Quasimorphisms, $\mathrm{Diff}_0(S, \text{area})$ and L^p-norm.

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joint work with M. Brandenbursky and E. Shelukhin

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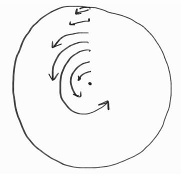
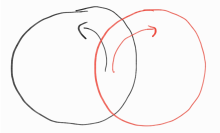
Isotopic do the identity means, that there exists a family $\{f_t\}$ of diffeomorphisms in $\mathrm{Diff}_0(S, \mathrm{area})$ such that $f_0 = Id$ and $f_1 = f$.

Exmaples

Pseudo-rotations on a disc: in the polar coordinates $f(\theta, r) = (\theta + \alpha(r), r)$, α is any function.





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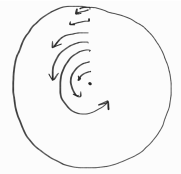
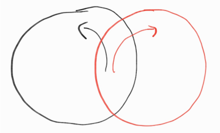
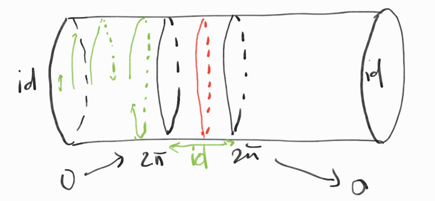




Point pushing maps along loops (or paths).



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In general we want to embed finitely generated subgroups in $\mathrm{Diff}_0(S, \mathrm{area})$ and we want to know what is the quality of this embedding.

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$$l_1(f) = \inf l_1(\{f_t\}),$$

where the infimum is taken over all isotopies $f_t \in \mathrm{Diff}_0(S, \mathrm{area})$ connecting the identity on S with f.

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Caveat: Let S_0 be a subsurface of S_1. It is an open question whether the natural inclusion $\mathrm{Diff}_0(S_0, \mathrm{area}) \to \mathrm{Diff}_0(S_1, \mathrm{area})$ is undistorted.

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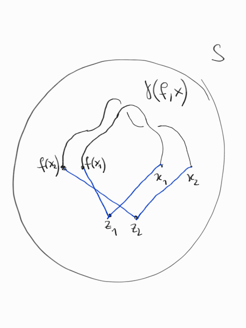

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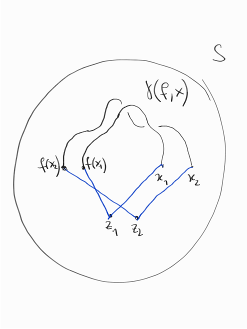

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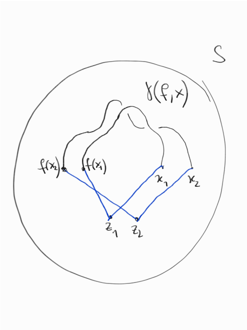

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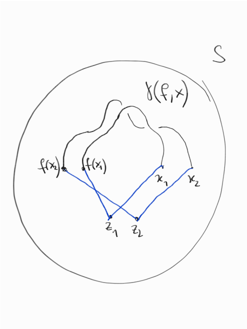

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(sometimes $\gamma(f)\colon C_n(S) \to P_n(S)/Z(P_n(S))$. Otherwise γ is not well defined.)

The image is finite. On $P_n(S)$ we look at the word norm.

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But there is many functions on $\mathrm{Diff}_0(S, \mathrm{area})$ that behave like homomorphisms.

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Let G be a group. We call $q\colon G \to \mathbb{R}$ a quasimorphism if there exists $D \in \mathbb{R}$ such that for all $a, b \in G$

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On finitely generated groups there is a lot of quasimorphisms. E.g., let $w \in F_n$,

$$q_w(x) = \#\{\ w \text{ is a subword of } x\} - \#\{\ w^{-1} \text{ is a subword of } x\}.$$

Remainder: $\gamma(f)\colon C_n(S) \to P_n(S)$.

Let $q\colon P_n(S) \to \mathbb{R}$ be a quasimorphisms.
Thus we have $q \circ \gamma(f)\colon C_n(S) \to \mathbb{R}$.

It induces a quasimorphism
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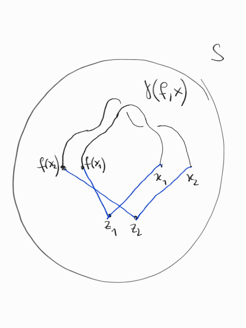

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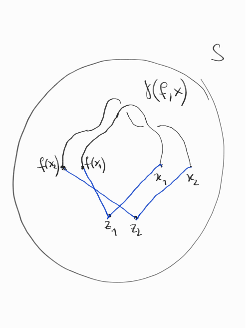

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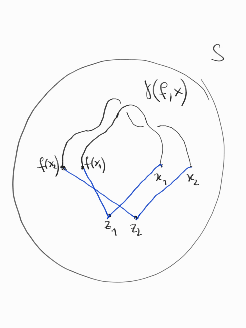

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E.g, if $n = 2$ and $S = D^2$, then $P_2(D^2) = \mathbb{Z}$. This construction gives us (after homogenisation) the Calabi homomorphism.

The lower bound

Theorem (Brandenbursky-M-Shelukhin)

Let S be a compact surface, $n \in \mathbb{N}$. There exist constants $A, B \in \mathbb{R}$ such that for every $f \in \mathrm{Diff}_0(S, \mathrm{area})$

$$\int_{C_n(S)} |\gamma(f, x)|_{P_n(S)} dx < A l_1(f) + B.$$

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Corollary

- *For every homogeneous GG_q quasimorphism: $GG_q(f) \leq A l_1(f)$.*
- *Every right angled Artin group can be embedded quasi-isometrically into $\mathrm{Diff}_0(S, \mathrm{area})$. E.g., $\mathbb{Z}^k$, F_k.*

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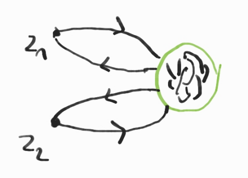

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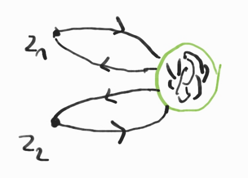

We need a different metric.

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But still we cannot compare $|\gamma|_{P_n(S)}$ to $l_{g_d}(\gamma)$.

Outline of the proof

$C_n(S)$ has an embedding to a high dimensional $\mathbb{R}^N$ (D. Sinha), such that the closure of the image $A_n(S)$ is a manifold with corners and such that $C_n(S)$ is the interior of $A_n(S)$ (so the π_1 does not change).

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$C_n(S)$ has an embedding to a high dimensional $\mathbb{R}^N$ (D. Sinha), such that the closure of the image $A_n(S)$ is a manifold with corners and such that $C_n(S)$ is the interior of $A_n(S)$ (so the π_1 does not change).

If we restrict the euc. metric from $\mathbb{R}^N$ to $A_n(S)$ (call it g_{comp}), then by Milnor-Schwartz we have $|\gamma(f,x)|_{P_n(S)} \sim l_{comp}(\gamma(f,x))$ and

$$\int_{C_n(S)} |\gamma(f,x)|_{P_n(S)} dx \sim \int_{C_n(S)} l_{comp}(\gamma(f,x)) dx$$

Outline of the proof

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It turns out that $g_{comp} \leq A' g_d$, thus

$$\int_{C_n(S)} l_{comp}(\gamma(f,x)) dx \leq A' \int_{C_n(S)} l_{g_d}(\gamma(f,x)) dx$$

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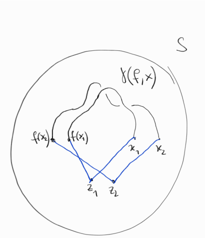
$$\leq A' \int_{C_n(S)} l_{g_d}(f_t^*(x))dx + B \leq AA' l_1(f) + B.$$

(we had: $\int_{C_n(S)} l_{g_d}(f_t^*(x))dx \leq A l_1(f)$)



Outline of the proof

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