Remarks on the non-commutative Khintchine inequalities

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Plan

The non-commutative Khintchine inequalities are mainly due to Françoise Lust-Piquard [LP1986] (see also [LPP1992])
They play a very important rôle in the recent developments in Operator Space Theory and non-commutative L_p-spaces,
central tool to understand unconditional convergence in non-commutative L_p (e.g. martingale ineq. P.-Xu).
also (cf. Maurey) in connection with Grothendieck's Theorem
The operator space analogues of Grothendieck's Theorem also close connection with some form of Khintchine inequalities (cf. e.g. P. -Shlyakhtenko, Invent. 2002, Haagerup-Musat 2008).

Further motivation in
Random Matrix Theory and
Free Probability.

The Rademacher functions in classical L_p (i.e. i.i.d. ± 1-valued independent random variables) satisfy the same inequalities as the freely independent ones in non-commutative L_p for $p < \infty$.

Let (ε_k) be the coordinate functions on $\{-1,1\}^{\mathbb{N}}$ equipped with the uniform probability μ on $\{-1,1\}^{\mathbb{N}}$ (equivalently the Rademacher functions on $\Omega = [0,1]$).
Recall the classical Khintchine inequalities : For any $0 < p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for any sequence $x = (x_n)$ in ℓ_2 we have

$$A_p \left(\sum |x_n|^2\right)^{1/2} \leq \left(\int \left|\sum x_n \varepsilon_n\right|^p d\mu(\varepsilon)\right)^{1/p} \leq B_p \left(\sum |x_n|^2\right)^{1/2}. \tag{1}$$

Note $A_p = 1$ if $p \geq 2$ and $B_p = 1$ if $p \leq 2$.
Note that the best possible A_p, B_p are known (Szarek, Haagerup,...) e.g. $A_1 = 1/\sqrt{2}$.

Suppose now that $x = (x_n)$ is a sequence in $L_p(\Omega, \nu)$ then after integration (1) yields

$$A_p \left\| (\sum |x_n|^2)^{1/2} \right\|_p \leq \left(\int \left\| \sum x_n \varepsilon_n \right\|_p^p d\mu \right)^{1/p} \leq B_p \left\| (\sum |x_n|^2)^{1/2} \right\|_p \tag{2}$$

where $\|.\|_p$ is relative to $L_p(\Omega, \nu)$.

Consider now a sequence $x = (x_n)$ in $L_p(M, \tau)$ where (M, τ) is a non-commutative semi-finite measure space. The non-commutative Khintchine inequalities say, when $1 \le p \le 2$, that there is a constant $A'_p > 0$ independent of $x = (x_n)$ such that

$$A'_p |||x|||_p \le \left(\int \left\| \sum \varepsilon_n x_n \right\|_p^p d\mu \right)^{1/p} \tag{3}$$

where

$$|||x|||_p \overset{\text{def}}{=} \inf_{x_n = a_n + b_n} \left\{ \left\| \left(\sum a_n^* a_n \right)^{1/2} \right\|_p + \left\| \left(\sum b_n b_n^* \right)^{1/2} \right\|_p \right\}. \tag{4}$$

Note : converse easy, with constant 1 (by operator concavity of $t \mapsto t^{p/2}$)

Ref : LP for $1 < p < 2$, LP and P for $p = 1$. Recently (JFA 08) Haagerup and Musat showed that $A'_1 = 1/\sqrt{2}$ (best) if one uses the complex version of (ε_n), i.e. the coordinates on $\mathbf{T}^{\mathbb{N}}$.

In the case $2 \le p < \infty$, the non-com Khintchine ineq. are different : we set

$$|||x|||_p \overset{\text{def}}{=} \max\left\{ \left\| \left(\sum x_n^* x_n\right)^{1/2} \right\|_p, \left\| \left(\sum x_n x_n^*\right)^{1/2} \right\|_p \right\}.$$

Then LP proved (see also Buchholz Math.Ann. 2001) that

$$\left(\int \left\|\sum \varepsilon_n x_n\right\|_p^p d\mu\right)^{1/p} \le B'_p |||x|||_p \le C\sqrt{p}|||x|||_p.$$

and again converse holds with constant $= 1$
The cases $1 < p < 2$ and $2 < p' < \infty$ are dual to each other.
The case $p = 1$ is essentially the dual equivalent (by LPP) to the little non-commutative Grothendieck ineq. ($p' = \infty$)
The case $0 < p < 1$ is still OPEN
but we will report partial progress

We will concentrate on the case $1 \le p < q < 2$.
Let us call (K_p) the validity of the non-commutative Khintchine ineq. for a given fixed sequence (ξ_n) of non-commutative "random variables" in $L_p(\varphi)$
Our main result is the implication ("extrapolation")

$$(\xi_n) \text{ satisfies } (K_q) \Rightarrow (\xi_n) \text{ satisfies } (K_p)$$

$$(K_p) \quad \begin{cases} \exists \beta_p \text{ such that for any finite sequence} \\ x = (x_n) \text{ in } L_p(\tau) \text{ we have} \\ |||x|||_p \le \beta_p \left\| \sum \xi_n \otimes x_n \right\|_{L_p(\varphi \times \tau)} \\ \text{where } ||| \cdot |||_p \text{ is defined as in (5).} \end{cases}$$

Examples

- If $\forall \pm 1, \quad \mathrm{dist}(\pm\xi_n) = \mathrm{dist}(\xi_n)$ then (K_p) for all $1 \leq p < \infty$
- (ε_n), i.i.d. real or complex Gaussian, Hadamard lacunary (or Sidon sets), free unitary, free circular or semi-circular, spin systems (CAR), q-free Gaussian ($-1 \leq q \leq 1$) all satisfy (K_p) for all $1 \leq p < \infty$
- Harcharras (Studia 1999) gave for each integer $k > 1$ examples that satisfy $(K_{p'})\&(K_p)$ when $p' = 2k$ but do not satisfy it for $p' > 2k$.
- Similar examples for arbitrary (non integer) $k > 1$ are an open problem (operator space analogue of the $\Lambda(p)$ set problem solved by Bourgain)
- Our main result shows in particular that the Harcharras examples satisfy (K_1) (Note : this special case can also be obtained by the Haagerup-Musat argument)

When $p < 2$, let us say that a sequence (f_n) in classical L_p satisfies the classical Khintchine inequality (KI_p) if $\exists c_p$ such that for all finite scalar sequences (a_n)

$$(\sum |a_n|^2)^{1/2} \leq c_p \| \sum a_n f_n \|_p.$$

Assume (f_n) is orthonormal in L_2, or merely that

$$\| \sum a_n f_n \|_2 \leq (\sum |a_n|^2)^{1/2}.$$

Then it is easy and well known that if $p < q < 2$,

$$(f_n) \text{ satisfies } (KI_q) \Rightarrow (\xi_n) \text{ satisfies } (KI_p)$$

Indeed, let $S = \sum a_n f_n$. Recall $p < q < 2$
Let θ be such that $1/q = (1-\theta)/p + \theta/2$.
By Hölder

$$(\sum |a_j|^2)^{1/2} \le c_q \|S\|_q \le c_q \|S\|_p^{1-\theta} \|S\|_2^{\theta} = c_q \|S\|_p^{1-\theta} (\sum |a_j|^2)^{\theta/2},$$

and hence after a suitable division we obtain

$$(\sum |a_j|^2)^{1/2} \le (c_q)^{1/(1-\theta)} \|S\|_p$$

and hence KI_p holds with $c_p = (c_q)^{1/(1-\theta)}$.

The heart of this simple argument is that for any $q < 2$

$$L_q - \mathrm{span}\{f_n \mid n \geq 1\} = L_2 - \mathrm{span}\{f_n \mid n \geq 1\}$$

with equivalent norms

In sharp contrast, the analogue of this fails for operator spaces, i.e. completely isomorphically

Nevertheless, the above extrapolation argument works, but requires a more sophisticated version of Hölder's inequality, that (apparently) forces us to restrict ourselves to $p \geq 1$.

In case $p < 1$ we are able to extend LP's result in special case

$$\sum_{i,j} \varepsilon_{i,j} x_{i,j} e_{i,j} \in S_p$$

Corollary

Let $\lambda_{ij} \in \mathbb{C}$ be arbitrary complex scalars. The following are equivalent.

(i) *The matrix $[\varepsilon_{ij}\lambda_{ij}]$ belongs to S_p for almost all choices of signs $\varepsilon_{ij} = \pm 1$.*

(ii) *Same as (i) for all choices of signs.*

(iii) *There is a decomposition $\lambda_{ij} = a_{ij} + b_{ij}$ with*

$$\sum_i \left(\sum_j |a_{ij}|^2\right)^{p/2} < \infty \quad \text{and} \quad \sum_j \left(\sum_i |b_{ij}|^2\right)^{p/2} < \infty$$

Corollary

Let $0 < p < 1$*. Let* r *be such that* $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$*. Consider a Schur multiplier*

$$u_\varphi : \ [x_{ij}] \to [x_{ij}\varphi_{ij}]$$

where $\varphi_{ij} \in \mathbb{C}$*. The following are equivalent :*

(i) u_φ *is bounded from* S_2 *to* S_p*.*

(ii) φ *admits a decomposition as* $\varphi = \psi + \chi$ *with* $\sum_i \sup_j |\psi_{ij}|^r < \infty$ *and* $\sum_j \sup_i |\chi_{ij}|^r < \infty$*.*

(iii) *There is a sequence* $f_i \geq 0$ *with* $\sum f_i < \infty$ *such that* $|\varphi_{ij}| \leq f_i^{1/r} + f_j^{1/r}$*.*

In Studia 1999 Harcharras proved that there are subsets of $\{e^{int} \mid 0 \leq n \leq N\}$ with size $\approx N^{1/2}$ that satisfy (K_4) and hence (by duality) $(K_{4/3})$. In particular, this holds for the random subset. Our result shows they automatically satisfy (K_1).

Operator Space interpretation :

$$R_{N^{1/2}} + C_{N^{1/2}} \subset \ell_1^N \text{ completely isomorphic embedding}$$

Conjecture : This holds with size $\approx N$ instead of $N^{1/2}$ and the random subset should work, yielding

$$R_{\delta N} + C_{\delta N} \subset \ell_1^N \quad (\delta > 0)$$

Kashin decomposition ?

Conjectures (more explicitly) :

- There are subsets of $\{e^{int} \mid 0 \leq n \leq N\}$ with size $\approx N^{1-\varepsilon}$ that satisfy (K_1) with a fixed constant $c(\varepsilon)$ (indep. of N).
Note : Probably randomly chosen subsets do the job !
cannot be proportional by Szemeredi's theorem (presence of arithmetic progressions)

- There is an orthonormal basis

$$\xi_1, \cdots, \xi_N$$

in L_2^N (equipped with uniform proba) such that both

$$\{\xi_1, \cdots, \xi_{N/2}\} \quad \text{and} \quad \{\xi_{N/2+1}, \cdots, \xi_N\}$$

satisfy (K_1) with a constant C (indep. of N) in L_1^N.
Note : Maybe a randomly chosen orthogonal decomposition does the job !

Corollary

There is a constant $c \geq 1$ such that, for any N, the usual "basis" $\{e_{ij}\}$ of S_1^N ($N \times N$ trace-class) contains a c-unconditional subset of size $\geq N^{3/2}$.

Conjecture : Same for $N^{2-\varepsilon}$...

Sketch of proof :

Assume (ξ_n) orthonormal in (non-commutative) $L_2(\varphi)$, or merely that

$$\|\sum a_n \xi_n\|_2 \le (\sum |a_n|^2)^{1/2}.$$

This guarantees that

$$\|\sum \xi_n \otimes x_n\|_2 \le (\sum \|x_n\|_2^2)^{1/2},$$

for all (x_n) in $L_2(M,\tau)$. Recall we want to prove :

Theorem

Let $1 < q < 2$.
Then (ξ_n) satisfies $(K_q) \Rightarrow (\xi_n)$ satisfies (K_p) *for all* $1 \le p < q$.

$$(K_p) \quad \begin{cases} \exists \beta_p \text{ such that for any finite sequence} \\ x = (x_n) \text{ in } L_p(\tau) \text{ we have} \\ |||x|||_p \le \beta_p \left\| \sum \xi_n \otimes x_n \right\|_{L_p(\varphi \times \tau)} \end{cases}$$

where

$$|||x|||_p \overset{\text{def}}{=} \inf_{x_n = a_n + b_n} \left\{ \left\| \left(\sum a_n^* a_n \right)^{1/2} \right\|_p + \left\| \left(\sum b_n b_n^* \right)^{1/2} \right\|_p \right\}. \tag{5}$$

Sketch of proof :

$$S = \sum \xi_n \otimes x_n.$$

Let $\mathcal{D}$ be the collection of all "densities," i.e. all f in $L_1(\tau)_+$ with $\tau(f) = 1$. Fix p with $0 < p \leq q$. Then we denote for $x = (x_n)$

$$C_q(x) = \inf\left\{\left\|\sum \xi_n \otimes y_n\right\|_q\right\}$$

where $\|\cdot\|_q$ is the norm in $L_q(\varphi \otimes \tau)$ and the infimum runs over all sequences $y = (y_n)$ in $L_q(\tau)$ for which there is f in $\mathcal{D}$ such that

$$2x_n = f^{\frac{1}{p}-\frac{1}{q}} y_n + y_n f^{\frac{1}{p}-\frac{1}{q}}.$$

Again

$$C_q(x) = \inf_{f \in \mathcal{D}} \left\{ \left\| \sum \xi_n \otimes T(f^{\frac{1}{p}-\frac{1}{q}})^{-1} x_n \right\|_{L_q(\varphi \otimes \tau)} \right\}$$

where

$$T(g)y = (gy + yg)/2$$

Observe that on one hand

$$C_p(x) = \|S\|_p$$

and on the other

$$C_2(x) \simeq |||x|||_p$$

Variant of “Maurey’s extrapolation principle”
we emphasize that here p remains fixed while the index q ($p < q \leq 2$) in $C_q(x)$ is “interpolating" between the cases $q = p$ and $q = 2$.

Recall

$$C_p(x) = \|S\|_p \quad \text{and} \quad C_2(x) \simeq |||x|||_p$$

Another rather simple observation is (p<q<2)

$$(K_q) \Rightarrow |||x|||_p \leq C' C_q(x).$$

Main point

(Ingredients Junge-Parcet Duke 08, + P. Pacific 92)

$$C_q(x) \leq C''' C_p(x)^{1-\theta} C_2(x)^{\theta}$$

where $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. (Recall $p < q < 2$ so that $0 < \theta < 1$.)
The preceding points put all together yield

$$|||x|||_p \leq C' C''' C_p(x)^{1-\theta} (C'' |||x|||_p)^{\theta}$$

and hence

$$|||x|||_p \leq C'''' C_p(x) = C'''' \|S\|_p.$$

Can assume f diagonal with monotone coefficients λ_n then

$$T(f^{\frac{1}{p}-\frac{1}{q}})^{-1}x = \left[\frac{1}{\lambda_i^{\frac{1}{p}-\frac{1}{q}} + \lambda_j^{\frac{1}{p}-\frac{1}{q}}} x_{ij}\right]$$

main step uses lower+upper triangular (UT) decomposition

$$T(f^{\frac{1}{p}-\frac{1}{q}})^{-1}x = \left[1_{i<j}\frac{1}{\lambda_i^{\frac{1}{p}-\frac{1}{q}} + \lambda_j^{\frac{1}{p}-\frac{1}{q}}} x_{ij}\right] + \left[1_{i\geq j}\frac{1}{\lambda_i^{\frac{1}{p}-\frac{1}{q}} + \lambda_j^{\frac{1}{p}-\frac{1}{q}}} x_{ij}\right]$$

$$\text{and } S_q/UT_q = (S_p/UT_p, S_2/UT_2)_\theta$$

Thank you !