Two dimensional non–commutative random vectors in terms of *APC* operators

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Abstract

First we introduce the joint annihilation, preservation, and creation operators (APC) of two random variables that are not necessarily commuting. We then define the notion of two dimensional non–degenerated Meixner random vectors and classify them up to an invertible linear transformation.

1 Commutative Background

$(\Omega,\ \mathcal{F},\ P)$ – probability space

$X_1,\ X_2,\ \ldots,\ X_d$ random var. on Ω.

$E[|X_i|^p] < \infty,\ \forall(p > 0,\ 1 \leq i \leq d)$.

For all $n \geq 0$, let:

$F_n := \{P(X_1, \ldots, X_d) \mid deg(P) \leq n\}$

$$\mathbf{C} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset L^2(\Omega, P)$$

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Let $G_0 := F_0$, and for all $n \geq 1$ let:

$G_n := F_n \ominus F_{n-1}$.

For all $1 \leq i \leq d$, let $X_i : F \to F$,

$P(X_1, \ldots, X_d) \mapsto X_i P(X_1, \ldots, X_d)$

Lemma: $\forall (1 \leq i \leq d \text{ and } n \geq 0)$,

$$X_i G_n \perp G_k, \qquad (1.1)$$

for all $k \neq n-1, n, n+1$.

Thus:

$$X_i G_n \subset G_{n-1} \oplus G_n \oplus G_{n+1}$$

If $f \in G_n$, then $\exists! f_{n-1} \in G_{n-1}$,

$f_n \in G_n$, and $f_{n+1} \in G_{n+1}$, s. t.:

$$X_i f = f_{n-1} + f_n + f_{n+1}.$$

We define:

$$D_n^-(i) f := f_{n-1},$$

$$D_n^0(i) f := f_n,$$

$$D_n^+(i) f := f_{n+1}$$

Observe that:

$$D_n^-(i) : G_n \to G_{n-1}$$

$D_n^-(i)$ – *annihilation operator*

$$D_n^0(i) : G_n \to G_n$$

$D_n^0(i)$ – *preservation operator*

$$D_n^+(i) : G_n \to G_{n+1}$$

$D_n^+(i)$ – *creation operator*

$$X_i|G_n = D_n^-(i) + D_n^0(i) + D_n^+(i)$$

If $F = \cup_{n\geq 0} F_n$, then for all $f \in F$,

there exist $f_0 \in G_0$, $f_1 \in G_1$, $\ldots$:

$$f \;=\; f_0 + f_1 + \cdots$$

We define:

$$a^-(i)f \;=\; D_0^-(i)f_0 + D_1^-(i)f_1 + \cdots$$

$$a^0(i)f \;=\; D_0^0(i)f_0 + D_1^0(i)f_1 + \cdots$$

$$a^+(i)f \;=\; D_0^+(i)f_0 + D_1^+(i)f_1 + \cdots$$

$$X_i \;=\; a^-(i) + a^0(i) + a^+(i).$$

If $d = 1$, we have only one r. v. X.

For all $n \geq 0$, $dim(G_n) \leq 1$.

For all $n \geq 0$, let $f_n(X)$ be the only polynomial random variable from G_n, having the leading coefficient 1.

There exist α_n and ω_n real numbers such that:

$$\begin{aligned} & X f_n(X) \\ = \; & f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X) \end{aligned}$$

$$X f_n = f_{n+1} + \alpha_n f_n + \omega_n f_{n-1}$$

$\{\alpha_n\}_{n\geq 0}$, $\{\omega_n\}_{n\geq 0}$ are called the *Szegö–Jacobi parameters* of X.

If $X_1, \ldots, X_d$ are random var., then:

$$X_i X_j = X_j X_i,$$

for all $1 \leq i, j \leq d$.

For all polynomial f, we have:

$$E[f(X_1, \ldots, X_d)] = \int_\Omega f(X_1, \ldots, X_d) dP$$
$$= \langle f(X_1, \ldots, X_d) 1, 1 \rangle.$$

2 Non–Commutative Background

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert sp. over $\mathbf{R}$.

Let $(H_c, \langle \cdot, \cdot \rangle)$ be its complexification.

Let $X_1, X_2, \ldots, X_d$ be d symmetric dens. def. linear operators on H.

Let $\mathcal{A}$ the complex algebra generated by $X_1, X_2, \ldots, X_d$.

We assume that there exists ϕ in H,

s.t. ϕ belongs to the domain of g,

for any $g \in \mathcal{A}$.

We fix ϕ and call it *vacuum vector*.

Definition 2.1 *We call any element g of $\mathcal{A}$, a* random variable. *For any g in $\mathcal{A}$, we define:*

$$E[g] := \langle g\phi, \phi \rangle \qquad (2.2)$$

and call the number $E[g]$ the expectation *of the random var. g. Finally, we call the pair $(\mathcal{A}, \phi)$ a* probability space supported by *H.*

Def. $(\mathcal{A}, \phi)$ and $(\mathcal{A}', \phi')$ prob. sp. supported by H and H'.

$X_1, X_2, \ldots, X_d$ operators from $\mathcal{A}$,

$X'_1, X'_2, \ldots, X'_d$ operators from $\mathcal{A}'$.

We say that $(X_1, X_2, \ldots, X_d)$ and

$(X'_1, X'_2, \ldots, X'_d)$ are *moment equal*

$(X_1, \ldots, X_d) \equiv (X'_1, \ldots, X'_d)$,

if for any pol. $p(x_1, x_2, \ldots, x_d)$ of

non–commutative variables, we have:

$$E\left[p\left(X_1, \ldots, X_d\right)\right] = E'\left[p\left(X'_1, \ldots, X'_d\right)\right].$$

We can do the same construction as before and get:

$$X_i = a^-(i) + a^0(i) + a^+(i).$$

The domain of X_i, $a^-(i)$, $a^0(i)$, and $a^+(i)$ is understood to be $\mathcal{A}\phi$.

If Y and Z are two operators, then we define their commutator $[Y, Z]$ as:

$$[Y, Z] := YZ - ZY.$$

The operators X_1, X_2, …, X_d commute among themselves if and only if the following three conditions hold, for any $i,\, j \in \{1,\, 2,\, \ldots,\, d\}$:

$$\left[a^-(i), a^-(j)\right] = 0$$

$$\left[a^-(i), a^0(j)\right] = \left[a^-(j), a^0(i)\right]$$

$$\left[a^-(i), a^+(j)\right] - \left[a^-(j), a^+(i)\right] = \left[a^0(j), a^0(i)\right]$$

$$[X_i, X_j] : G_n \to G_{n-2} \oplus G_{n-1} \oplus G_n \oplus G_{n+1} \oplus G_{n+2},$$

Definition 2.2 *We define the* number operator $\mathcal{N}$ *as the linear operator whose domain is* $\mathcal{A}\phi$*, defined by the formula:*

$$\mathcal{N}u = nu, \tag{2.3}$$

for all $n \geq 0$ *and all* $u \in G_n$.

3 Meixner random vectors of class $\mathcal{M}_L$

A *Meixner random variable* X,

with infinite support,

has the Szegö–Jacobi parameters:

$$\alpha_n = \alpha n + \alpha_0, \qquad (3.1)$$

$$\omega_n = \beta n^2 + (t - \beta)n, \qquad (3.2)$$

for all $n \geq 1$,

where α, β, and t are real numbers,

such that $\beta \geq 0$ and $t > 0$.

The Meixner r.v.,

with infinite support, are divided,

up to a re–scaling, into five sub–classes:

Gaussian,

Poisson,

negative binomial,

gamma,

two parameter hyperbolic secant r.v..

Since $\alpha_n = \alpha n + \alpha_0$, and

$a^0 f_n(X) = \alpha_n f_n(X)$, we have:

$$a^0 = \alpha\mathcal{N} + \alpha_0 I$$

Since $\omega_n = \beta n^2 + (t - \beta)n$, for all $n \geq 1$,

$$[a^-, a^+] = 2\beta\mathcal{N} + tI,$$

Also,

$$[a^-, a^0] = \alpha a^-.$$

If $\alpha \neq 0$, then:

$$\mathcal{N} = \frac{1}{\alpha} a^0 - \frac{\alpha_0}{\alpha} I.$$

Let $\mathcal{M}_L$ be the class of Meixner

random variables with $\alpha \neq 0$ or $\alpha = \beta = 0$.

Proposition 3.1 *The Meixner random variables of class $\mathcal{M}_L$, are exactly those random variables $X = a^- + a^0 + a^+$, having finite moments of all orders and infinite support, for which the vector space W spanned by a^-, a^0, a^+, and I, equipped with the commutator $[\cdot, \cdot]$, forms a Lie algebra, where I denotes the identity operator.*

Definition 3.2 *Let* $(\mathcal{A}, \phi)$ *be a non–commutative probability space and* X *and* Y *two random variables from* $\mathcal{A}$. *Let* $\{a_u^-, a_u^0, a_u^+\}_{u\in\{x,y\}}$ *be their joint (APC) decomposition. We say that the pair* (X, Y) *is a* Meixner random vector of class $\mathcal{M}_L$ *if the real vector space* W *spanned by the operators* $a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, a_y^+$, *and* I, *equipped with the bracket*

$[\cdot, \cdot]$ *given by the commutator, forms a Lie algebra.*

Definition 3.3 *Let* X *and* Y *be two random variables in a non–commutative probability space* $(\mathcal{A}, \phi)$ *supported by the Hilbert space* H. *We say that the random vector* (X, Y) *is* non–degenerated *if the vectors* $X\phi$, $Y\phi$, *and* ϕ *are linearly independent in* H.

Proposition 3.4 *If (X, Y) is a non–degenerated random vector, then the annihilation operators a_x^- and a_y^-, of X and Y, are linearly independent.*

Example 1. Let X and Y be two independent centered Meixner random variables of class $\mathcal{M}_L$ defined on the same probability space $(\Omega, \mathcal{F}, P)$.

$X := a_x^- + a_x^0 + a_x^+$

$Y := a_y^- + a_y^0 + a_y^+.$

Since X and Y are independent, we know from [1], that $[a_x^{\epsilon_1}, a_y^{\epsilon_2}] = 0$, for all $(\epsilon_1, \epsilon_2) \in \{-, 0, +\}^2$.

Moreover, one can see that:

$$
\begin{aligned}
\left[a_x^-, a_x^+\right] &\in \mathbf{R}I + \mathbf{R}a_x^0, \\
\left[a_y^-, a_y^+\right] &\in \mathbf{R}I + \mathbf{R}a_y^0, \\
\left[a_x^-, a_x^0\right] &\in \mathbf{R}a_x^-, \\
\left[a_x^0, a_x^+\right] &\in \mathbf{R}a_x^+, \\
\left[a_y^-, a_y^0\right] &\in \mathbf{R}a_y^-, \\
\left[a_y^0, a_y^+\right] &\in \mathbf{R}a_y^+.
\end{aligned}
$$

Hence $(W, [\cdot, \cdot])$ is a Lie algebra, where W is the real vector space spanned by I, a_x^-, a_x^0, a_x^+, a_y^-, a_y^0, and a_y^+.

Thus (X, Y) is a commutative Meixner random vector of class $\mathcal{M}_L$.

Example 2. Let T and Z be two independent centered Meixner r.v. of class $\mathcal{M}_L$, having the same numbers $\alpha = 1$ and $t = 1$, and

$\beta_T := (1/2)(cp + dr)$, and

$\beta_Z := (1/2)(js' + kv)$, where

$$cs' + dv \, = \, 0$$

and

$$jp + kr \, = \, 0.$$

That means

$$\begin{aligned}
\beta_T &= \frac{1}{2}(c,d)\cdot(p,r),\\
\beta_Z &= \frac{1}{2}(j,k)\cdot(s',v),\\
(c,d) &\perp (s',v),\\
(j,k) &\perp (p,r).
\end{aligned}$$

$T = a_t^- + a_t^0 + a_t^+$ and

$Z = a_z^- + a_z^0 + a_z^+$.

Let us consider the following symmetric operators:

$$X := a_t^- + (pa_t^0 + s'a_z^0) + a_t^+$$

and

$$Y := a_z^- + (ra_t^0 + va_z^0) + a_z^+.$$

Then (X, Y) is a non–commutative Meixner random vector of class $\mathcal{M}_L$.

We call X and Y *independent Meixner random variables with mixed preservation operators.*

Theorem 3.5 *If* (X, Y) *is a non–degenerated centered Meixner random vector, then there exists an invertible linear transformation* $S : \mathbf{R}^2 \to \mathbf{R}^2$*, such that the random vector* $(X', Y') := S(X, Y)$ *is equivalent (moment equal) to a random vector of two independent Meixner random variables with mixed* *preservation operators of class $\mathcal{M}_L$. Equivalently, the vector space spanned by the identity operator and the joint (APC) operators of X and Y is isomorphic, as a Lie algebra, to the vector space spanned by the identity operator and the joint (APC) operators of two independent Meixner random variables of class $\mathcal{M}_L$, with mixed preservation operators.*

In particular if X and Y commute, then X' and Y' are independent.

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