# THE FREE MEIXNER CLASS FOR PAIRS OF MEASURES

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ABSTRACT. We investigate in more detail the two-state free convolution semigroups $\{(\widetilde{\mu}_t, \mu_t)\}$ of pairs of measures whose Jacobi parameters are linear in the convolution parameter $t$. These semigroups were constructed in [AM10], where we also showed that measures with the analogous property for the usual and free convolution are exactly the classical, resp. free Meixner classes. The class of measures in this paper has not been considered explicitly before, but we show that it also has Meixner-type properties. Specifically, it appears in limit theorems, has a Laha-Lukacs-type characterization, and is related to the $q = 0$ case of quadratic harnesses.

## 1. INTRODUCTION.

The Meixner class of measures consists of the normal, Poisson, gamma, binomial, negative binomial, and hyperbolic tangent distributions. Numerous characterizations of this class are known, starting with [Mei34], see [Mor82] for a survey. In [AM10], we obtained a new "dynamical" characterization of the Meixner class: it consists exactly of convolution semigroups $\{\mu_t\}$ whose Jacobi parameters $\{\beta_n(t), \gamma_n(t)\}$ are polynomial functions of the convolution parameter $t$. Here the Jacobi parameters of a measure are the entries in the tridiagonal matrix

$$\begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & \ddots \\ 1 & \beta_1 & \gamma_1 & 0 & \ddots \\ 0 & 1 & \beta_2 & \gamma_2 & \ddots \\ 0 & 0 & 1 & \beta_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

for which $\mu$ is the spectral measure; see also Section 2.1. In fact, for the Meixner class the Jacobi parameters depend on $t$ linearly:

$$\beta_n(t) = \beta_0 t + nb, \qquad \gamma_n(t) = (n+1)[\gamma_0 t + nc].$$

In the same paper, we considered the analogous question for other convolution operations coming from non-commutative probability theories. For the most important of these, the free convolution (coming from free probability [NS06]), there is a well-understood family of measures called the free Meixner class. The measures in this class are very different from those in the classical Meixner class, see Section 2.4. Surprisingly, many characterizations of the classical Meixner class have precise analogs for the free Meixner class, see [Ans03, BB06] and other sources. We showed that

*Date*: March 21, 2010.

2000 *Mathematics Subject Classification.* Primary 46L54; Secondary 33C45, 46L53, 60J25.

M.A. was supported in part by NSF grant DMS-0900935. W.M. was supported by MNiSW: N N201 364436.

for free convolution semigroups, the property similar to the one above holds precisely for the free Meixner class.

Yet another convolution operation was introduced in [BLS96] in relation to what the authors called "conditionally free probability", but is better called two-state free probability theory. As the name indicates, this is a convolution operation $\boxplus_c$ on pairs of measures $(\widetilde{\mu}, \mu)$, and as such does not really have a classical analogue. The techniques from [Mło09] allowed us to find all pairs $(\widetilde{\mu}, \mu)$ of measures such that if $(\widetilde{\mu}_t, \mu_t) := (\widetilde{\mu}, \mu)^{\boxplus_c t}$ then the Jacobi parameters of $\widetilde{\mu}_t$ and $\mu_t$ are linear with respect to this convolution. However, unlike in the two cases described above, the class of such $\widetilde{\mu}$ has not been explicitly described before. It consists of measures whose Jacobi parameters do not depend on $n$ for $n \geq 3$. In this paper, we show that, without being so identified, this class has in fact appeared in applications: in the two-state free Poisson limit theorem, in the two-state Laha-Lukacs characterization [BB09], and as a subclass of the $q = 0$ case of quadratic harnesses [BW05].

**Acknowledgements.** The paper was started during the 12th workshop on Non-commutative Harmonic Analysis with Applications to Probability at the Banach center. M.A. would like to thank the organizers for an enjoyable conference. He would also like to thank Włodek Bryc and Jacek Wesołowski for explaining their work to him.

## 2. BACKGROUND.

2.1. **Jacobi parameters.** Throughout the paper, $\mu$ will be a probability measure on $\mathbb{R}$ all of whose *moments*

$$s_m(\mu) := \int_{\mathbb{R}} x^m \, d\mu(x) \tag{1}$$

are finite. Then there is a sequence $\{P_m\}_{m=0}^{\infty}$ of monic polynomials, with $\deg P_m = m$, which are orthogonal with respect to $\mu$. They satisfy a recurrence relation: $P_0(x) = 1$ and for $m \geq 0$

$$xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1} P_{m-1}(x), \tag{2}$$

under convention that $P_{-1}(x) = 0$, where the *Jacobi parameters* [Chi78] satisfy $\beta_m \in \mathbb{R}$ and $\gamma_m \geq 0$. Then we will write

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \dots \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \dots \end{pmatrix}.$$

$\{P_m\}$ are unique for $m \leq |\mathrm{supp}(\mu)|$. Moreover $N := |\mathrm{supp}(\mu)| < \infty$ if and only if $\gamma_{N-1} = 0$ and $\gamma_m > 0$ for $m < N - 1$. In this case for $m \geq N$, $P_{m+1}$ are not uniquely determined. Our convention in this case will be that $\beta_m, \gamma_m$ can be chosen in an arbitrary convenient way.

2.2. **Generating functions, cumulants, and convolutions.** We will define cumulants and convolution operations via generating functions; see [AM10] for the definitions involving set partitions.

The ordinary moment generating function of $\mu$ is

$$M^{\mu}(z) = \sum_{k=1}^{\infty} s_m(\mu) z^m.$$

The $R$-transform (free cumulant generating function) $R^{\mu}$ is determined by

$$M^{\mu}(z) = R^{\mu}((1 + M^{\mu}(z))z),$$

the eta-transform (Boolean cumulant generating function) $\eta^\mu$ by

$$\eta^\mu(z) = 1 - (1 + M^\mu(z))^{-1},$$

and the two-state free cumulant generating function $R^{\widetilde{\mu},\mu}$ by

$$\eta^{\widetilde{\mu}}(z) = (1 + M^\mu(z))^{-1} R^{\widetilde{\mu},\mu}((1 + M^\mu(z))z).$$

Now expanding into (formal) power series

$$R^\mu(z) = \sum_{k=1}^\infty r_k(\mu) z^k$$

and

$$R^{\widetilde{\mu},\mu}(z) = \sum_{k=1}^\infty R_k(\widetilde{\mu},\mu) z^k,$$

we obtain the free cumulants $r_k(\mu)$ and the two-state free cumulants $R_k(\widetilde{\mu},\mu)$.

We now define the free convolution $\mu \boxplus \nu$ via

$$R^{\mu \boxplus \nu}(z) = R^\mu(z) + R^\nu(z).$$

The pair $(\widetilde{\tau},\tau)$ is the two-state free convolution $(\widetilde{\mu},\mu) \boxplus_c (\widetilde{\nu},\nu)$ if $\tau = \mu \boxplus \nu$ and also

$$R^{(\widetilde{\mu},\mu)\boxplus_c(\widetilde{\nu},\nu)}(z) = R^{\widetilde{\mu},\mu}(z) + R^{\widetilde{\nu},\nu}(z).$$

The Boolean convolution $\uplus$ is defined similarly using $\eta$.

A *free convolution semigroup* generated by $\mu$ is a family of measures $\{\mu_t\}$ such that $\mu_1 = \mu$ and $\mu_t \boxplus \mu_s = \mu_{t+s}$. The two-state free convolution semigroups $\{(\widetilde{\mu}_t,\mu_t)\} = \left\{(\widetilde{\mu},\mu)^{\boxplus_c t}\right\}$ are defined similarly. A priori, our semigroups are indexed by $t \in \mathbb{N}$, but in fact, one can always extend the semigroup to $t \in [1,\infty)$. If $\mu$ is infinitely divisible in the appropriate sense, one has a full convolution semigroup for $t \in [0,\infty)$.

2.3. **Other operations.** We will occasionally use the following notation. $\mathbb{B}$ is the Boolean-to-free Bercovici-Pata bijection defined in [BN08]; it is determined by

$$\eta^\mu(z) = R^{\mathbb{B}[\mu]}(z).$$

$\Phi$ is the Jacobi shift, considered for example in [BN09], determined by

$$\eta^{\Phi[\mu]}(z) = z^2(1 + M^\mu(z)).$$

The name comes from the observation that

$$J(\Phi(\mu)) = \begin{pmatrix} 0, & \beta_0(\mu), & \beta_1(\mu), & \beta_2(\mu), & \ldots \\ 1, & \gamma_0(\mu), & \gamma_1(\mu), & \gamma_2(\mu), & \ldots \end{pmatrix}.$$

Finally, the two-variable map $\Phi : (\omega,\mu) \mapsto \widetilde{\mu}$ from [Ans10] is determined by

$$\eta^{\Phi[\omega,\mu]}(z) = (1 + M^\mu(z))^{-1} R^\omega\Big((1 + M^\mu(z))z\Big)$$

and corresponds to $R^{\widetilde{\mu},\mu} = R^\omega$.

2.4. **Free Meixner distributions.** The free Meixner distributions $\mu_{b,c}$ with mean zero, variance one, and parameters $b \in \mathbb{R}$, $c \geq -1$ are probability measures with

$$J(\mu_{b,c}) = \begin{pmatrix} 0, & b, & b, & b, & \dots \\ 1, & 1+c, & 1+c, & 1+c, & \dots \end{pmatrix}.$$

The general free Meixner distributions are affine transformations of these. More explicitly, the distribution with parameters $b, c$ is

$$\frac{1}{2\pi} \cdot \frac{\sqrt{\left(4(1+c)-(x-b)^2\right)_+}}{1+bx+cx^2}\, dx + 0, 1, \text{or } 2 \text{ atoms},$$

see [SY01, Ans03, BB06]. There are numerous characterizations of this class in free probability.

## 3. THE TWO-STATE FREE MEIXNER CLASS.

Using the techniques from [Mło09], in [AM10] we proved the following result.

**Theorem 1.** *Let $(\widetilde{\mu}, \mu)$ be a pair of measures with Jacobi parameters $\widetilde{\beta}_m, \widetilde{\gamma}_m$ and $\beta_m, \gamma_m$ respectively.*

*Assume that neither $\widetilde{\mu}$ nor $\mu$ is a point mass and that for the conditionally free powers $(\widetilde{\mu}_t, \mu_t) := (\widetilde{\mu}, \mu)^{\boxplus_c t}$, $t \in \mathbb{N}$, all the Jacobi parameters of $\widetilde{\mu}_t$ are polynomials on $t$. Then*

$$\beta_1 = \widetilde{\beta}_2 = \beta_2 = \widetilde{\beta}_3 = \beta_3 = \widetilde{\beta}_4 = \dots, \tag{3}$$

$$\gamma_1 = \widetilde{\gamma}_2 = \gamma_2 = \widetilde{\gamma}_3 = \gamma_3 = \widetilde{\gamma}_4 = \dots, \tag{4}$$

*so that $\widetilde{\mu}$ is a general distribution whose Jacobi parameters do not depend on $n$ for $n \geq 3$, and $\mu$ is the corresponding free Meixner distribution.*

*On the other hand, if (3) and (4) hold then, putting*

$$\widetilde{b} = \widetilde{\beta}_1 - \beta_0,\ b = \beta_1 - \beta_0,\ \widetilde{c} = \widetilde{\gamma}_1 - \gamma_0,\ c = \gamma_1 - \gamma_0, \tag{5}$$

*the conditionally free power $(\widetilde{\mu}_t, \mu_t) := (\widetilde{\mu}, \mu)^{\boxplus_c t}$ exists for $t \geq 0$, $c + \gamma_0 t \geq 0$, $\widetilde{c} + \gamma_0 t \geq 0$ and we have*

$$J(\widetilde{\mu}_t) = \begin{pmatrix} \widetilde{\beta}_0 t, & \widetilde{b} + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ \widetilde{\gamma}_0 t, & \widetilde{c} + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix} \tag{6}$$

*and*

$$J(\mu_t) = \begin{pmatrix} \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix}. \tag{7}$$

*In particular, the pair $(\widetilde{\mu}, \mu)$ is $\boxplus_c$-infinitely divisible if and only if $c \geq 0$ and $\widetilde{c} \geq 0$. In this case,*

$$R^{\widetilde{\mu}, \mu}(z) = \widetilde{\beta}_0 z + \widetilde{\gamma}_0 z^2 \int_{\mathbb{R}} \frac{d\widetilde{\rho}(x)}{1 - xz}, \tag{8}$$

*where $\widetilde{\rho}$ is the free Meixner probability measure which satisfies*

$$J(\widetilde{\rho}) = \begin{pmatrix} \widetilde{b}, & b, & b, & b, & \dots \\ \widetilde{c}, & c, & c, & c, & \dots \end{pmatrix}. \tag{9}$$

**Remark 1.** An explicit formula for $\widetilde{\mu}_t$ can be obtained from the continued fraction expansion of its Cauchy transform:

$$G_{\widetilde{\mu}_t}(z) = \cfrac{1}{z - \widetilde{\beta}_0 t - \cfrac{\widetilde{\gamma}_0 t}{z - \beta_0 t - \widetilde{b} - (\gamma_0 t + \widetilde{c}) G_{\sigma_t}(z)}},$$

where $\sigma_t$ is the semicircular distribution with mean $\beta_0 t + b$ and variance $\gamma_0 t + c$. The corresponding measure belongs to the Bernstein-Szegő class, and has the form

$$\widetilde{\mu}_t = \frac{\sqrt{4(\gamma_0 t + c) - (x - \beta_0 t - b)^2}}{\text{cubic polynomial}}\, dx + \text{ at most 3 atoms.}$$

**Proposition 2.** *Let $\widetilde{\rho}$ be a probability measure, $a, c \in \mathbb{R}$, $b \geq 0$, $d \geq 0$. Define $\mu$ and $\omega$ via*

$$\mathbb{B}[\widetilde{\rho}] = \mu^{\boxplus b} \boxplus \delta_a$$

*and*

$$\omega = \mathbb{B}[\Phi[\widetilde{\rho}]]^{\boxplus d} \boxplus \delta_c.$$

*Denote $\omega_t = \omega^{\boxplus t}$, $\mu_t = \mu^{\boxplus t}$, and define*

$$\widetilde{\mu}_t = \Phi[\omega_t, \mu_t],$$

*so that $R^{\widetilde{\mu}_t, \mu_t} = R^{\omega_t}$ and the pairs $\{(\widetilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup. Suppose that*

$$J(\mu_t) = \begin{pmatrix} \beta_0(t), & \beta_1(t), & \beta_2(t), & \dots \\ \gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \dots \end{pmatrix}.$$

*Then*

$$R^{\widetilde{\mu}, \mu}(z) = R^{\omega}(z) = cz + \frac{dz^2}{1 - az - bR^{\mu}(z)}$$

*and*

$$J(\widetilde{\mu}_t) = \begin{pmatrix} ct, & a + \left(1 + \frac{b}{t}\right) \beta_0(t), & \beta_1(t), & \beta_2(t), & \dots \\ dt, & \left(1 + \frac{b}{t}\right) \gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \dots \end{pmatrix}.$$

*Proof.* Using definitions of $\omega, \mathbb{B}, \Phi, \eta, \mu$ in succession, we transform

$$\begin{aligned} R^{\omega}(z) &= cz + dR^{\mathbb{B}[\Phi[\widetilde{\rho}]]}(z) = cz + d\eta^{\Phi[\widetilde{\rho}]}(z) = cz + dz^2(1 + M^{\widetilde{\rho}}(z)) \\ &= cz + \frac{dz^2}{1 - \eta^{\widetilde{\rho}}(z)} = cz + \frac{dz^2}{1 - R^{\mathbb{B}[\widetilde{\rho}]}(z)} = cz + \frac{dz^2}{1 - az - bR^{\mu}(z)}. \end{aligned}$$

Therefore by definition of $\widetilde{\mu}_t$ and $\Phi[\cdot, \cdot]$,

$$\begin{aligned} \eta^{\widetilde{\mu}_t}(z) &= t(1 + M^{\mu_t}(z))^{-1} R^{\omega}\Big((1 + M^{\mu_t}(z))z\Big) \\ &= ctz + \frac{dtz^2(1 + M^{\mu_t}(z))}{1 - a((1 + M^{\mu_t}(z))z) - \frac{b}{t} R^{\mu_t}\Big((1 + M^{\mu_t}(z))z\Big)} \\ &= ctz + \frac{dtz^2(1 + M^{\mu_t}(z))}{1 - a((1 + M^{\mu_t}(z))z) - \frac{b}{t} M^{\mu_t}(z)} \\ &= ctz + \frac{dtz^2}{1 - az - \left(1 + \frac{b}{t}\right) \eta^{\mu_t}(z)}. \end{aligned}$$

So writing

$$\eta^{\mu_t}(z) = \beta_0(t) z + \gamma_0(t) z^2 F,$$

we get

$$\begin{aligned}\eta^{\widetilde{\mu}_t}(z) &= ctz + \frac{dtz^2}{1 - az - \left(1 + \frac{b}{t}\right)(\beta_0(t) z + \gamma_0(t) z^2 F)} \\ &= ctz + \frac{dtz^2}{1 - \left(a + \left(1 + \frac{b}{t}\right)\beta_0(t)\right) z - \left(1 + \frac{b}{t}\right)\gamma_0(t) z^2 F}.\end{aligned}$$

The conclusion follows. □

**Corollary 3.** *Let $\mu$ be a free Meixner distribution,*

$$\mu = \mu_{b,c}^{\boxplus \gamma_0} \boxplus \delta_{\beta_0}$$

*and*

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_0 + b, & \beta_0 + b, & \dots \\ \gamma_0, & \gamma_0 + c, & \gamma_0 + c, & \dots \end{pmatrix}.$$

(a) *$\mu_t$ has the Jacobi parameters in equation* (7).

(b) *If $c \geq 0$ and $\widetilde{c} \geq 0$, then*

$$\widetilde{\rho} = \mu_{b,c-1}^{\uplus \widetilde{c}} \uplus \delta_{\widetilde{b}}$$

*is a free Meixner distribution with*

$$J(\widetilde{\rho}) = \begin{pmatrix} \widetilde{b}, & b, & b, & \dots \\ \widetilde{c}, & c, & c, & \dots \end{pmatrix}.$$

*For*

$$\mathbb{B}[\widetilde{\rho}] = \mu^{\boxplus(\widetilde{c}/\gamma_0)} \boxplus \delta_{(\widetilde{b}\gamma_0 - \beta_0 \widetilde{c})/\gamma_0}, \quad \omega = \mathbb{B}[\Phi[\widetilde{\rho}]]^{\boxplus \widetilde{\gamma}_0} \boxplus \delta_{\widetilde{\beta}_0}, \quad \widetilde{\mu}_t = \Phi[\omega_t, \mu_t],$$

*$\omega$ is freely infinitely divisible, and $\widetilde{\mu}_t$ has the Jacobi parameters in equation* (6).

*In the case that $c$ or $\widetilde{c}$ are negative, note that the operations $\boxplus, \mathbb{B}, \Phi$ can be defined purely combinatorially (in terms of moments), so they are well-defined operations on (not necessarily positive) functionals, in which case $\mathbb{B}$ is even a bijection. So the construction remains valid, as long as $\mu_t, \widetilde{\mu}_t$ themselves are positive.*

(c) *If $\widetilde{b} = b$ and $\widetilde{c} = c$, so that $\widetilde{\rho}$ is a semicircular distribution, then*

$$\omega = \mu_{b,c}^{\boxplus \widetilde{\gamma}_0} \boxplus \delta_{\widetilde{\beta}_0}$$

*and*

$$J(\widetilde{\mu}_t) = \begin{pmatrix} \widetilde{\beta}_0 t, & \beta_0 t + b, & \beta_0 t + b, & \dots \\ \widetilde{\gamma}_0 t, & \gamma_0 t + c, & \gamma_0 t + c, & \dots \end{pmatrix}$$

*so that $\omega, \widetilde{\mu}_t$ are also free Meixner distributions.*

## 4. Probabilistic appearances

Free Meixner distributions arise in many results in free and Boolean probability theories. In this section we describe a number of appearances of the family from Theorem 1 in the two-state-free probability theory, which justify the name "two-state free Meixner class". Other places where distributions with Jacobi parameters constant after step 3 were encountered include Theorems 11 and 12 of [KW05], examples in [Len07], as well as [HM07, HM08] and [HKM09]. See also Proposition 7 of [AM10].

### 4.1. **Limit theorems.**

(a) In the two-state free central limit theorem, one gets pairs of distributions $(\widetilde{\mu}, \mu)$ such that

$$R^{\mu}(z) = uz^2, \qquad R^{\widetilde{\mu},\mu}(z) = vz^2.$$

Thus $\mu = \mu_{0,0}^{\boxplus u}$ and $\omega = \mu_{0,0}^{\boxplus v}$ are semicircular distributions. In this case

$$\widetilde{b} = b = 0, \quad \widetilde{c} = c = 0, \quad \widetilde{\beta}_0 = \beta_0 = 0, \quad \widetilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

Therefore $\widetilde{\rho} = \delta_0$ and

$$\widetilde{\mu} = \mu_{0,u-v}^{\boxplus v}$$

with

$$J(\widetilde{\mu}) = \begin{pmatrix} 0, & 0, & 0 & \dots \\ v, & u, & u, & \dots \end{pmatrix}$$

is a symmetric free Meixner distribution.

(b) In the (centered) two-state free Poisson limit theorem, one gets pairs of distributions $(\widetilde{\mu}, \mu)$ such that

$$R^{\mu}(z) = \frac{uz^2}{1 - pz}, \qquad R^{\widetilde{\mu},\mu}(z) = \frac{vz^2}{1 - rz},$$

Thus $\mu = \mu_{p,0}^{\boxplus u}$ and $\omega = \mu_{r,0}^{\boxplus v}$ are free Poisson distributions. Assuming that $p = r$, in this case

$$\widetilde{b} = b = p, \quad \widetilde{c} = c = 0, \quad \widetilde{\beta}_0 = \beta_0 = 0, \quad \widetilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

Therefore

$$\widetilde{\rho} = \delta_p$$

with

$$J(\widetilde{\rho}) = \begin{pmatrix} p, & p, & p, & \dots \\ 0, & 0, & 0, & \dots \end{pmatrix}$$

and

$$\widetilde{\mu} = \mu_{p,u-v}^{\boxplus v}$$

with

$$J(\widetilde{\mu}) = \begin{pmatrix} 0, & p, & p, & \dots \\ v, & u, & u, & \dots \end{pmatrix}$$

is a free Meixner distribution.

(c) If $p \neq r$, then

$$\widetilde{b} = r, \quad b = p, \quad \widetilde{c} = c = 0, \quad \widetilde{\beta}_0 = \beta_0 = 0, \quad \widetilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

In this case $\widetilde{\rho} = \delta_r$ still, but now we identify

$$J(\widetilde{\rho}) = \begin{pmatrix} r, & p, & p, & \dots \\ 0, & 0, & 0, & \dots \end{pmatrix}$$

and not with a degenerate semicircular distribution. Finally,

$$J(\widetilde{\mu}) = \begin{pmatrix} 0, & r, & p, & p, & \dots \\ v, & u, & u, & u, & \dots \end{pmatrix}$$

has Jacobi parameters constant after step 3.

4.2. **Laha-Lukacs characterization.** A classical paper [LL60] characterizes Meixner distributions in terms of certain conditional expectations. In [BB06], the authors obtained a similar characterization of free Meixner distributions. The following is their result for the two-state free independence. Recall that the distribution of an operator $X$ with respect to a state $\varphi$ is the measure $\mu$ such that for all $n$,

$$\varphi[X^n] = s_n(\mu) = \int_{\mathbb{R}} x^n \, d\mu(x).$$

Also, one says that $\mathbb{X}, \mathbb{Y}$ are $(\varphi|\psi)$-free if all their mixed two-state free cumulants are zero. See the paper quoted below for the rest of the terminology.

**Theorem** (Theorem 2.1 in [BB09])**.** *Suppose $\mathbb{X}, \mathbb{Y}$ are self-adjoint $(\varphi|\psi)$-free and*

$$\varphi[\mathbb{X}^n] = \varphi[\mathbb{Y}^n], \quad \psi[\mathbb{X}^n] = \psi[\mathbb{Y}^n]$$

*for all $n$. Furthermore, assume that $\varphi[\mathbb{X}] = 0$, $\varphi[\mathbb{X}^2] = 1$. (This can always be achieved by a shift and dilation, as long as $\varphi[\mathbb{X}^2] \neq 0$.) Let $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ and suppose that there are $b, C \in \mathbb{R}$ and $c > -2$ such that*

$$\varphi[(\mathbb{X} - \mathbb{Y})^2 \mathbb{S}^n] = C\varphi[(4\mathbb{I} + 2b\mathbb{S} + c\mathbb{S}^2)\mathbb{S}^n], n = 0, 1, 2 \dots. \tag{10}$$

*Then the $\varphi$ and $\psi$-moment generating functions*

$$M_{\mathbb{S}}(z) := \sum_{k=1}^{\infty} z^k \varphi[\mathbb{S}^k] \text{ and } m_{\mathbb{S}}(z) := \sum_{k=1}^{\infty} z^k \psi[\mathbb{S}^k],$$

*which are defined as formal power series, are related as follows:*

$$1 + M_{\mathbb{S}}(z) = \frac{2 + c - (2bz + c)(1 + m_{\mathbb{S}}(z))}{2 + c - (4z^2 + 2bz + c)(1 + m_{\mathbb{S}}(z))}. \tag{11}$$

Bożejko and Bryc described the corresponding distributions more explicitly in particular cases corresponding to the Gaussian and Poisson regressions. We now provide a complete description. Recall that if $\varphi = \psi$, then the $\psi$-distributions of $\mathbb{X}, \mathbb{Y}, \mathbb{S}$ are free Meixner distributions.

**Proposition 4.** *Denote by $\widetilde{\mu}_{\mathbb{S}}, \mu_{\mathbb{S}}$ the distributions of $\mathbb{S}$ with respect to $\varphi$, $\psi$, respectively, and by $\widetilde{\mu}, \mu$ the corresponding distributions of $\mathbb{X}$. Then*

$$J(\widetilde{\mu}_{\mathbb{S}}) = \begin{pmatrix} 0, & b + (1 + c/2)\beta_0(\mu_{\mathbb{S}}), & \beta_1(\mu_{\mathbb{S}}), & \dots \\ 2, & (1 + c/2)\gamma_0(\mu_{\mathbb{S}}), & \gamma_1(\mu_{\mathbb{S}}), & \dots \end{pmatrix}.$$

*If* (10) *also holds with* $\psi$ *in place of* $\varphi$, *for the same* $b, c$, *then* $\mu = \mu_{b,c}^{\boxplus\gamma_0} \boxplus \delta_{\beta_0}$ *is a free Meixner distribution, and*

$$J(\widetilde{\mu}) = \begin{pmatrix} 0, & (b + c\beta_0) + \beta_0, & b + \beta_0, & b + \beta_0, & \dots \\ 1, & c\gamma_0 + \gamma_0, & c + \gamma_0, & c + \gamma_0, & \dots \end{pmatrix},$$

*so that* $\widetilde{\mu}$ *is a general (up to normalization) distribution with Jacobi parameters constant after step* 3.

*Proof.* From equation (11),

$$\eta^{\widetilde{\mu}_{\mathbb{S}}} = -\frac{4z^2}{c + 2bz - (2 + c)\left(M^{\mu_{\mathbb{S}}}\right)^{-1}},$$

so that

$$\begin{aligned} \eta^{\widetilde{\mu}_{\mathbb{S}}} &= -\frac{4z^2}{c + 2bz - (2 + c)\left(M^{\mu_{\mathbb{S}}}\right)^{-1}} \\ &= -\frac{4z^2}{c + 2bz - (2 + c)(1 - \eta^{\mu_{\mathbb{S}}})} \\ &= -\frac{4z^2}{-2 + 2bz + (2 + c)\eta^{\mu_{\mathbb{S}}}} \\ &= \frac{2z^2}{1 - bz - (1 + c/2)\eta^{\mu_{\mathbb{S}}}} \end{aligned}$$

Comparing with the general formula in Proposition 2, we conclude that in that proposition, $c = 0$, $d = 1$, $t = 2$, $a = b$ and $b = c$. Therefore in terms of the Jacobi parameters of $\mu_{\mathbb{S}}$,

$$J(\widetilde{\mu}_{\mathbb{S}}) = \begin{pmatrix} 0, & b + (1 + c/2)\beta_0(\mu_{\mathbb{S}}), & \beta_1(\mu_{\mathbb{S}}), & \dots \\ 2, & (1 + c/2)\gamma_0(\mu_{\mathbb{S}}), & \gamma_1(\mu_{\mathbb{S}}), & \dots \end{pmatrix}.$$

If $\mathbb{X}$ has, with respect to $\psi$, a free Meixner distribution $\mu = \mu_{b,c}^{\boxplus\gamma_0} \boxplus \delta_{\beta_0}$, then

$$\mu_{\mathbb{S}} = \mu_{b,c}^{\boxplus 2\gamma_0} \boxplus \delta_{2\beta_0}$$

and

$$J(\mu_{\mathbb{S}}) = \begin{pmatrix} 2\beta_0, & b + 2\beta_0, & b + 2\beta_0, & \dots \\ 2\gamma_0, & c + 2\gamma_0, & c + 2\gamma_0, & \dots \end{pmatrix}.$$

So

$$J(\widetilde{\mu}_{\mathbb{S}}) = \begin{pmatrix} 0, & (b + c\beta_0) + 2\beta_0, & b + 2\beta_0, & b + 2\beta_0, & \dots \\ 2, & c\gamma_0 + 2\gamma_0, & c + 2\gamma_0, & c + 2\gamma_0, & \dots \end{pmatrix}.$$

This is the $t = 2$ case of

$$J(\widetilde{\mu}_t) = \begin{pmatrix} 0, & (b + c\beta_0) + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ t, & c\gamma_0 + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix},$$

with $\widetilde{b} = b + c\beta_0$ and $\widetilde{c} = c\gamma_0$. By setting $t = 1$ instead, we get

$$J(\widetilde{\mu}) = \begin{pmatrix} 0, & (b + c\beta_0) + \beta_0, & b + \beta_0, & b + \beta_0, & \dots \\ 1, & c\gamma_0 + \gamma_0, & c + \gamma_0, & c + \gamma_0, & \dots \end{pmatrix}.$$

□

4.3. **Free quadratic harnesses.** In a series of papers starting with [BW05], Bryc and Wesołowski (along with Matysiak and Szabłowski) have investigated quadratic harnesses. These are square-integrable processes $(X_t)_{t\geq 0}$, with normalization $\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t X_s] = \min(t, s)$, such that $\mathbb{E}[X_t|\mathcal{F}_{s,u}]$ is a linear function of $X_s, X_u$ and $\mathrm{Var}[X_t|\mathcal{F}_{s,u}]$ is a quadratic function of $X_s, X_u$. Here $\mathcal{F}_{s,u}$ is the two-sided $\sigma$-field generated by $\{X_r : r \in [0, s] \cup [u, \infty)\}$. Then

$$\mathbb{E}[X_t|\mathcal{F}_{s,u}] = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u$$

and under certain technical assumptions (see [BMW07]),

$$\begin{aligned}\mathrm{Var}[X_t|\mathcal{F}_{s,u}] = \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau-\gamma s}\Bigg(&1 + \sigma\frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau\frac{(X_u - X_s)^2}{(u-s)^2} \\ &+ \eta\frac{uX_s - sX_u}{u-s} + \theta\frac{X_u - X_s}{u-s} - (1-\gamma)\frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2}\Bigg).\end{aligned}$$

The authors proved the existence of such processes for a large range of parameters $\sigma, \tau, \eta, \theta, \gamma$, in particular connecting the analysis to the Askey-Wilson distributions in [BW09] (the standard Askey-Wilson parameter $q = \gamma + \sigma\tau$). One reason for the interest in this analysis comes from numerous particular cases.

(a) If $\gamma = 1$ and $\sigma = \eta = 0$, the processes automatically have classically independent increments, and each $X_t$ has a Meixner distribution, see [Wes93].
(b) For $\gamma = \sigma = \eta = 0$, the processes are classical versions of processes with free independent increments, and have free Meixner distributions.
(c) For $\sigma = \eta = 0$ and $-1 \leq \gamma = q < 1$, the corresponding orthogonal martingale polynomials have Jacobi parameters

$$\begin{pmatrix} \beta_n(t) & = \theta[n]_q \\ \gamma_n(t) & = [n+1]_q(t + \tau[n]_q) \end{pmatrix},$$

where $[n]_q := 1 + q + \ldots + q^{n-1}$ is the $q$-integer. If $\tau = 0$, the process is a (classical version of a) $q$-Poisson process from [Ans01]. The case where in addition, $\theta = 0$ was considered even earlier in [Bry01] and corresponds to the $q$-Brownian motion [BKS97]. The challenge of interpreting the general processes with $\sigma = \eta = 0$ as "processes with $q$-independent increments" remains open.
(d) Finally, for $\gamma = \sigma = \tau = 0$, the free bi-Poisson processes from [BW07] are shown, in Section 4 of that paper, to have increments freely independent with respect to a pair of states.

We will now extend the last result above. Proposition 4.3 of [BMW07] states that for

$$q = \gamma + \sigma\tau = 0,$$

there exist orthogonal martingale polynomials for the process. They satisfy recursion relations

$$
\begin{aligned}
xP_0(x,t) &= P_1(x,t),\\
xP_1(x,t) &= (1+\sigma t)P_2(x,t) + (ut+v)P_1(x,t) + tP_0(x,t),\\
xP_2(x,t) &= (1+\sigma t)P_3(x,t) + \left(\frac{u+\sigma v}{1-\sigma\tau}t + \frac{v+\tau u}{1-\sigma\tau}\right)P_2(x,t) + \frac{1+uv}{1-\sigma\tau}(t+\tau)P_1(x,t),\\
xP_n(x,t) &= (1+\sigma t)P_{n+1}(x,t) + \left(\frac{u+\sigma v}{1-\sigma\tau}t + \frac{v+\tau u}{1-\sigma\tau}\right)P_n(x,t) + \frac{1+uv}{(1-\sigma\tau)^2}(t+\tau)P_{n-1}(x,t)
\end{aligned}
$$

for $n \geq 3$, where

$$u = \frac{\eta + \sigma\theta}{1-\sigma\tau}, \quad v = \frac{\tau\eta+\theta}{1-\sigma\tau}$$

and as long as

$$1 + uv > 0, 0 \leq \sigma\tau < 1.$$

The coefficients in this recursion are linear in $t$ and constant for $t \geq 3$, but the corresponding polynomials are not monic. It follows that the Jacobi parameters for the monic orthogonal polynomials for this process (which are not martingale polynomials) are quadratic in $t$. Therefore they do not form a semigroup with respect to any of the convolutions considered in this paper, unless $\sigma = 0$.

**Proposition 5.** *Let $q = 0$ and $\sigma = 0$. Denoting by $\widetilde{\mu}_t$ the distribution of $X_t$, for some $\{\mu_t\}$, the pairs $\{(\widetilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup. Also in this case, $\widetilde{\rho}$ is a free Poisson distribution.*

*Proof.* Since $\sigma = 0$, we have $u = \eta$, $v = \tau\eta + \theta$, and the identification with parameters in our Theorem 1 gives

$$\widetilde{\beta}_0 = 0, \quad \beta_0 = \eta, \quad \widetilde{b} = (\tau\eta+\theta), \quad b = 2\tau\eta+\theta$$

and

$$\widetilde{\gamma}_0 = 1, \quad \gamma_0 = 1 + \eta(\tau\eta+\theta), \quad \widetilde{c} = c = \tau(1+\eta(\tau\eta+\theta)).$$

Note that this is a distribution with Jacobi parameters constant after step 3, but not the most general one.

Thus if

$$\mu_t = \left(\mu_{2\tau\eta+\theta,\tau(1+\eta(\tau\eta+\theta))}^{\boxplus(1+\eta(\tau\eta+\theta))} \boxplus \delta_\eta\right)^{\boxplus t}$$

are free Meixner distributions, then the pairs $\{(\widetilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup. Also,

$$J(\widetilde{\rho}) = \begin{pmatrix} (\tau\eta+\theta), & 2\tau\eta+\theta, & 2\tau\eta+\theta, & \ldots \\ \tau(1+\eta(\tau\eta+\theta)), & \tau(1+\eta(\tau\eta+\theta)), & \tau(1+\eta(\tau\eta+\theta)), & \ldots \end{pmatrix}$$

and $\widetilde{\rho}$ is a free Poisson distribution. □

**Remark 2.** Restricting to the $\tau = 0$ case of the free bi-Poisson process gives

$$\mu_t = \left(\mu_{\theta,0}^{\boxplus(1+\eta\theta)} \boxplus \delta_\eta\right)^{\boxplus t}$$

a free Poisson distribution, $\widetilde{\rho} = \delta_\theta$, $\omega = \mu_{\theta,0}$ is a free Poisson distribution, and $\widetilde{\mu}_t = \mu_{\eta t+\theta,(1+\eta\theta)t}$ is a free Meixner distribution. Further restriction to $\theta = 0$ gives $\mu$ a (non-centered) semicircular distribution, $\omega$ the standard semicircular distribution, and $\widetilde{\mu}_t = \mu_{\eta t,t}$ a free Meixner distribution.

On the other hand, restriction to $\eta = 0$ gives

$$\mu = \widetilde{\mu} = \omega = \mu_{\theta,\tau}$$

and $\widetilde{\rho}$ is a semicircular distribution.

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