A simple proof that the L^p-diameter of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite

Michał Marcinkowski

IMUWr

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Thus $l_1(\{f_t\})$ is the average of the lengths of all paths $f_t(x)$.

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Hölder inequality: $l_p(f) \geq C * l_1(f)$.

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• Eliashberg and Ratiu: L^p-diameter ($p \geq 1$) of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite if S is a surface with boundary. They show that the Calabi homomorphism is Lipschitz with respect to the L^p-norm.

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• Gambaudo and Lagrange: similar result for a huge class of quasimorphisms on $\mathrm{Diff}_0(S, \mathrm{area})$ if S is the closed disc. Their proof makes use of the braid group of the disc and inequalities relating the geometric intersection number of a braid and its word-length.

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• The last unsolved case was the sphere. Recently Brandenbursky and Shelukhin showed that in this case the diameter is as well infinite. They show as well that e.g. right angled Artin groups embed quiasi-isometrically (as well Kim-Koberda, Crisp-Wiest).

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Theorem 2 is new for hyperbolic surfaces. For disc, sphere and torus it was shown by Kim-Koberda and Brandenbursky-Shelukhin.

Warm-up: hyperbolic surface

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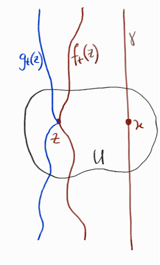

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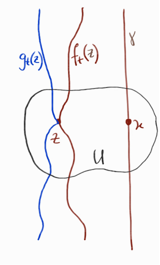

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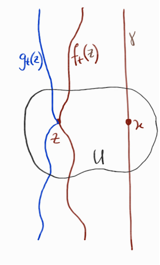

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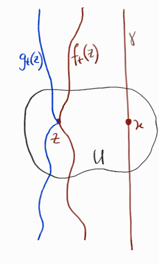

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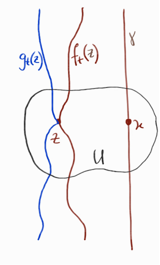

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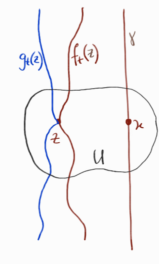

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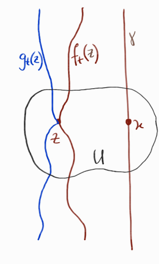

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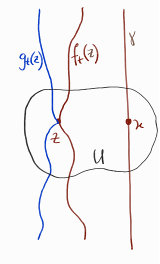

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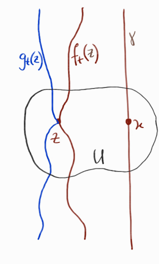

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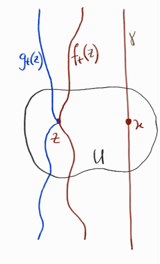

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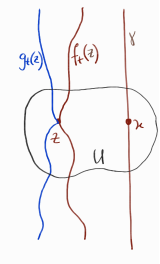

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For every $z \in U$, the trajectories $g_t(z)$ and $f_t(z)$ are homotopic, thus $g_t(z)$ has length at least L.
Hence $l_1(\{g_t\}) \geq \text{area}(U)L$ and $l_1(f_1) \geq \text{area}(U)L$.

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Problem: There exists C, such that every element of $P_n(S)$ can be realized as a loop of length $< C$.

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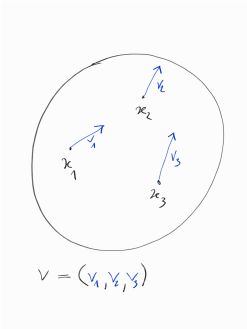

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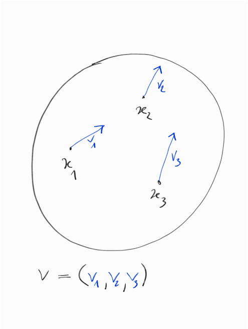

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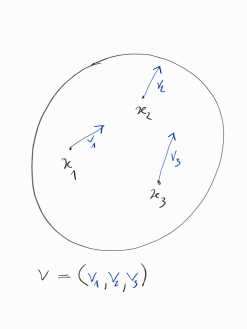

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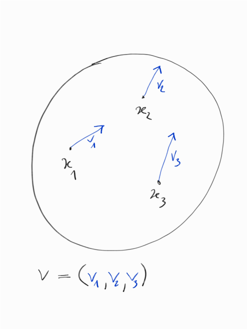

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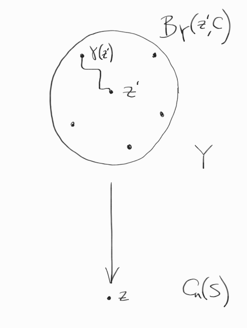

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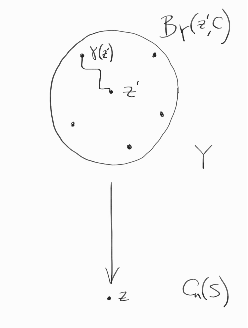

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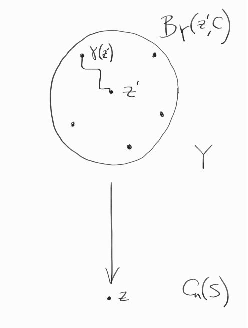

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For every C, the closed ball $B_Y(z', C)$ is compact.

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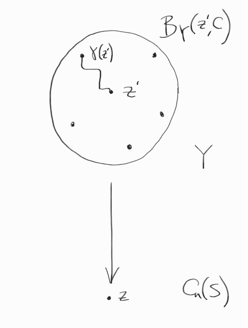

Since $C_n(S)$ is complete, so is Y.
Let $z' \in Y$ be in the pre-image of $z \in C_n(S)$.
For every C, the closed ball $B_Y(z', C)$ is compact.

Thus there is only finitely $\gamma \in \pi_1(C_n(S), z)$ such that $\gamma(z') \in B_Y(z', C)$.

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Proof: certain integral of $\frac{1}{d(x)}$ is finite.

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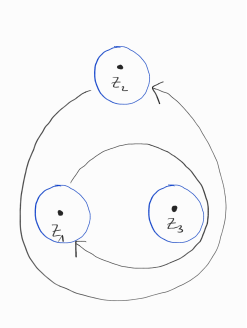

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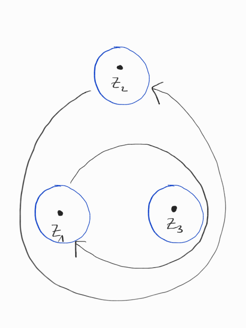

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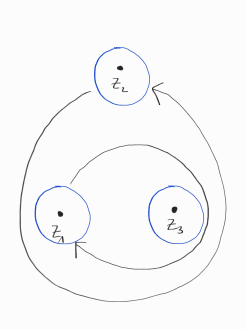

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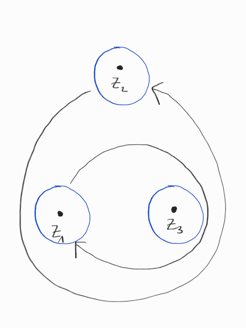

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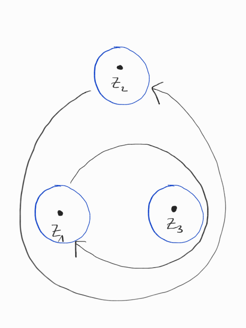

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$\{g_t\}$ was arbitrary, so $area(U) D_\gamma \leq l_1(f_1^*)$.

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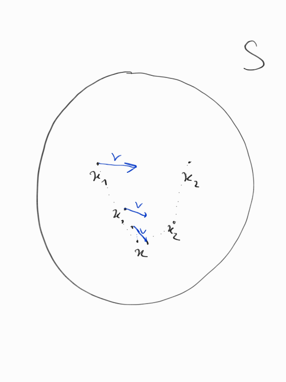

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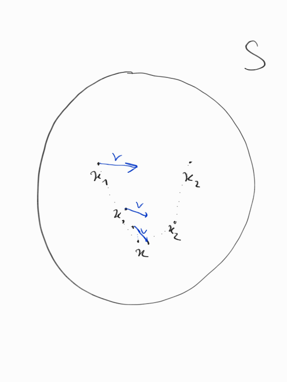

In general, there is a compactification $C_n(S) \subset \overline{C}_n(S)$ such that the inclusion is a homotopy equivalence. Moreover, $\overline{C}_n(S)$ is a manifold with corners (D. Sinha).

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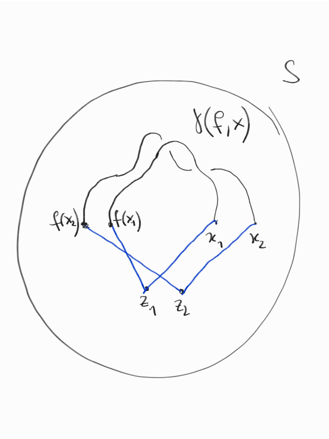

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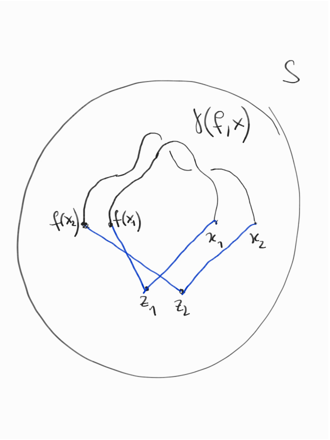

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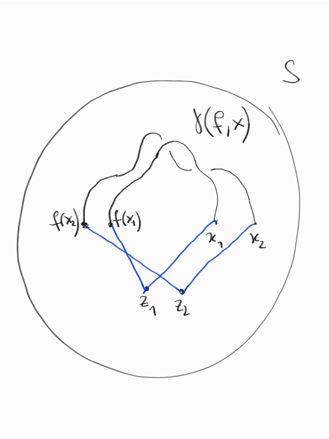

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Connect points in x with z and $f(x)$ with z. We get an element $\gamma(f, x) \in P_n(S)$.

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Corollary 2.

$$\int_{S^n} |\gamma(f,x)|_{P_n} dx \leq C|f|_1 + D.$$

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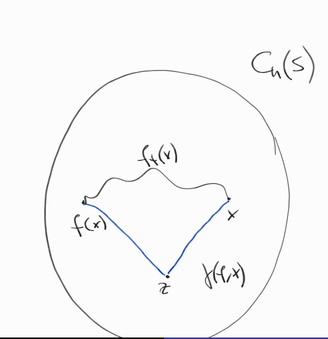

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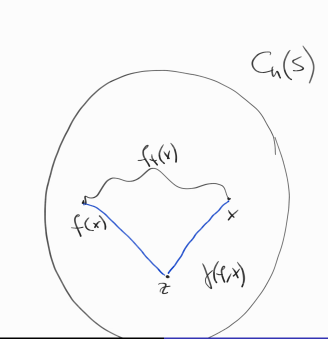

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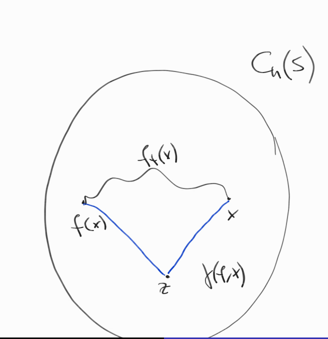

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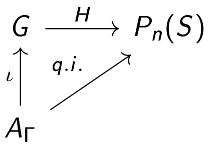

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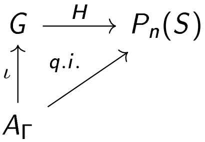

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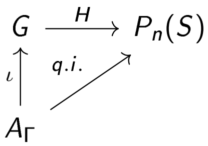

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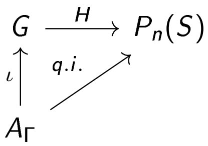

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Let A_Γ be a RAAG
(Kim-Koberda: Γ^c is a tree)

The End