Spectral density of generalized Wishart matrices and free multiplicative convolution

Wojciech Młotkowski[1], Maciej A. Nowak[2], Karol A. Penson[3], Karol Życzkowski[2,4]

[1]*Institute of Mathematics, University of Wrocław,*
pl. Grunwaldzki 2/4, PL 50-284, Wrocław, Poland.
[2]*Marian Smoluchowski Institute of Physics, Jagiellonian University, ul. Reymonta 4, PL 30-059 Kraków, Poland*
[3] *Sorbonne Universités, Université Paris VI,*
Laboratoire de Physique de la Matière Condensée (LPTMC), CNRS UMR 7600,
t.13, 5ème ét. BC.121, 4 pl. Jussieu, F 75252 Paris Cedex 05, France
[4]*Center for Theoretical Physics, Polish Academy of Sciences,*
al. Lotników 32/46 PL 02-668 Warszawa, Poland
(Dated: July 19, 2014)

We investigate level density for several ensembles of positive random matrices of a Wishart–like structure, $W = XX^\dagger$, where X stands for a nonhermitian random matrix. In particular, making use of the Cauchy transform we study free multiplicative powers of the Marchenko-Pastur (MP) distribution, $\mathrm{MP}^{\boxtimes s}$, which for an integer s yield Fuss-Catalan distributions corresponding to a product of s independent square random matrices, $X = X_1 \cdots X_s$. Known formulae for the level densities are rederived in the case $s = 2$ and $s = 1/2$ and explicit distributions are obtained for $s = 3$ and $s = 1/3$. Moreover, the level density generated by a product of two rectangular random matrices $X = X_1 X_2$ is obtained and the generalized Bures distribution given by the free convolution of arcsine and MP distributions is derived. The technique proposed here allows us to obtain level densities for several related cases.

I. INTRODUCTION

Ensembles of nonhermitian random matrices are of considerable scientific interest [1] in view of their numerous applications in several fields of statistical and quantum physics [2]. On the other hand, any ensemble of nonhermitian matrices X allows us to write a positive, hermitian matrix of the *Wishart* form,

$$X \to W = \frac{XX^\dagger}{\mathrm{Tr}\, XX^\dagger} \; . \tag{1}$$

Normalization implies that the random matrix satisfies a fixed trace condition, $\mathrm{Tr}\, W = 1$, so it can be interpreted as a density matrix.

Ensembles of such random density matrices analyzed in [3] can be obtained by taking a random pure state on a bipartite system and performing partial trace over a single subsystem. In such a case of an isotropic, structureless ensemble of random pure states generated according to the unique, unitarily invariant measure, the asymptotic level density of the corresponding quantum states is described by the the Marchenko–Pastur (MP) distribution $P_{1,c}$ [4], with its parameter c determined by the ratio of the dimensions of the auxiliary and the principal quantum systems.

If the global unitary symmetry of the measure defining the ensemble of pure random states is broken, the partial trace yields *structured* ensembles of random density matrices. They can be constructed combining products of non-hermitian random Ginibre matrices and sums of random unitary matrices distributed according to the Haar measure. Investigation of these ensembles initiated in [5] was further developed by Jarosz [6, 7].

Random matrices described by the Wishart ensemble corresponding to the product of s Ginibre matrices, $X = G_1 G_2 \cdots G_s$, were found useful to describe

level density of mixed quantum states associated to a graph [8] and states obtained by projection onto the maximally entangled states of a multi–partite system [5]. Hence these distributions describe asymptotic statistics of the Schmidt coefficients characterizing entanglement of a random pure state [3].

As the moments of the level density $P_s(x)$ for such ensembles are known to be asymptotically described by the Fuss–Catalan numbers [9, 10],

$$C_s(n) = \frac{1}{sn+1}\binom{sn+n}{n}, \tag{2}$$

these distributions are called *Fuss-Catalan*. These distributions describe singular values of products of independent Ginibre matrices – see [12–14], but they are also known [15] to describe asymptotic distribution of singular values of the s–power of a single random Ginibre matrix G^s.

These distributions may be considered as a generalization of the Marchenko-Pastur distribution for square random matrices, $P_1(x)$, which corresponds to the case $s = 1$. The Fuss–Catalan distributions can be interpreted as the free multiplicative convolution product [9] of s copies of the MP distribution $P_1(x)$, written as $P_s(x) = [P_1(x)]^{\boxtimes s}$. Spectral distribution of $P_s(x)$ for a product of an arbitrary number of s random Ginibre matrices was analyzed by Burda et al. [11] also in the general case of rectangular matrices, see also [13, 16]. This distribution was expressed as a solution of a polynomial equation and it was conjectured that the finite size effects can be described by a simple multiplicative correction.

An explicit form of $P_2(x)$ was derived in [17] in context of construction of generalized coherent states from combinatorial sequences. An exact form of the Fuss-Catalan

distributions for any integer s was derived in [18] in terms of hypergeometric functions $_sF_{s-1}$. These results were extended in [19] in which the Mellin transform was used to derive analogous distributions for a rational values of the exponent $s = p/q$ in terms of special functions. Free multiplicative powers of the MP distributions were recently investigated by Haagerup and Möller [20], while generalized Fuss–Catalan distributions were also studied in [12, 21].

In this work we obtain some complementary results basing on the resolvent method, Cauchy transform and Green functions defined on the complex plane. Writing down the Voiculescu S–transform [22] for the multiplicative free convolution of the Marchenko-Pastur distribution $[P_1(x)]^{\boxtimes s}$ with integer s, we arrive at a Green function in terms of a polynomial of order $s + 1$. The inverse S transform can be analytically performed e.g. for $s = 1/2, 2$, and $s = 1/3, 3$. In the latter case we obtain an expression for the higher order Fuss–Catalan (FC) distribution $P_3(x)$ in terms of elementary functions. The power series expansions for the FC distributions were recently obtained in [16].

The same technique works also for other ensembles of random matrices defined by free convolution of the Arcsine distribution (AS) and the Fuss-Catalan distributions $P_s(x)$. In the case $s = 1$ one obtains the Bures distribution [23, 24], while higher values of s lead to its generalization referred as s–Bures distribution. It is worth to mention that these distributions belong to the broader class of *Raney distributions* studied in [18, 19].

This paper is organized as follows. In section II we review basic properties of the Cauchy transform and recall how the level density can be derived from the Green functions. As an exemplary application we discuss the Marchenko-Pastur distribution $P_{1,c}$ with an arbitrary rectangularity parameter c and the arcsine distributions, for which the Green function is given as a solution of a quadratic equation. Furthermore we discuss the generalized Fuss-Catalan distribution $P_{2,c}$ and the generalized Bures distribution, for which the Green function is given by a Cardano solution of a cubic equation. The third order generalized Fuss-Catalan distribution $P_{3,c}$ and the 2-Bures distribution is studied in subsequent subsection. In these cases the Green function is given by a Ferrari solutions of a quartic equation, which allows us to express corresponding level density in terms of elementary functions. Some technical details of the derivations are relegated to the Appendix.

II. CAUCHY FUNCTIONS AND LEVEL DENSITIES

To derive the level density corresponding to certain ensembles of random matrices, and more generally, to some free convolutions of the Marchenko-Pastur (MP) distribution, we will use the Voiculescu S-transform and the Cauchy functions.

Consider a square random matrix X of size N pertaining to the Ginibre ensemble of non-hermitian random matrices. The Wishart matrix $W = XX^\dagger$ is positive, and its level density is asymptotically, $N \to \infty$, described by the Marchenko–Pastur distributions [4], with the rectangularity parameter c set to unity,

$$P_1(x) = \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}}, \quad x \in [0, 4]. \tag{3}$$

Variable x denotes a suitably rescaled eigenvalue λ of W. If a random Wishart matrix is normalized according to the trace condition $\text{Tr} W = 1$, the rescaled variable reads $x = \lambda N$, which implies that the mean value $\langle x \rangle$ is set to unity. Thus the MP distribution describes asymptotically the level density of random quantum states generated with the measure induced by the Hilbert-Schmidt metric [5].

In order to analyze convolutions of the MP distribution it is convenient to use its Voiculescu S–transform [22] defined as a function of a complex variable w,

$$S_{MP}(w) = \frac{1}{1 + w}. \tag{4}$$

We are looking for the free multiplicative case for which the S–transform of the convolution is given by the product of the S–transforms. For instance, the Fuss–Catalan distribution P_s of an integer order s [9, 18], which corresponds to a product of s independent non-hermitian random matrices, $X = X_1 \cdots X_s$, can be written as a multiplicative free convolution of the Marchenko–Pastur distribution, $P_s(X) = [P_1(x)]^{\boxtimes s}$. Hence the corresponding S transform reads $S_{C_s}(w) = [S_{MP}(w)]^s$.

Assume now we are given an S–transform $S(w)$, which corresponds to an unknown probability measure at the real axis. To infer this measure and the spectral density $\rho(\lambda)$ we write the S–transform as $S(w) = \frac{1+w}{w}\chi(w)$, where

$$\frac{1}{\chi(w)} G\left(\frac{1}{\chi(w)}\right) - 1 = w. \tag{5}$$

To recover the resolvent, we put

$$\frac{1}{\chi(w)} = z, \tag{6}$$

what allows us to write an implicit solution of the Green function $G(z)$, known also as the *Cauchy* function in the mathematical literature,

$$G(z) \equiv \frac{1}{N} \left\langle \text{tr} \frac{1}{z \mathbf{1}_N - M} \right\rangle = \frac{1 + w(z)}{z}. \tag{7}$$

Here M represents a random matrix from the ensemble investigated. In other words, for any given S–transform $S(w)$ the corresponding Green function $G(z)$ defined on the complex plane is given as a solution of the following algebraic equation

$$z w(z) \, S\big(w(z)\big) = 1 + w(z). \tag{8}$$

Note that Green's function (7) acts as a generating function for the spectral moments $m_k = \frac{1}{N}\left\langle \mathrm{Tr} M^k \right\rangle = \int d\lambda \lambda^k \rho(\lambda)$, i.e. $G(z) = \sum_{k=0}^{\infty} m_k/z^{k+1}$, as seen by expanding the Green's function at $z = \infty$. Another useful function is the Voiculescu R-transform, defined as a generating function for the free cumulants κ_k, i.e. $R(z) = \sum_{k=1}^{\infty} \kappa_k z^{k-1}$. Both functions G and R are related by functional relations $R(G(z)) + 1/G(z) = z$ (or equivalently $G(R(z) + 1/z) = z$). Finally, R and S transforms can be also related. They form a pair of mutually invertible maps $z = yS(y)$ and $y = zR(z)$, provided $R(0) \neq 0$ [25].

In several cases equation (8) can be solved analytically with respect to w. For instance, this is the case for the Fuss–Catalan distribution, as $S(w)$ reads $(1+w)^{-s}$ and Eq.(8) yields a polynomial equation of order $s+1$. It can be solved analytically for $s = 2$ and $s = 3$.

Thus to obtain the spectral density we apply the Stieltjes inversion formula. One needs to analyze all solutions of Eq.(8) to extract the desired information. In the case $s = 2$ the corresponding polynomial has three solutions, one of which is real, the remaining pair is mutually complex-conjugated. On the basis of Sochocki-Plemelj formula, $\frac{1}{\lambda \pm i\epsilon} = \mathrm{P.V.}\frac{1}{\lambda} \mp i\pi\delta(\lambda)$, the negative imaginary part of the Green's function yields the spectral function

$$\rho(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0}\Im G(z)|_{z=\lambda+i\epsilon}. \qquad (9)$$

As analytical solutions of equations of order three and four contain square roots raised to power $1/3$ and $-1/3$, so a care has to be taken by evaluation of the imaginary part of a complex solution along the real axis - for more details see Appendix A.

We would like to mention, that the relevant spectral function can be as well recovered from the real part of the resolvent. In this case one uses the maximal entropy argument, yielding

$$\lim_{\epsilon\to 0}[G(\lambda + i\epsilon) + G(\lambda - i\epsilon)] = \frac{\partial V(\lambda)}{\partial \lambda}, \qquad (10)$$

where V is the random matrix potential defining the measure, i.e. $d\mu(M) = dM\exp(-N\mathrm{tr}V(M))$ – see Eynard [26].

On the basis of aforementioned Sochocki-Plemelj formula, resulting equation is a singular integral-differential equation. In the case of the spectral support localized on a single, finite interval, one can solve the equation e.g. by methods developed by Tricomi [27]. Interestingly, one can also view (10) as an equation for potential V, provided spectral density $\rho(\lambda)$ is known. Then the calculation of the Hilbert transform of the spectral density according to eq. (10) yields the derivative of the potential, which after integrating the derivative and using the rotational invariance allows to infer the form of $V(M)$. Above procedure, although well-defined, is complicated at the technical level. In particular, in the case of the spectral functions resulting from the solution of cubic or quartic algebraic equations, integration yields complicated expressions for $V(M)$, which in general are non-polynomial.

III. GENERALIZED WISHART MATRICES AND THEIR SPECTRAL DENSITIES

A. Quadratic equation

As a warm-up exercise we start recalling simple problems which correspond to a quadratic equation. Consider first the Green's function corresponding to the free binomial distribution, where $\rho(\lambda) = \frac{1}{2}(\delta(\lambda) + \delta(\lambda - 1))$. The Green's function reads therefore $G(z) = \frac{1}{2}(\frac{1}{z} + \frac{1}{z-1})$. Straightforward manipulations yield the R-transform and S-transform, given respectively by $R(z) = (z - 1 + \sqrt{z^2 + 1})/(2z)$ and $S(z) = 2(1+y)/(1+2y)$. Anticipating the results needed for the further part of this work, we consider not the free sum of two binomial distributions. Since R-transform is additive, we get $R_{AS}(z) = 2R(z) = (z - 1 + \sqrt{z^2 + 1})/z$. Then the corresponding S-transform reads $S_{AS}(z) = (z + 2)/(2 + 2z)$. Substituting it into Eq. (8) we get

$$wz(w + 2) = 2(1 + w)^2. \qquad (11)$$

Solving it with respect to w we obtain two conjugated solutions. Selecting the one with negative imaginary part and plugging it into Eq. (9) yields the *arcsine distribution*,

$$AS(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(2 - x)}}, \quad x \in [0, 2]. \qquad (12)$$

This distribution gives us the level density of the suitably normalized sum of a random unitary matrix U and its adjoint $U^{\dagger}$. It describes the ensemble of quantum states obtained by reduction of a coherent combination of maximally entangled states [5] and will be used here to construct other distributions.

Before passing to the cubic equation and more complicated cases, let us recall how to obtain in this way the general form of the Marchenko–Pastur distribution. It describes the asymptotic level density of random states $\rho = XX^{\dagger}/\mathrm{Tr}XX^{\dagger}$, where X is a rectangular complex Ginibre matrix of size $N \times M$. We choose the rectangularity parameter $c = M/N \leq 1$. The case $c > 1$ yields the same nonzero eigenvalues and additional $N - M$ zero eigenvalues. Let us then start with the corresponding S–transform, $S_c(w) = 1/(1 + cw)$, which reduces to (4) for $c = 1$. Plugging this expression into (8) leads to a quadratic equation

$$zw = (1 + w)(1 + cw). \qquad (13)$$

Its solution with respect to w with a negative imaginary part together with Eq. (9) allows one to obtain

$$P_{1,c}(x) = \frac{1}{2\pi xc}\sqrt{(x - x_-)(x_+ - x)}, \qquad (14)$$

where $x \in [x_-, x_+]$, with the edges of the support $x_\pm = 1 + c \pm 2\sqrt{c}$. In the case $c \to 0$, Marchenko-Pastur distribution reduces to $\rho(\lambda) = \delta(\lambda - 1)$.

B. Cubic equation and Cardano solutions

We are going to present here solutions of problems motivated by ensembles of random matrices, for which equation (8) becomes a cubic polynomial in $w = w(z)$.

1. Fuss–Catalan distribution of order two

To show the presented method in action we rederive the Fuss-Catalan distribution $P_2(x) = [P_1(x)]^{\boxtimes 2}$, which describes ensemble (1) with X being a product of two independent square Ginibre matrices. As a starting point we thus take the square of the S transform of MP distribution, $S_{FC_2}(w) = [S_{MP}(w)]^2 = (1+w)^{-2}$. Putting this form into (8) we get a cubic equation

$$wz = (1+w)^3. \tag{15}$$

Calculating the Green function (7) and making use of (9) one obtains the Fuss–Catalan distribution of order two,

$$P_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\left[\sqrt[3]{2}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}\right]}{x^{\frac{2}{3}}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{1}{3}}}, \tag{16}$$

where $x \in [0, 27/4]$. This result was first obtained in [17] in context of construction of generalized coherent states from combinatorial sequences, and later used in [8] to describe asymptotic level density of mixed quantum states related to certain graphs.

2. Generalized Fuss–Catalan distribution $P_{2,c}$

In a an analogous way we can treat the case of a product of two independent rectangular Ginibre matrices characterized by an rectangularity parameter $c = M/N$. The corresponding S–transform $S_{2,c} = 1/(1 + cw)^2$ leads to a modified equation of the third order,

$$wz = (1+w)(1+cw)^2. \tag{17}$$

Solving it with respect to w and computing the corresponding Green function (7) and its imaginary part one obtains a level density. A particular case of the generalized Fuss-Catalan distribution of order two obtained for $c = 1/2$ is shown in Fig. 1. This very case was very recently studied in [16], where an explicit density was provided.
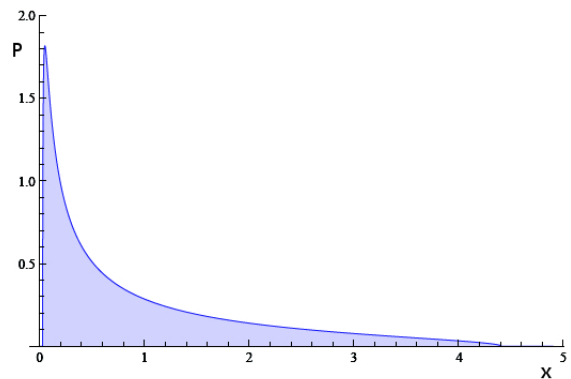

FIG. 1: Generalized Fuss-Catalan distribution of order two $P_{2,c}(x)$ plotted for rectangularity parameter $c = 1/2$.

3. Free–square root of the Marchenko–Pastur distribution

To derive this distribution we consider the square root of the S transform of the MP distribution, $S_{1/2}(w) = [S_{MP}(w)]^{1/2}$, which used in (8) yields a Cardano cubic equation,

$$w^3 + (3 - z^2)w^2 + 3w + 1 = 0. \tag{18}$$

Writing down the Green function (7) we use eq. (9) to get an explicit form of the free multiplicative square root of the Marchenko–Pastur distribution, $P_{1/2}(x) := [P_1(x)]^{\boxtimes 1/2}$,

$$P_{1/2}(x) = x^{-1/3}\frac{(9+Y)^{1/3} - (9-Y)^{1/3}}{2^{4/3}3^{1/6}\pi} + \\ + x^{1/3}\frac{(9+Y)^{2/3} - (9-Y)^{2/3}}{2^{4/3}3^{5/6}\pi}, \tag{19}$$

where $Y(x) = \sqrt{81 - 12x^2}$ and x belongs to $[0, \sqrt{27/4}]$.

This distribution was derived in [19] using the inverse Mellin transform and the Meijer G functions. We are not aware of any method to generate an ensemble of random matrices characterized asymptotically by the above level density.

4. Bures distribution

The Bures distribution describes the asymptotic level density of random mixed states distributed according to the measure [23] induced by the Bures metric [28]. To generate random states with respect to this measure it is sufficient [24] to take $X = (\mathbb{1} + U)G$, where U is a Haar random unitary matrix and G is a square random Ginibre matrix of the same size and substitute it into (1). This procedure follows from the fact that the Bures distribution can be represented as the multiplicative free product of the positive arcsine law and the Marchenko-Pastur law: $B_1 = AS \boxtimes MP$. The free S-transform of B_1 reads

$$S_{B_1}(w) = \frac{w+2}{2(w+1)^2} = \frac{w+2}{2w+2} \cdot \frac{1}{1+w}. \tag{20}$$

Observe that the first factor is the S-transform of AS while the second one, $1/(1+w)$, is the S-transform of MP, which implies the above law of free multiplication. The S transform (20) together with eq. (8) leads to an equation of order three, $wz(w+2) = 2(1+w)^3$, which can be explicitly solved with respect to the complex variable w. Making use of (7) and (9) one arrives at the Bures density

$$B_1(x) = C\left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1}\right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1}\right)^{2/3}\right]$$
(21)

where $C = 1/4\pi\sqrt{3}$ and $a = 3\sqrt{3}$. This distribution, first obtained in [23], is defined on a support larger than the standard MP distribution, $x \in [0, a]$ and it diverges for $x \to 0$ as $x^{-2/3}$.

5. Generalized Bures distributions

Generalized Bures distribution can be defined by a convolution of arcsine and the Marchenko–Pastur distribution with rectangularity parameter c, namely $B_{1,c} = AS \boxtimes P_{1,c}$. The corresponding ensemble of random matrices can be obtained writing $X = (1+U)G$ where U stands for a random unitary matrix of size N generated according to the Haar measure on $U(N)$, while G denotes a rectangular non-hermitian random Ginibre matrix of order $N \times K$ with $c = K/N$. Similar ensembles of random matrices were recently studied by Jarosz [7], while to get the corresponding ensemble of density matrices one may use superpositions of pure states of a four-party systems followed by projection on maximally entangled states and partial trace [5].

Multiplying the corresponding S–transforms we get $S_{B_{1,c}}(w) = (w+2)/(2(1+w)(1+cw))$ which leads to the following cubic equation $wz(w+2) = 2(1+cw)(1+w)^2$. In the special case $c = 1/2$ the above equation simplifies to the quadratic one, $wz = (1+w)^2$, corresponding to the Marchenko–Pastur distribution. The generalized Bures distribution $B_{1,c}(x)$ for $c \in [1/2, 1]$ can be thus interpreted as an interpolation between MP and Bures distributions. In the case $c \le 1$ this distribution is absolutely continuous. In the case $c > 1$, presented in Figs. 2 and 3, the distribution consists of a Dirac delta, $\delta(x)$ with weight $(1-1/c)$ and a continuous part – see Th. 4.1 in [29].

We shall conclude this section emphasizing that the method discussed here is not limited to the cases presented. For instance, analyzing the free multiplicative square root of the arcsine distribution, $AS^{\boxtimes 1/2}$, or its free square, $AS^{\boxtimes 2}$, one arrives at similar cubic equations, $(w+2)w^2z^2 = 2(w+1)^3$, and $(w+2)^2wz = 4(w+1)^3$, respectively, which allow one to derive corresponding level densities.
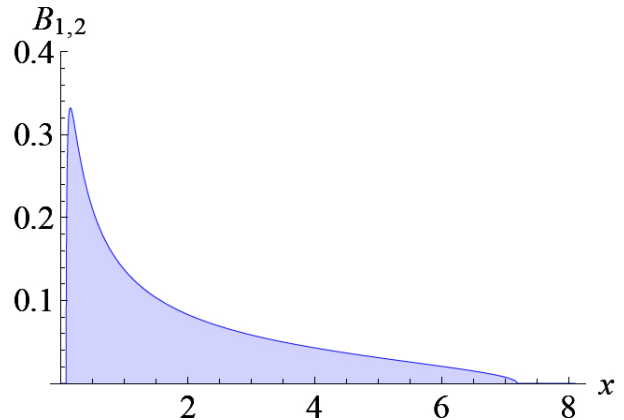

FIG. 2: Continuous part of the generalized Bures distribution $B_{1,c}(x)$ plotted for rectangularity parameter $c = 2$, so the shaded area equals $1/2$.
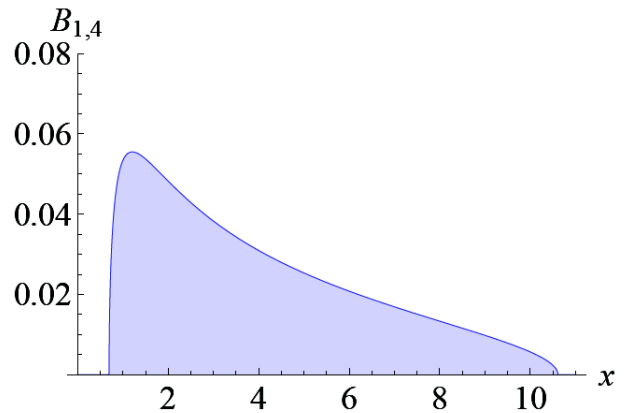

FIG. 3: As in Fig. 2 for $c = 4$ so the area under the curve is $1/4$.

C. Quartic equation and Ferrari solutions

The list of cases for which equation (8) forms a quartic equation contains for instance, the third order Fuss-Catalan distribution P_3, the third root of the Marchenko Pastur distribution, $P_{1/3}$, and the higher order Bures distribution.

1. Fuss–Catalan distribution of order three

To find an analytical expression for the Fuss–Catalan distribution, $P_3 = [P_1(x)]^{\boxtimes 3}$, describing the asymptotic level density of normalized Wishart matrix $XX^\dagger$, where X is a product of three independent Ginibre matrices, we start with the third power of the S–transform corresponding to the Marchenko Pastur distribution, $S_3(w) =$

$S_{MP}^3 = 1/(1 + w)^3$. Equation (8) leads then to the following quartic equation

$$w^4 + 4w^3 + 6w^2 + w(4 - z) + 1 = 0. \qquad (22)$$

Making use of the standard Ferrari formulae we obtain four explicit solutions of this equation given as square roots of expressions which contain polynomials of z in power $1/3$ and $-1/3$. Analyzing the imaginary part of the corresponding Green function (7), as discussed in the Appendix A, we arrive at an explicit expression for the Fuss-Catalan distribution of order three,

$$P_3(x) = \frac{x^{-3/4}}{2 \cdot 3^{1/4}\pi} \sqrt{4Y - \frac{3^{3/4}x^{1/4}}{\sqrt{Y}}}, \qquad (23)$$

where $Y(x) = \cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{16}\sqrt{x}\right)\right]$ and x belongs to $[0, 256/27]$. Interestingly, the same distribution, shown in Fig. 4 was derived earlier in [18] and expressed in terms of combinations of hypergeometric functions $_3F_2(x)$, which in this specific case admits an elementary representation.
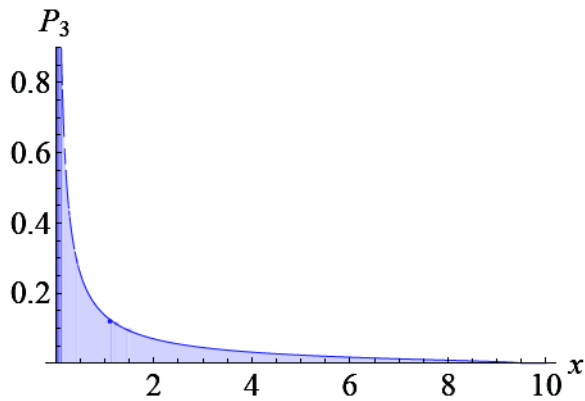

FIG. 4: Fuss-Catalan distribution $P_3(x) = [P_1(x)]^{\boxtimes 3}$ given in Eq. (23).

Note that in an analogous way it is also possible to obtain expressions for the generalized Fuss-Catalan distributions of order three, $P_{3,c}$ which correspond to the S–transform $S_{2,c} = 1/(1 + cw)^3$. This distribution, representing asymptotic level density of Wishart matrices obtained from a product of three independent rectangular Ginibre matrices with rectangularity parameter $c = N/K$, may in principle be further generalized for three different rectangularity parameters, so that the S–transform reads $S_{2,c} = 1/(1 + c_1w)(1 + c_2w)(1 + c_3w)$ – see also [16].

2. Free–third root of Marchenko–Pastur

Consider third root of the S transform corresponding to the MP distribution, $S_{1/3}(w) = [S_{MP}(w)]^{1/3}$. This

choice applied to (8) leads again to a quartic equation in terms of w,

$$w^4 + (4 - z^3)w^3 + 6w^2 + 4w + 1 = 0. \qquad (24)$$

Solving analytically this equation for w, evaluating the Green function (7) and applying (9) we arrive at the following form of the third free multiplicative root of the Marchenko–Pastur distribution, $P_{1/3}(x) := [P_1(x)]^{\boxtimes 1/3}$,

$$P_{1/3}(x) = \frac{1}{2\pi x}\left[Y + 4x^3 - \frac{1}{2}x^6 + \left|\frac{x^3(24 - 12x^3 + x^6)}{4\sqrt{Y - 2x^3 + \frac{1}{4}x^6}}\right|\right]^{1/2}, \qquad (25)$$

where $Y(x) = (4/\sqrt{3})x^{3/2}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{16}x^{3/2}\right)\right]$ and x belongs to $[0, (256/27)^{1/3}]$.

3. 2-Bures distributions

The higher order s-Bures distribution can be defined as a free convolution of arcsine and the s–Fuss–Catalan distribution, $B_s = AS \boxtimes P_s$. It describes asymptotic level density of Wishart matrices $XX^\dagger$ where $X = (\mathbb{1} + U)G_1 \cdots G_s$. Here U denotes a random unitary matrix distributed according to the Haar measure while $G_1, \ldots G_s$ are independent square complex Ginibre matrices. In the case $s = 1$ one obtains back the standard Bures ensemble [24]. Note that these distributions coincide with $\mu((s+2)/2, 1/2)$ from [10], up to dilation by 2. Indeed, the free S-transform of s-Bures is

$$S(w) = \frac{w + 2}{2(w + 1)^{s+1}}, \qquad (26)$$

which can be compared with (4.11) in [10] for $p = (s + 2)/2$ and $r = 1/2$.

Consider the case $s = 2$ for which the Cauchy function $S_{B_2}(w) = (w + 2)/(2(1 + w)^3)$ leads to the quartic equation

$$wz(w + 2) = 2(1 + w)^4.$$

Out of four analytical Ferrari solutions select the one, $w(z) = -1 + (z + i\sqrt{(z - 8)z})^{1/2}/2$ which can be rewritten as $w = -1 + \sqrt{8z}\exp[i\arccos(\sqrt{z/8})]$. Plugging this into (7) we get the Green function, which used in (9) yields the desired density, $B_{2,1}(x) = \sin[\frac{1}{2}\arccos(\sqrt{2x}/4)]/(2^{1/4}\pi x^{3/4})$. Making use of the known formula of the sine of the half angle, $\sin(x/2) = \sqrt{(1 - \cos(x))/2}$ we can get rid of arc cosine and arrive at the result

$$B_2(x) = \frac{1}{\pi \, 2^{5/4}x^{3/4}}\sqrt{2 - \sqrt{x/2}}, \qquad (27)$$

for $x \in [0, 8]$, see [19]. It is worth to add, that other recent representations of Fuss-Catalan, Raney and related

distributions [20, 30, 31], also contain sine functions, the argument of which is an inverse trigonometric function of the rescaled argument.

In a similar way one obtains results for the generalized 2–Bures distribution $B_{2,c}(x)$, corresponding to the product $X = (\mathbb{1} + U)G_1 G_2$ with rectangular matrices G_1 and G_2. For any rectangularity parameter $c = N/M$ the corresponding quartic equation reads now $wz(w + 2) = 2(1 + cw)(1 + w)^3$ and can be solved analytically. Corresponding level densities are too lengthy to reproduce them here. However, in the special case $c = 1/2$ this equation reduces to the case (15), so the generalized 2–Bures distribution with rectangularity parameter $c = 1/2$ coincides with the Fuss–Catalan distribution (16), $B_{2,1/2}(x) = P_2(x)$.

The list of other interesting cases, which lead to quartic equations includes, for instance, the free multiplicative convolution of arc-sine and Bures, $AS \boxtimes B = AS^{\boxtimes 2} \boxtimes MP$, or the free multiplicative square root of the Bures distribution, $B^{\boxtimes 1/2} = AS^{\boxtimes 1/2} \boxtimes MP^{\boxtimes 1/2}$. The corresponding level densities can be obtained by solving quartic equations $(w+2)^2 wz = 4(w+1)^4$ and $(w+2)w^2 z^2 = 2(w+1)^4$, respectively.

IV. CONCLUDING REMARKS

Making use of the S–transform and the Cauchy (Green) function it is possible to write down an explicit form of probability measures defined by free multiplicative convolution of Marchenko-Pastur (MP) distribution P_1 and other probability measures with known S-transform. For instance, multiplicative convolution of the Arcsine distribution and P_1 raised in the free multiplicative sense to an integer power leads to an algebraic equation for the argument of the S–transform. We studied some relevant cases for which this algebraic equation is of the third or fourth order, so basing on the known Cardano and Ferrari solutions one can derive analytically an explicit form of the required probability measures. This is the case, for instance, for free multiplicative powers of Marchenko-Pastur distribution, $[P_1(x)]^{\boxtimes s}$, with exponent s equal to $2, 3$ and also $1/2$ and $1/3$ and for the convolution of P_1 and $P_2(x) = [P_1(x)]^{\boxtimes 2}$ with the Arcsine distribution (AS).

Several distributions derived in this paper are of a direct use for the theory of random matrices and their numerous applications in physics. Integer multiplicative powers of MP, called Fuss–Catalan distributions, describe asymptotic level density of generalized Wishart random matrices, $W = XX^\dagger$, where X represents a product of s independent nonhermitian random square Ginibre matrices, $X = X_1 \cdots X_s$. We obtained here an explicit expression for $P_3 = [P_1(x)]^{\boxtimes 3}$ in terms of elementary functions and analyzed also the extension of the problem for the case of rectangular Ginibre matrices. Furthermore, the case of the multiplicative convolution of AS with P_1 and P_2 corresponds to the Bures distribution

B_1, generalized Bures distribution $B_{1,c}$ and higher order Bures distribution B_2 which describe level distributions of generalized Wishart matrices, for which X is a product of a sum of two random Haar unitary matrices and a product of s random Ginibre matrices. These results are applicable to describe asymptotic level density of certain ensembles o random quantum states [5].

As a by-product of our analysis we derived explicit results for the probability measure corresponding to the free multiplicative square/cubic root of the Marchenko-Pastur distribution, written $P_{1/2} = [MP]^{\boxtimes 1/2}$ and $P_{1/3} = [MP]^{\boxtimes 1/3}$, respectively. Note that for $p < 1$ the distribution $[MP]^{\boxtimes p}$ is not infinitely divisible with respect to the additive free convolution $\boxplus$, so the method of Cabanal–Duvillard [32] is not applicable. In fact, a stronger statement is true: if $p < 1$ then the additive free power $\left([MP]^{\boxtimes p}\right)^{\boxplus t}$ exists if and only if $t \geq 1$, see the recent result of Arizmendi and Hasebe [33]. It is thus unlikely to expect that there exists a random matrix model which corresponds to the level density described e.g. by the multiplicative free square root of the Marchenko–Pastur distribution.

Note added. After completing the paper, we became aware of two recent works, where related issues of Raney-type distributions have been addressed using either differential equations [34] or combinatorial analysis [31].

Appendix A: On imaginary part of solution of a quartic equation

To demonstrate the derivation of the spectral density we treat in this appendix an exemplary case corresponding to the Fuss–Catalan distribution of order three (23). Writing down the Ferrari solutions of the quartic equation (22) we identify the one with an imaginary part, denoted by w_3, so that the imaginary part of the corresponding Green function (7) yields the desired spectral density(9).

The full expression for this solution consists of two terms, $w_3(z) = a_1 + a_2$. We may omit the real term a_1,

as it does not contribute to the imaginary part of the Green function. The relevant term reads then

$$a_2 = -\frac{6^{2/3}}{12}\sqrt{-A - B + \frac{12z}{\sqrt{A+B}}}$$

where z–dependent symbols $A = \left(\frac{8z}{3z^2 + T/\sqrt{3}}\right)^{1/3}$ and $B = (18z^2 + \sqrt{12}T)^{1/3}$ contain a square root $T = \sqrt{z^3(-256 + 27z)}$. Its argument is negative for $z \in [0, 256/27]$, so T can be rewritten as $T = i\sqrt{z^3(256 - 27z)} = it$, where t is a real number. Let us now write the argument of the cubic root in B in polar form, $Z = re^{i\phi}$, with radius $r = 32\sqrt{3}z^{3/2}$ and phase

$\phi = \arccos(3\sqrt{3}\sqrt{z}/16)$. Then the key term reads

$$a_2 = -\frac{6^{2/3}}{12}\sqrt{-\frac{8z}{(Z/6)^{1/3}} - Z^{1/3} + \frac{12z}{\sqrt{\frac{8z}{(Z/6)^{1/3}} + Z^{1/3}}}}.$$

We can take the third root of Z represented in polar form, $Z^{1/3} = r^{1/3}\exp(i\phi/3)$, group terms $y\exp(i\phi/3)$ and $y\exp(-i\phi/3)$ and replace them by $2y\cos(\phi/3)$. Simplifying this expression we arrive eventually at the final form of the Green function (7) and by taking its imaginary part (9) we arrive at the Fuss–Catalan distribution of order three (23), defined for $x \in [0, 256/27]$.

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