an example of a graph G with values of $x \in \mathbb{R}^V$ (inside circles) obtained in the described procedure.



Proposition 4.3.13. $x \in \mathrm{Ext}(Z_G)$.

Proof. Suppose there is $z \neq 0$ such that $\|x \pm z\|_G \leq 1$. For $p \in \{0, \frac{1}{2}, 1\}$ let V_p denotes $x^{-1}[\{p\}]$. One can see that $\mathrm{supp}(z) \cap V_1 = \mathrm{supp}(z) \cap V_0 = \emptyset$. Indeed, every vertex v with $x(v) = 0$ is either adjacent to w for which $x(w) = 1$, or forms an 3-clique with $v_0, v_1 \in V_{\frac{1}{2}}$. It is easy to verify that in both cases $z(v) = 0$. Similarly, every v with $x(v) = 1$ is either adjacent to w, for which $x(w) = 0$ or v is isolated, and so $z(v)$ needs to be 0.
Thus $\mathrm{supp}(z) \subseteq V_{\frac{1}{2}}$. Let $V_{\frac{1}{2}}^0 = C_k$ and for every $n \geq 1$ put

$$V_{\frac{1}{2}}^n = \{v \in V_{\frac{1}{2}} : \text{ there exists } w \in V_{\frac{1}{2}}^{n-1} \text{ such that } \{v, w\} \in E\}.$$

For every $n \in \mathbb{N}$, fix a vertex v_n with the following properties:

- $v_n \in V_{\frac{1}{2}}^n$,
- $\{v_n, v_{n+1}\} \in E$.

Put $\varepsilon_n = z(v_n)$ for every n. Since $\{v_n, v_{n+1}\} \in E$, then we obtain $\varepsilon_{n+1} = -\varepsilon_n$ and, what follows, $\varepsilon_n = (-1)^n \varepsilon_0$ for every $n \geq 1$. Since $v_0 \in C_k$, then by Proposition 4.3.7 $\varepsilon_0 = 0$, and thus $\varepsilon_n = 0$ for every n. Since the *chain* of vertices (v_n) was chosen arbitrarily, it implies that $z = 0$ and it finishes the proof. □

Remark 4.3.14. By Proposition 4.3.13, we can informally say that, starting from an odd hole G and an extreme point x in Z_G, and following the described procedure, we can always extend G to any graph $\tilde{G}$ and x to some $\tilde{x}$ in such a way that $\tilde{x} \in \mathrm{Ext}(Z_{\tilde{G}})$. However, it is not clear whether a similar procedure exists when starting with an odd antihole. We are able to construct some simple extensions, but no clear pattern emerges. This may suggest that if such an algorithm exists, it is highly likely to be more complicated than the one described above.

Bibliography

[1] Fernando Albiac, José L. Ansorena, and Ben Wallis. “Garling sequence spaces”. *J. Lond. Math. Soc. (2)* 98.1 (2018), pp. 204–222.

[2] Fernando Albiac and Nigel J. Kalton. *Topics in Banach space theory*. Second. Vol. 233. Graduate Texts in Mathematics. Springer, [Cham], 2016.

[3] Dale E. Alspach and Spiros A. Argyros. “Complexity of weakly null sequences”. *Dissertationes Math. (Rozprawy Mat.)* 321 (1992), p. 44.

[4] Leandro Antunes and Kevin Beanland. “Surjective isometries on Banach sequence spaces: A survey”. *Concrete Operators* 9.1 (2022), pp. 19–40.

[5] Leandro Antunes, Kevin Beanland, and Hùng Viêt Chu. “On the geometry of higher order Schreier spaces”. *Illinois J. Math.* 65.1 (2021), pp. 47–69.

[6] Spiros A. Argyros, Sophocles K. Mercourakis, and Athanasios Tsarpalias. “Convex unconditionality and summability of weakly null sequences”. *Israel J. Math.* 107 (1998), pp. 157–193.

[7] Parviz Azimi and James N. Hagler. “Examples of hereditarily l^1 Banach spaces failing the Schur property”. *Pacific J. Math.* 122.2 (1986), pp. 287–297.

[8] Albert Baernstein II. “On reflexivity and summability”. *Studia Math.* 42 (1972), pp. 91–94.

[9] Stefan Banach. *Théorie des opérations linéaries*. Warszawa, 1932.

[10] Stefan Banach and Stanisław Saks. “Sur la convergence forte dans les champs L^p”. *Studia Mathematica* 2.1 (1930), pp. 51–57.

[11] Hyunsoo Bang and Edward W. Odell. “On the stopping time Banach space”. *Quart. J. Math. Oxford Ser. (2)* 40.159 (1989), pp. 257–273.

[12] Kevin Beanland, Noah Duncan, Michael Holt, and James Quigley. “Extreme Points for Combinatorial Banach Spaces”. *Glasgow Mathematical Journal* 61.2 (2019), pp. 487–500.

[13] Bernard Beauzamy. “Banach-Saks properties and spreading models.” *Mathematica Scandinavica* 44 (1979), pp. 357–384.

[14] Claude Berge. “Färbung von Graphen, deren sämtliche beziehungsweise deren ungerade Kreise starr sind”. *Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg, Mathematisch-naturwissenschaftliche Reihe* 10.1 (1961), pp. 114–115.

[15] Czesław Bessaga and Aleksander Pełczyński. “On bases and unconditional convergence of series in Banach spaces”. *Studia Math.* 17 (1958), pp. 151–164.

[16] Alistair Bird and Niels J. Laustsen. “An amalgamation of the Banach spaces associated with James and Schreier, Part I: Banach-space structure”. *Banach Center Publications* 91 (2010), pp. 45–76.

[17] Piotr Borodulin-Nadzieja and Barnabás Farkas. “Analytic P-ideals and Banach spaces”. *J. Funct. Anal.* 279.8 (2020), pp. 108702, 31.

[18] Piotr Borodulin-Nadzieja, Barnabás Farkas, Sebastian Jachimek, and Anna Pelczar-Barwacz. “The zoo of combinatorial Banach spaces”. *Studia Mathematica* 282 (2025).

[19] Piotr Borodulin-Nadzieja, Sebastian Jachimek, and Anna Pelczar-Barwacz. “On the spaces dual to combinatorial Banach spaces”. *Mathematische Nachrichten* 298 (2025), pp. 998–1017.

[20] Jean Bourgain. “ℓ^1 subspaces of Banach spaces”. Lecture Notes. Free University of Brussels.

[21] Christina Brech, Valentin Ferenczi, and Adi Tcaciuc. “Isometries of combinatorial Banach spaces”. *Proc. Amer. Math. Soc.* 148.11 (2020), pp. 4845–4854.

[22] Peter G. Casazza and Thaddeus J. Shura. *Tsirelson’s space*. Vol. 1363. Lecture Notes in Mathematics. With an appendix by J. Baker, O. Slotterbeck and R. Aron. Springer-Verlag, Berlin, 1989.

[23] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. “The strong perfect graph theorem”. *Ann. of Math. (2)* 164.1 (2006), pp. 51–229.

[24] Vašek Chvátal. “On certain polytopes associated with graphs”. *Journal of Combinatorial Theory, Series B* 18.2 (1975), pp. 138–154.

[25] Neil Dew. “Asymptotic structure of Banach spaces”. *Ph.D. thesis* University of Oxford (2002).

[26] Per H. Enflo. “A counterexample to the approximation problem in Banach spaces”. *Acta Math.* 130 (1973), pp. 309–317.

[27] Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos, and Václav Zizler. *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*. CMS Books in Mathematics. Springer New York, 2010.

[28] Ilijas Farah. “Analytic quotients: theory of liftings for quotients over analytic ideals on the integers”. *Mem. Amer. Math. Soc.* 148.702 (2000), pp. xvi+177.

[29] Tadeusz Figiel and William B. Johnson. “A uniformly convex Banach space which contains no l_p”. eng. *Compositio Mathematica* 29.2 (1974), pp. 179–190.

[30] Elói M. Galego, Manuel González, and Javier Pello. “On subprojectivity and superprojectivity of Banach spaces”. *Results Math.* 71.3-4 (2017), pp. 1191–1205.

[31] William T. Gowers. “Must an ’explicitly defined’ Banach space contain c_0 or ℓ_p?” *Gowers’s Weblog* (2009).

[32] William T. Gowers and Bernard Maurey. “The unconditional basic sequence problem”. *J. Amer. Math. Soc.* 6.4 (1993), pp. 851–874.

[33] Robert Gray. “Toeplitz and Circulant Matrices: A Review”. *Foundations and Trends® in Communications and Information Theory* 2 (2001).

[34] Robert C. James. “A non-reflexive Banach space isometric with its second conjugate space”. *Proc. Nat. Acad. Sci. U.S.A.* 37 (1951), pp. 174–177.

[35] Robert C. James. “Bases and reflexivity of Banach spaces”. *Ann. of Math. (2)* 52 (1950), pp. 518–527.

[36] Nigel J. Kalton. “Quasi-Banach spaces”. In: *Handbook of the geometry of Banach spaces, Vol. 2*. North-Holland, Amsterdam, 2003, pp. 1099–1130.

[37] Nigel J. Kalton, N. Tenney Peck, and James W. Roberts. *An F-space sampler*. Vol. 89. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1984, pp. xii+240.

[38] Joram Lindenstrauss and Aleksander Pełczyński. “Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications”. eng. *Studia Mathematica* 29.3 (1968), pp. 275–326.

[39] Joram Lindenstrauss and Lior Tzafriri. *Classical Banach spaces. I.* Vol. Band 92. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. Sequence spaces. Springer-Verlag, Berlin-New York, 1977.

[40] Jordi López-Abad and Stevo Todorcevic. “Positional graphs and conditional structure of weakly null sequences”. *Advances in Mathematics* 242 (2013), pp. 163–186.

[41] László Lovász. “Normal hypergraphs and the perfect graph conjecture”. *Discrete Mathematics* 2.3 (1972), pp. 253–267.

[42] Togo Nishiura and Daniel Waterman. “Reflexivity and summability”. *Studia Mathematica* 23.1 (1963), pp. 53–57.

[43] Edward W. Odell. “On quotients of Banach spaces having shrinking unconditional bases”. *Illinois J. Math.* 36.4 (1992), pp. 681–695.

[44] Diana Ojeda-Aristizabal. “A norm for Tsirelson’s Banach space”. *Extracta Math.* 28.2 (2013), pp. 235–245.

[45] Aleksander Pełczyński. “Projections in certain Banach spaces”. *Studia Mathematica* 19.2 (1960), pp. 209–228.

[46] Aleksander Pełczyński. “Universal bases”. *Studia Mathematica* 32.3 (1969), pp. 247–268.

[47] Mikhail M. Popov. “A hereditarily l_1 subspace of L_1 without the Schur property”. *Proc. Amer. Math. Soc.* 133.7 (2005), pp. 2023–2028.

[48] Haskell P. Rosenthal. “A characterization of Banach spaces containing l^1”. *Proc. Nat. Acad. Sci. U.S.A.* 71 (1974), pp. 2411–2413.

[49] Józef Schreier. “Ein Gegenbeispiel zur Theorie der schwachen Konvergenz”. *Studia Mathematica* 2.1 (1930), pp. 58–62.

[50] Thaddeus J. Shura and David A. Trautman. “The λ-property in Schreier’s space S and the Lorentz space $d(a, 1)$”. *Glasgow Mathematical Journal* 32.3 (1990), pp. 277–284.

[51] Wiesław Szlenk. “The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces”. *Studia Mathematica* 30.1 (1968), pp. 53–61.

[52] Marek Wójtowicz. “On the premutative equivalence of unconditional bases in F-spaces”. *Functiones et Approximatio* 16 (1988), pp. 51–54.