A LUSIN TYPE MEASURABILITY PROPERTY FOR VECTOR-VALUED FUNCTIONS

Kirill Naralenkov

MGIMO University,
Moscow, Russian Federation

Będlewo, Poland 2014
In a Banach space, there are two basic notions of function measurability — the notions of Bochner (or strong) measurability and scalar (or weak) measurability.

The Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued.

Why do we need another notion of function measurability to deal with Riemann type integration theories, such as those of McShane and Henstock, in a Banach space?

The above notions of function measurability diverge sharply for non-separable range spaces. Two classical examples illustrate some of the difficulties:
In a Banach space, there are two basic notions of function measurability — the notions of *Bochner* (or *strong*) measurability and *scalar* (or *weak*) measurability.

The Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued.

Why do we need another notion of function measurability to deal with Riemann type integration theories, such as those of McShane and Henstock, in a Banach space?

The above notions of function measurability diverge sharply for *non-separable* range spaces. Two classical examples illustrate some of the difficulties:
Introduction

- In a Banach space, there are two basic notions of function measurability — the notions of Bochner (or strong) measurability and scalar (or weak) measurability.

- The Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued.

- Why do we need another notion of function measurability to deal with Riemann type integration theories, such as those of McShane and Henstock, in a Banach space?

- The above notions of function measurability diverge sharply for non-separable range spaces. Two classical examples illustrate some of the difficulties:
In a Banach space, there are two basic notions of function measurability — the notions of Bochner (or strong) measurability and scalar (or weak) measurability.

The Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued.

Why do we need another notion of function measurability to deal with Riemann type integration theories, such as those of McShane and Henstock, in a Banach space?

The above notions of function measurability diverge sharply for non-separable range spaces. Two classical examples illustrate some of the difficulties:
Graves (1927) Define $\varphi : [0, 1] \to \ell^\infty[0, 1]$ by $\varphi(t) = \chi_{[t,1]}$ for each $t$ in $[0, 1]$. Then $\varphi$ is Riemann integrable but not Bochner measurable on $[0, 1]$.

Phillips (1940) Under the Continuum Hypothesis, there exists a bounded scalarly measurable function $\varphi : [0, 1] \to \ell^\infty[0, 1]$ such that Pettis’ theory does not assign any integral to $\varphi$ on $[0, 1]$.

It is well-known that the McShane and Henstock integrals can be defined without the use of Lebesgue measure as well as of any notion of function measurability.

Which other integration theories are based on Riemann type sums?
Graves (1927) Define $\varphi : [0, 1] \rightarrow \ell^\infty[0, 1]$ by $\varphi(t) = \chi_{[t,1]}$ for each $t$ in $[0, 1]$. Then $\varphi$ is Riemann integrable but not Bochner measurable on $[0, 1]$.

Phillips (1940) Under the Continuum Hypothesis, there exists a bounded scalarly measurable function $\varphi : [0, 1] \rightarrow \ell^\infty[0, 1]$ such that Pettis’ theory does not assign any integral to $\varphi$ on $[0, 1]$.

It is well-known that the McShane and Henstock integrals can be defined without the use of Lebesgue measure as well as of any notion of function measurability.

Which other integration theories are based on Riemann type sums?
Graves (1927) Define $\varphi : [0, 1] \to \ell^\infty[0, 1]$ by $\varphi(t) = \chi_{[t,1]}$ for each $t$ in $[0, 1]$. Then $\varphi$ is Riemann integrable but not Bochner measurable on $[0, 1]$.

Phillips (1940) Under the Continuum Hypothesis, there exists a bounded scalarly measurable function $\varphi : [0, 1] \to \ell^\infty[0, 1]$ such that Pettis’ theory does not assign any integral to $\varphi$ on $[0, 1]$.

It is well-known that the McShane and Henstock integrals can be defined without the use of Lebesgue measure as well as of any notion of function measurability.

Which other integration theories are based on Riemann type sums?
Graves (1927) Define $\varphi : [0, 1] \to \ell^\infty[0, 1]$ by $\varphi(t) = \chi_{[t,1]}$ for each $t$ in $[0, 1]$. Then $\varphi$ is Riemann integrable but not Bochner measurable on $[0, 1]$.

Phillips (1940) Under the Continuum Hypothesis, there exists a bounded scalarly measurable function $\varphi : [0, 1] \to \ell^\infty[0, 1]$ such that Pettis’ theory does not assign any integral to $\varphi$ on $[0, 1]$.

It is well-known that the McShane and Henstock integrals can be defined without the use of Lebesgue measure as well as of any notion of function measurability.

Which other integration theories are based on Riemann type sums?
Kolmogorov (1930) (for real-valued functions); Birkhoff (1935) (for vector-valued functions): A function $f$ from $[a, b]$ into a real Banach space $X$ is said to be \textit{(Birkhoff) integrable} on $[a, b]$ to a vector $w \in X$ if for each $\varepsilon > 0$ there exists a partition of $[a, b]$ into Lebesgue measurable sets $\{E_n\}$ such that the series $\sum_n f(t_n) \lambda(E_n)$ ($\lambda$ denotes Lebesgue measure) is unconditionally summable for all $t_n$ in $E_n$ and

$$\left\| \sum_n f(t_n) \lambda(E_n) - w \right\| < \varepsilon.$$

The Kolmogorov-Birkhoff theory of integration is not as simple and as useful as the Riemann type integration theories: the above definition uses Lebesgue measurable partitions as well as the notion of unconditional convergence of an infinite series of elements in a Banach space.
Kolmogorov (1930) (for real-valued functions); Birkhoff (1935) (for vector-valued functions): A function $f$ from $[a, b]$ into a real Banach space $X$ is said to be (Birkhoff) integrable on $[a, b]$ to a vector $w \in X$ if for each $\varepsilon > 0$ there exists a partition of $[a, b]$ into Lebesgue measurable sets $\{E_n\}$ such that the series $\sum_n f(t_n) \lambda(E_n)$ ($\lambda$ denotes Lebesgue measure) is unconditionally summable for all $t_n$ in $E_n$ and
\[
\left\| \sum_n f(t_n) \lambda(E_n) - w \right\| < \varepsilon.
\]

The Kolmogorov-Birkhoff theory of integration is not as simple and as useful as the Riemann type integration theories: the above definition uses Lebesgue measurable partitions as well as the notion of unconditional convergence of an infinite series of elements in a Banach space.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- Jeffery (1940) ‘measurable’ functions;
- Kunisawa (1943) ∗-measurable functions;
- Snow (1958) almost-Riemann-integrable functions;
- Cascales and Rodríguez (2005) the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- **Jeffery (1940)** ‘measurable’ functions;
- **Kunisawa (1943)** *-measurable functions;
- **Snow (1958)** almost-Riemann-integrable functions;
- **Cascales and Rodríguez (2005)** the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- Jeffery (1940) ‘measurable’ functions;
- Kunisawa (1943) \(*\)-measurable functions;
- Snow (1958) almost-Riemann-integrable functions;
- Cascales and Rodríguez (2005) the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- **Jeffery (1940)** ‘measurable’ functions;
- **Kunisawa (1943)** $\ast$-measurable functions;
- **Snow (1958)** almost-Riemann-integrable functions;
- **Cascales and Rodríguez (2005)** the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- Jeffery (1940) ‘measurable’ functions;
- Kunisawa (1943) \(*\)-measurable functions;
- Snow (1958) \(almost\)-Riemann-integrable functions;
- Cascales and Rodríguez (2005) the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
In connection with some of later investigations of the Kolmogorov-Birkhoff construction several classes of ‘measurable’ functions were defined that included the collection of Bochner measurable functions as a subclass:

- **Jeffery (1940)** ‘measurable’ functions;
- **Kunisawa (1943)** *-measurable functions;
- **Snow (1958)** almost-Riemann-integrable functions;
- **Cascales and Rodríguez (2005)** the Bourgain property.

These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.
Jeffery (1940) A bounded function $f : [a, b] \to X$ is restricted on $[a, b]$ if there exists $R > 0$ such that to each $\varepsilon > 0$ there corresponds a sequence of disjoint Lebesgue measurable sets $\{E_n\}$ with $\lambda(E_n) < \varepsilon$, $\sum_n \lambda(E_n) = b - a$, and

$$\left\| \sum_n \{f(t_n) - f(t'_n)\} \right\| \leq R,$$

where $t_n$ and $t'_n$ are any two points in $E_n$.

It is easily seen that all countably-valued Bochner measurable functions are restricted.

On the other hand, the function of Graves’ example is restricted but not Bochner measurable on $[0, 1]$. 

**Jeffery (1940)** A bounded function $f : [a, b] \rightarrow X$ is *restricted* on $[a, b]$ if there exists $R > 0$ such that to each $\varepsilon > 0$ there corresponds a sequence of disjoint Lebesgue measurable sets $\{E_n\}$ with $\lambda(E_n) < \varepsilon$, $\sum_n \lambda(E_n) = b - a$, and

$$\left\| \sum_n \{f(t_n) - f(t'_n)\} \right\| \leq R,$$

where $t_n$ and $t'_n$ are any two points in $E_n$.

It is easily seen that all countably-valued Bochner measurable functions are restricted.

On the other hand, the function of Graves’ example is restricted but not Bochner measurable on $[0, 1]$.
Jeffery (1940) A bounded function $f : [a, b] \to X$ is restricted on $[a, b]$ if there exists $R > 0$ such that to each $\varepsilon > 0$ there corresponds a sequence of disjoint Lebesgue measurable sets $\{E_n\}$ with $\lambda(E_n) < \varepsilon$, $\sum_n \lambda(E_n) = b - a$, and

$$\left\| \sum_n \{f(t_n) - f(t'_n)\} \right\| \leq R,$$

where $t_n$ and $t'_n$ are any two points in $E_n$.

It is easily seen that all countably-valued Bochner measurable functions are restricted.

On the other hand, the function of Graves’ example is restricted but not Bochner measurable on $[0, 1]$. 

Jeffery (1940) A function $f : [a, b] \rightarrow X$ is Jeffery-measurable on $[a, b]$ if there exists a sequence $\{f_n\}$ of restricted functions on $[a, b]$ such that $f_n \rightarrow f$ almost everywhere on $[a, b]$.

It is clear that all Bochner measurable functions are Jeffery-measurable.

Jeffery (1940) If $f : [a, b] \rightarrow X$ is both bounded and Jeffery-measurable $[a, b]$, then $f$ is Birkhoff integrable on $[a, b]$.

Which notion of function measurability is more relevant to the Riemann type integration theories than that of Jeffery-measurability?

Such a notion of function measurability should be formulated without the use of partitions into Lebesgue measurable sets or considering the relation of the function to any special function sequence.
Jeffery (1940) A function $f : [a, b] \rightarrow X$ is *Jeffery-measurable* on $[a, b]$ if there exists a sequence $\{f_n\}$ of restricted functions on $[a, b]$ such that $f_n \rightarrow f$ almost everywhere on $[a, b]$.

It is clear that all Bochner measurable functions are Jeffery-measurable.

Jeffery (1940) If $f : [a, b] \rightarrow X$ is both bounded and Jeffery-measurable $[a, b]$, then $f$ is Birkhoff integrable on $[a, b]$.

Which notion of function measurability is more relevant to the Riemann type integration theories than that of Jeffery-measurability?

Such a notion of function measurability should be formulated without the use of partitions into Lebesgue measurable sets or considering the relation of the function to any special function sequence.
Jeffery (1940) A function \( f : [a, b] \rightarrow X \) is *Jeffery-measurable* on \([a, b]\) if there exists a sequence \( \{f_n\} \) of restricted functions on \([a, b]\) such that \( f_n \rightarrow f \) almost everywhere on \([a, b]\).

It is clear that all Bochner measurable functions are Jeffery-measurable.

Jeffery (1940) If \( f : [a, b] \rightarrow X \) is both bounded and Jeffery-measurable \([a, b]\), then \( f \) is Birkhoff integrable on \([a, b]\).

Which notion of function measurability is more relevant to the Riemann type integration theories than that of Jeffery-measurability?

Such a notion of function measurability should be formulated without the use of partitions into Lebesgue measurable sets or considering the relation of the function to any special function sequence.
Jeffery (1940) A function $f : [a, b] \rightarrow X$ is *Jeffery-measurable* on $[a, b]$ if there exists a sequence $\{f_n\}$ of restricted functions on $[a, b]$ such that $f_n \rightarrow f$ almost everywhere on $[a, b]$.

It is clear that all Bochner measurable functions are Jeffery-measurable.

Jeffery (1940) If $f : [a, b] \rightarrow X$ is both bounded and Jeffery-measurable $[a, b]$, then $f$ is Birkhoff integrable on $[a, b]$.

Which notion of function measurability is more relevant to the Riemann type integration theories than that of Jeffery-measurability?

Such a notion of function measurability should be formulated without the use of partitions into Lebesgue measurable sets or considering the relation of the function to any special function sequence.
Introduction

- **Jeffery (1940)** A function \( f : [a, b] \to X \) is
  *Jeffery-measurable* on \([a, b]\) if there exists a sequence \( \{f_n\} \) of
  restricted functions on \([a, b]\) such that \( f_n \to f \) almost
  everywhere on \([a, b]\).

- It is clear that all Bochner measurable functions are
  Jeffery-measurable.

- **Jeffery (1940)** If \( f : [a, b] \to X \) is both bounded and
  Jeffery-measurable \([a, b]\), then \( f \) is Birkhoff integrable on \([a, b]\).

- Which notion of function measurability is more relevant to the
  Riemann type integration theories than that of
  Jeffery-measurability?

- Such a notion of function measurability should be formulated
  without the use of partitions into Lebesgue measurable sets or
  considering the relation of the function to any special function
  sequence.
[a, b] will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval.

A positive function defined on [a, b] will be called a gauge on [a, b].

A McShane partition of [a, b] is a finite collection $P = \{(I_k, t_k)\}_{k=1}^K$ of interval-point pairs such that $\{I_k\}_{k=1}^K$ is a collection of pairwise non-overlapping intervals, $t_k \in [a, b]$ for each $k$, and $\{I_k\}_{k=1}^K$ covers [a, b]. $P$ is subordinate to a gauge $\delta$ on [a, b] if $I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each $k$.

A Henstock partition of [a, b] is a McShane partition $P = \{(I_k, t_k)\}_{k=1}^K$ of [a, b] with $t_k \in I_k$ for each $k$. 
[a, b] will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval.

A positive function defined on [a, b] will be called a gauge on [a, b].

A McShane partition of [a, b] is a finite collection \( \mathcal{P} = \{(I_k, t_k)\}_{k=1}^K \) of interval-point pairs such that \( \{I_k\}_{k=1}^K \) is a collection of pairwise non-overlapping intervals, \( t_k \in [a, b] \) for each \( k \), and \( \{I_k\}_{k=1}^K \) covers [a, b]. \( \mathcal{P} \) is subordinate to a gauge \( \delta \) on [a, b] if \( I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k)) \) for each \( k \).

A Henstock partition of [a, b] is a McShane partition \( \mathcal{P} = \{(I_k, t_k)\}_{k=1}^K \) of [a, b] with \( t_k \in I_k \) for each \( k \).
[a, b] will denote a fixed nondegenerate interval of the real line and \( I \) its closed nondegenerate subinterval.

A positive function defined on \([a, b]\) will be called a gauge on \([a, b]\).

A **McShane partition** of \([a, b]\) is a finite collection \( \mathcal{P} = \{(I_k, t_k)\}_{k=1}^{K} \) of interval-point pairs such that \( \{I_k\}_{k=1}^{K} \) is a collection of pairwise non-overlapping intervals, \( t_k \in [a, b] \) for each \( k \), and \( \{I_k\}_{k=1}^{K} \) covers \([a, b]\). \( \mathcal{P} \) is subordinate to a gauge \( \delta \) on \([a, b]\) if \( I_k \subseteq (t_k - \delta(t_k), t_k + \delta(t_k)) \) for each \( k \).

A **Henstock partition** of \([a, b]\) is a McShane partition \( \mathcal{P} = \{(I_k, t_k)\}_{k=1}^{K} \) of \([a, b]\) with \( t_k \in I_k \) for each \( k \).
[a, b] will denote a fixed nondegenerate interval of the real line and $I$ its closed nondegenerate subinterval.

A positive function defined on $[a, b]$ will be called a gauge on $[a, b]$.

A McShane partition of $[a, b]$ is a finite collection $\mathcal{P} = \{(I_k, t_k)\}_{k=1}^K$ of interval-point pairs such that $\{I_k\}_{k=1}^K$ is a collection of pairwise non-overlapping intervals, $t_k \in [a, b]$ for each $k$, and $\{I_k\}_{k=1}^K$ covers $[a, b]$. $\mathcal{P}$ is subordinate to a gauge $\delta$ on $[a, b]$ if $I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each $k$.

A Henstock partition of $[a, b]$ is a McShane partition $\mathcal{P} = \{(I_k, t_k)\}_{k=1}^K$ of $[a, b]$ with $t_k \in I_k$ for each $k$. 
The McShane and Henstock integrals

Definition

A function \( f : [a, b] \rightarrow X \) is **McShane integrable** (Henstock integrable) on \([a, b]\), with **McShane integral** (Henstock integral) \( w \in X \), if for each \( \varepsilon > 0 \) there is a gauge \( \delta \) on \([a, b]\) such that

\[
\left\| \sum_{k=1}^{K} f(t_k) \lambda(I_k) - w \right\| < \varepsilon
\]

whenever \( \{(I_k, t_k)\}_{k=1}^{K} \) is a McShane partition (Henstock partition) of \([a, b]\) subordinate to \( \delta \).
The restricted versions

Definition

A function $f : [a, b] \to X$ is said to be $\mathcal{M}$-integrable ($\mathcal{H}$-integrable) on $[a, b]$ if it is McShane (Henstock) integrable on $[a, b]$ and for each $\varepsilon > 0$ there exists a Lebesgue measurable gauge $\delta$ on $[a, b]$ that corresponds to $\varepsilon$ in the definition of the McShane (Henstock) integral of $f$ on $[a, b]$.

- **Solodov (2005)** The $\mathcal{M}$-integral was first introduced for vector-valued functions.
- **Solodov (2005)** The $\mathcal{M}$-integral and the Kolmogorov-Birkhoff integral are equivalent.
The restricted versions

**Definition**

A function $f : [a, b] \rightarrow X$ is said to be $\mathcal{M}$-integrable ($\mathcal{H}$-integrable) on $[a, b]$ if it is McShane (Henstock) integrable on $[a, b]$ and for each $\varepsilon > 0$ there exists a Lebesgue measurable gauge $\delta$ on $[a, b]$ that corresponds to $\varepsilon$ in the definition of the McShane (Henstock) integral of $f$ on $[a, b]$.

- **Solodov (2005)** The $\mathcal{M}$-integral was first introduced for vector-valued functions.

- **Solodov (2005)** The $\mathcal{M}$-integral and the Kolmogorov-Birkhoff integral are equivalent.
The restricted versions

Definition

A function \( f : [a, b] \rightarrow X \) is said to be \( M \)-integrable (\( H \)-integrable) on \([a, b]\) if it is McShane (Henstock) integrable on \([a, b]\) and for each \( \varepsilon > 0 \) there exists a Lebesgue measurable gauge \( \delta \) on \([a, b]\) that corresponds to \( \varepsilon \) in the definition of the McShane (Henstock) integral of \( f \) on \([a, b]\).

- **Solodov (2005)** The \( M \)-integral was first introduced for vector-valued functions.

- **Solodov (2005)** The \( M \)-integral and the Kolmogorov-Birkhoff integral are equivalent.
A function $f : [a, b] \to X$ is said to be *Lusin measurable* on $[a, b]$ if for each $\varepsilon > 0$ there exists a closed set $F \subset [a, b]$ with $\lambda([a, b] \setminus F) < \varepsilon$ such that the function $f|_F$ is continuous.

- Lusin measurability is equivalent to Bochner measurability.
Definitions of measurability

Definition
A function \( f : [a, b] \rightarrow X \) is said to be Lusin measurable on \([a, b]\) if for each \( \varepsilon > 0 \) there exists a closed set \( F \subset [a, b] \) with \( \lambda([a, b] \setminus F) < \varepsilon \) such that the function \( f|_F \) is continuous.

- Lusin measurability is equivalent to Bochner measurability.
A function $f : [a, b] \to X$ is said to be Riemann measurable on $[a, b]$ if for each $\varepsilon > 0$ there exist a closed set $F \subset [a, b]$ with $\lambda([a, b] \setminus F) < \varepsilon$ and $\delta > 0$ such that

$$
\left\| \sum_{k=1}^{K} \{ f(t_k) - f(t'_k) \} \cdot \lambda(I_k) \right\| < \varepsilon
$$

whenever $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise non-overlapping intervals with $\max_k \lambda(I_k) < \delta$ and $t_k, t'_k$ are any two points in $I_k \cap F$.

- Lusin measurability implies Riemann measurability.
- Riemann integrability implies Riemann measurability.
Definitions of measurability

**Definition**

A function $f : [a, b] \rightarrow X$ is said to be *Riemann measurable* on $[a, b]$ if for each $\varepsilon > 0$ there exist a closed set $F \subset [a, b]$ with $\lambda([a, b] \setminus F) < \varepsilon$ and $\delta > 0$ such that

$$\left\| \sum_{k=1}^{K} \{f(t_k) - f(t'_k)\} \cdot \lambda(I_k) \right\| < \varepsilon$$

whenever $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise non-overlapping intervals with $\max_k \lambda(I_k) < \delta$ and $t_k, t'_k$ are any two points in $I_k \cap F$.

- Lusin measurability implies Riemann measurability.
- Riemann integrability implies Riemann measurability.
Definitions of measurability

**Definition**

A function \( f : [a, b] \to X \) is said to be *Riemann measurable* on \([a, b]\) if for each \( \varepsilon > 0 \) there exist a closed set \( F \subset [a, b] \) with \( \lambda([a, b] \setminus F) < \varepsilon \) and \( \delta > 0 \) such that

\[
\left\| \sum_{k=1}^{K} \left\{ f(t_k) - f(t'_k) \right\} \cdot \lambda(I_k) \right\| < \varepsilon
\]

whenever \( \{I_k\}_{k=1}^{K} \) is a finite collection of pairwise non-overlapping intervals with \( \max_k \lambda(I_k) < \delta \) and \( t_k, t'_k \) are any two points in \( I_k \cap F \).

- Lusin measurability implies Riemann measurability.
- Riemann integrability implies Riemann measurability.
The main results

**Theorem**

If \( f : [a, b] \rightarrow X \) is \( \mathcal{H} \)-integrable on \([a, b]\), then \( f \) is Riemann measurable on \([a, b]\).

**Theorem**

If \( f : [a, b] \rightarrow X \) is both bounded and Riemann measurable on \([a, b]\), then \( f \) is \( \mathcal{M} \)-integrable on \([a, b]\).

**Corollary**

If \( f : [a, b] \rightarrow X \) is Riemann measurable on \([a, b]\), then \( f \) is scalarly measurable on \([a, b]\).
The main results

**Theorem**

If $f : [a, b] \rightarrow X$ is $\mathcal{H}$-integrable on $[a, b]$, then $f$ is Riemann measurable on $[a, b]$.

**Theorem**

If $f : [a, b] \rightarrow X$ is both bounded and Riemann measurable on $[a, b]$, then $f$ is $\mathcal{M}$-integrable on $[a, b]$.

**Corollary**

If $f : [a, b] \rightarrow X$ is Riemann measurable on $[a, b]$, then $f$ is scalarly measurable on $[a, b]$.
The main results

Theorem

If \( f : [a, b] \rightarrow X \) is \( \mathcal{H} \)-integrable on \([a, b]\), then \( f \) is Riemann measurable on \([a, b]\).

Theorem

If \( f : [a, b] \rightarrow X \) is both bounded and Riemann measurable on \([a, b]\), then \( f \) is \( \mathcal{M} \)-integrable on \([a, b]\).

Corollary

If \( f : [a, b] \rightarrow X \) is Riemann measurable on \([a, b]\), then \( f \) is scalarly measurable on \([a, b]\).
The main results

**Theorem**

If $f : [a, b] \to X$ is both McShane (Henstock) integrable and Riemann measurable on $[a, b]$, then $f$ is $M$-integrable ($\mathcal{H}$-integrable) on $[a, b]$.

**Corollary**

Let $f : [a, b] \to X$. Then $f$ is Kolmogorov-Birkhoff integrable on $[a, b]$ if and only if $f$ is both Pettis integrable and Riemann measurable on $[a, b]$.

**Theorem**

Let $f : [a, b] \to X$. Suppose that $X$ is separable. If $f$ is McShane (Henstock) integrable on $[a, b]$, then $f$ is $M$-integrable ($\mathcal{H}$-integrable) on $[a, b]$. 
The main results

**Theorem**

If $f : [a, b] \rightarrow X$ is both McShane (Henstock) integrable and Riemann measurable on $[a, b]$, then $f$ is $\mathcal{M}$-integrable ($\mathcal{H}$-integrable) on $[a, b]$.

**Corollary**

Let $f : [a, b] \rightarrow X$. Then $f$ is Kolmogorov-Birkhoff integrable on $[a, b]$ if and only if $f$ is both Pettis integrable and Riemann measurable on $[a, b]$.

**Theorem**

Let $f : [a, b] \rightarrow X$. Suppose that $X$ is separable. If $f$ is McShane (Henstock) integrable on $[a, b]$, then $f$ is $\mathcal{M}$-integrable ($\mathcal{H}$-integrable) on $[a, b]$. 
The main results

**Theorem**

If \( f : [a, b] \to X \) is both McShane (Henstock) integrable and Riemann measurable on \([a, b]\), then \( f \) is \( \mathcal{M} \)-integrable (\( \mathcal{H} \)-integrable) on \([a, b]\).

**Corollary**

Let \( f : [a, b] \to X \). Then \( f \) is Kolmogorov-Birkhoff integrable on \([a, b]\) if and only if \( f \) is both Pettis integrable and Riemann measurable on \([a, b]\).

**Theorem**

Let \( f : [a, b] \to X \). Suppose that \( X \) is separable. If \( f \) is McShane (Henstock) integrable on \([a, b]\), then \( f \) is \( \mathcal{M} \)-integrable (\( \mathcal{H} \)-integrable) on \([a, b]\).
Concluding remarks

How wide the Riemann measurable function class for a non-separable range space may be?

Let \( f : [a, b] \rightarrow X \) be bounded on \([a, b]\). Then each of the following statements about \( f \) implies all the others.

(i) \( f \) is \( M \)-integrable on \([a, b]\).
(ii) \( f \) is Riemann measurable on \([a, b]\).
(iii) \( f \) is Jeffery-measurable on \([a, b]\).
(iv) \( f \) is \( * \)-measurable on \([a, b]\).
(v) \( f \) is almost-Riemann-integrable on \([a, b]\).
(vi) \( Z_f = \{ x^* f : x^* \in X^*, \|x^*\| \leq 1 \} \) has the Bourgain property.

Fremlin (2007) There exists a bounded function \( \varphi : [0, 1] \rightarrow \ell^\infty(c) \) that is McShane integrable but not Birkhoff integrable on \([0, 1]\). As a result, \( \varphi \) is neither \( M \)-integrable nor Riemann measurable on \([0, 1]\).
How wide the Riemann measurable function class for a non-separable range space may be?

Let \( f : [a, b] \to X \) be bounded on \([a, b]\). Then each of the following statements about \( f \) implies all the others.

(i) \( f \) is \( \mathcal{M} \)-integrable on \([a, b]\).
(ii) \( f \) is Riemann measurable on \([a, b]\).
(iii) \( f \) is Jeffery-measurable on \([a, b]\).
(iv) \( f \) is \( \ast \)-measurable on \([a, b]\).
(v) \( f \) is almost-Riemann-integrable on \([a, b]\).
(vi) \( Z_f = \{x^* f : x^* \in X^*, \|x^*\| \leq 1\} \) has the Bourgain property.

Fremlin (2007) There exists a bounded function \( \varphi : [0, 1] \to \ell^\infty(c) \) that is McShane integrable but not Birkhoff integrable on \([0, 1]\). As a result, \( \varphi \) is neither \( \mathcal{M} \)-integrable nor Riemann measurable on \([0, 1]\).
How wide the Riemann measurable function class for a non-separable range space may be?

Let \( f : [a, b] \to X \) be \textit{bounded} on \([a, b]\). Then each of the following statements about \( f \) implies all the others.

(i) \( f \) is \( \mathcal{M} \)-integrable on \([a, b]\).
(ii) \( f \) is Riemann measurable on \([a, b]\).
(iii) \( f \) is Jeffery-measurable on \([a, b]\).
(iv) \( f \) is \( * \)-measurable on \([a, b]\).
(v) \( f \) is \textit{almost-Riemann-integrable} on \([a, b]\).
(vi) \( \mathcal{Z}_f = \{x^* f : x^* \in X^*, \|x^*\| \leq 1\} \) has the Bourgain property.

\textbf{Fremlin (2007)} There exists a bounded function \( \varphi : [0, 1] \to \ell^\infty(c) \) that is McShane integrable but not Birkhoff integrable on \([0, 1]\). As a result, \( \varphi \) is neither \( \mathcal{M} \)-integrable nor Riemann measurable on \([0, 1]\).
Fremlin (2007) Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Then each McShane integrable $X$-valued function is Birkhoff integrable. Consequently, if $f : [a, b] \to X$ is McShane integrable on $[a, b]$, then $f$ is Riemann measurable on $[a, b]$.

Fremlin and Mendoza (1994) There exists a bounded function $\varphi : [0, 1] \to \ell^\infty$ that is Pettis integrable but not McShane integrable. In particular, $\varphi$ is not Riemann measurable on $[0, 1]$.

Question: Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Is any Henstock integrable $X$-valued function necessarily Riemann measurable?
Concluding remarks

- **Fremlin (2007)** Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Then each McShane integrable $X$-valued function is Birkhoff integrable. Consequently, if $f : [a, b] \to X$ is McShane integrable on $[a, b]$, then $f$ is Riemann measurable on $[a, b]$.

- **Fremlin and Mendoza (1994)** There exists a bounded function $\varphi : [0, 1] \to \ell^\infty$ that is Pettis integrable but not McShane integrable. In particular, $\varphi$ is not Riemann measurable on $[0, 1]$.

- **Question:** Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Is any Henstock integrable $X$-valued function necessarily Riemann measurable?
Concluding remarks

- **Fremlin (2007)** Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Then each McShane integrable $X$-valued function is Birkhoff integrable. Consequently, if $f : [a, b] \to X$ is McShane integrable on $[a, b]$, then $f$ is Riemann measurable on $[a, b]$.

- **Fremlin and Mendoza (1994)** There exists a bounded function $\varphi : [0, 1] \to \ell^\infty$ that is Pettis integrable but not McShane integrable. In particular, $\varphi$ is not Riemann measurable on $[0, 1]$.

- **Question**: Suppose that $X$ is linearly isometric to a subspace of $\ell^\infty$. Is any Henstock integrable $X$-valued function necessarily Riemann measurable?
References


