

The exact asymptotic for the stationary distribution of some unreliable systems.

Paweł Lorek
University of Wrocław

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Abstract

In this paper we find asymptotic distribution for some unreliable networks.

We investigate following models: Model 1 is $M/M/1$ system with the unreliable server. Customers arriving while the server is in *Down* status join the queue. Using Markov Additive Structure and Adan, Foley, McDonald [1] method, we find the exact asymptotic for the stationary distribution. With the help of MA structure and matrix geometric approach, we also investigate the asymptotic when breakdown intensity α is small. In particular, two different asymptotic regimes for small α suggest two different patterns of large deviations, which is confirmed by simulation study.

Model 2 is the following network of 2 servers: Customers arrive only to server 2 which is typical $M/M/1$ queue and then proceed to unreliable (as described above) server 1. After being served customer leaves the network with probability p or returns to server 1 with probability $1 - p$. Using the same method we give the exact asymptotic for the system. We show, that the method does not allow to conclude the asymptotic of the “limiting system” as $\alpha \rightarrow 0$. In other words, in some cases having only second largest eigenvalue of the transition matrix is not enough for asymptotic of such “limiting systems”.

Keywords: Queueing systems, unreliable server, h -transform, Feynman-Kac kernel, matrix-geometric approach, large deviations, exact asymptotic.

1 Introduction

In this paper we consider problem of finding an exact asymptotic of some non-standard queueing systems. We consider two models: Model 1 is an unreliable $M/M/1$ system, and Model 2 is a system consisting of 2 servers: server 1 being unreliable and server 2 reliable one with a possible feedback from server 1 to 2. There are a lot of practical problems modelled as Markov chains using 2 or 3 variables. Explicit formulas for stationary distribution can be found only in some special cases. That is why studies on asymptotic for such stationary distributions have been actively conducted by theoretical- and application-oriented researches. There are several techniques available, starting with matrix geometric approach (Neuts, [9]), matrix analytic method (for recent related work see Miyazawa, Zhao [8], Liu, Miyazawa, Zhao [6] and Tang, Zhao [12]) or a method of Adan, Foley, McDonald [1] which we are going to use widely in this paper.

To describe our results, let us start with a brief description of the models. In the unreliable $M/M/1$ system, the customers arrive according to a Poisson process with intensity λ and are served with intensity μ . Moreover, there is an external Markov process which governs the breakdowns and repairs: with intensity α the server can change status from *Up* to *Down*, and with intensity

β vice-versa. While server is in *Down* status, customers are no longer served, but new ones can join the queue at the server.

Model 2 consists of two servers: Customers arrive to the server 2, which is reliable, according to a Poisson process with intensity λ and after being served they are directed to the unreliable server 1 which is as the one described above. The service rate at both servers is μ . After being served at server 1 the customer leaves the network with probability p and with probability $1 - p$ is rerouted back to the queue at server 1.

The situation described above is different than *the loss regime*. In this regime, customers arriving when a server is in *Down* status, are lost (to the *Down* server). In [11], Sauer and Daduna showed that under this regime the stationary distribution of network of unreliable servers is of product form: the stationary distribution of a pure Jackson network and the stationary distribution of the breakdowns/repairs process. In such a network when customer arrives while server is in *Down* status, it is lost to the server, but not to the network: it is rerouted - according to some routing regime - to some other server which is in *Up* status.

However, if a customer can join the queue while the server is broken, then the stationary distribution is not of product form, as can be seen in White and Christie [13]. There, the authors give the stationary distribution of Model 1 only. For the Model 2 we are not aware of any results, neither exact distribution, nor asymptotic one.

In our paper, we give exact asymptotic for both models, following the method of Adan, Foley, McDonald [1]. Using Markov Additive Process approach we can clearly show all the differences between two models.

We also consider the behaviour of the “limiting system”, i.e. the system in which the breaking probability α goes to 0. From the method of the above authors we are able to conclude the exact asymptotic of such “limiting system”, but only for Model 1 and for some set of parameters: when $\mu < \lambda + \beta$. It turns out to be the same (up to a constant) as the stationary distribution of a $M/M/1$ queue. For the other set of parameters ($\mu > \lambda + \beta$), this method does not lead to a valid asymptotic. However, using the matrix geometric approach (Neuts, [9]) we show that then the “limiting system” for Model 1 still has the same (up to a constant) stationary distribution as $M/M/1$ queue. The matrix geometric approach, however, does not give us any information about constants. Nevertheless we conclude two different ways in which the system can accumulate a big number of customers. When $\mu < \lambda + \beta$, then in most cases a path leading to a big queue is to be in *Up* status, and to accumulate a big number of customers, exactly like in standard $M/M/1$ queue (the system does not manage to service customers). The breakdowns/repairs of the system do not have big influence on large deviations. However, if $\mu > \lambda + \beta$, then in most cases a big number of customers is accumulated while the system is in *Down* status. We illustrate it with simulations (for small α), see Figure 1 and description on page 6 for details.

Furthermore, for Model 2 the method of Adan, Foley, McDonald [1] does not lead to a valid asymptotic for the “limiting system”. Also, matrix geometric approach is not applicable.

For the related work, but using different technique see for example Miyazawa, Zhao [8], Liu, Miyazawa, Zhao [6] and Tang, Zhao [12]. Authors therein use matrix analytic method. It uses the fact that some stationary distributions can be presented in matrix form and shown to be solutions of Markov renewal equation, this way decay rates are considered.

In Section 2 we give detailed description of both models, present and discuss all the results. The proofs are in Section 3.

2 Unreliable server systems and results

2.1 Description of systems

Model 1 is a following system consisting of 1 server: customers arrive according to an external Poisson arrival stream with intensity λ and are served according to the First Come First Served (FCFS) regime. Each of them requests a service which is exponentially distributed with mean 1. Service is provided with intensity μ . There is an external process on the state space $\{Up, Down\}$: with intensity α the server changes status form Up to $Down$ and with intensity β from $Down$ to Up ; $Down$ -to- Up and Up -to- $Down$ times are exponentially distributed. When the server is broken it immediately stops service, the customer being served is redirected back to the queue. When a new customer arrives while the server is in the $Down$ status, it joins the queue at the server. We assume that all service times, inter-arrivals time, $Down$ -to- Up and Up -to- $Down$ times constitute an independent family of random variables. If number of customers is strictly positive, then the transition intensities are as depicted in Figure 1.

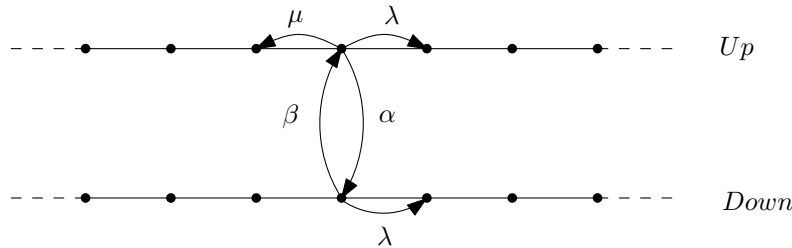


Figure 1: Transitions of Model 1: Unreliable single server.

Otherwise, if the number of customers is 0, then the transition intensities are similar, except there is no transition from $(0, Up)$ to $(-1, Up)$.

Model 2 consists of 2 servers. The customers arrive to the reliable server 2 according to a Poisson process with intensity λ and are served there with intensity μ . After being served they are directed to the unreliable server 1, which is exactly unreliable single server system described in Model 1. The service intensity at both systems is μ . After being served at server 2 the customer leaves the network with probability p and with probability $1 - p$ it is rerouted back to join the queue at server 2. The system is depicted in Figure 2.

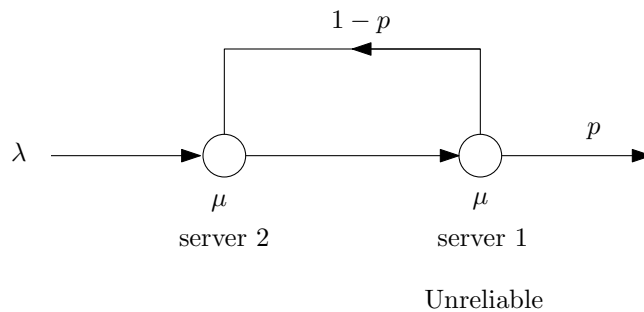


Figure 2: Model 2.

Let $X^{(1)}(t)$ denotes the number of customers present at the server in Model 1 at time $t \geq 0$, either waiting or in service, and let $\sigma(t) \in \{Up, Down\}$ denotes the status of the server. Similarly for Model 2: $X^{(2)}(t)$ denotes the number of customers present at server 1, $Y^{(2)}(t)$ the number of customers at server 2 and $\sigma(t)$ the status of the unreliable server 1, all at time $t \geq 0$. We denote the process of Model 1 by $\mathbf{Z}^{(1)} = \{(X(t)^{(1)}, \sigma(t)), t \geq 0\}$ and the process of Model 2 by $\mathbf{Z}^{(2)} = \{(X(t)^{(2)}, Y^{(2)}(t), \sigma(t)), t \geq 0\}$. The state space of $\mathbf{Z}^{(1)}$ is $E^{(1)} = \{(x, \sigma), x \in \mathbb{N}, \sigma \in \{Up, Down\}\}$ and the state space of $\mathbf{Z}^{(2)}$ is $E^{(2)} = \{(x, y, \sigma), x, y \in \mathbb{N}, \sigma \in \{Up, Down\}\}$. In the following, the superscripts $^{(1)}, ^{(2)}$ denote that constant/number is associated with Model 1 or Model 2 respectively. To have concise notation, we identify $\{Up, Down\}$ with $\{U, D\}$. Throughout the paper we assume that $p > 0$ and that the system is not trivial, i.e.

$$\lambda > 0, \quad \mu > 0, \quad \alpha > 0, \quad \beta > 0.$$

Moreover we assume that

$$\lambda < \frac{\beta}{\alpha + \beta} \mu p, \tag{1}$$

which implies that both systems are stable (for Model 1 we mean that condition holds with $p = 1$). Actually for Model 1 and for Model 2 with $p = 1$ it is ‘‘if and only if’’ condition. See Lemma 3.1 for details.

We can consider $\frac{\beta}{\alpha + \beta} \mu$ as the effective service rate of the unreliable server. Stability in this case means that we have the unique stationary distribution, which we denote by π . It will be clear from the context whether π is associated with Model 1 or Model 2.

By $n_k \sim m_k$ we mean that $n_k/m_k \rightarrow 1$ as $k \rightarrow \infty$. In this paper, ‘‘the exact asymptotic of π ’’ means deriving an asymptotic expression for $\pi(k, \sigma)$ (Model 1) or $\pi(k, y, \sigma)$ (Model 2), that is, deriving an expression of the form $\pi(k, \sigma) \sim f_k$ or $\pi(k, y, \sigma) \sim g_k$.

It is convenient to define some constants in this place. Let $s_p = (\mu p - \lambda - \beta - \alpha)^2 + 4\alpha\mu p$.

Define also

$$\gamma_p = \frac{2\lambda}{\lambda + \beta + \mu p + \alpha - \sqrt{s_p}} \in (0, 1)$$

and

$$G = \left(\frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} + \frac{2\alpha\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}} \right).$$

2.2 Results for Model 1

Our first result is following.

Proposition 2.1. *Assume that (1) holds with $p = 1$. For the unreliable server system (Model 1) we have*

$$\begin{aligned} \pi(k, Up) &\sim C^{(1)}(Up)\gamma_1^k, \\ \pi(k, Down) &\sim C^{(1)}(Down)\gamma_1^k, \end{aligned}$$

where

$$C^{(1)}(Up) = \frac{\eta^{(1)}}{\tilde{d}^{(1)}} \frac{1}{G} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} \neq 0, \quad C^{(1)}(Down) = \frac{\eta^{(1)}}{\tilde{d}^{(1)}} \frac{\alpha}{G} \neq 0,$$

$\tilde{d}^{(1)}$ is defined in (11) and $\eta^{(1)}$ is equal to η in (6) defined for appropriate process.

Remark: Comparison with standard $M/M/1$. Consider standard $M/M/1$ queue with arrival and service intensities λ_0, μ_0 given by $\lambda_0 = \lambda$, $\mu_0 = \frac{\beta}{\alpha+\beta}\mu$, i.e. both systems have the same effective rates. We can compare the behaviour of both system for large number of customers k . For the $M/M/1$ system, the stationary distribution π_0 is known exactly:

$$\pi_0(l) = \pi_0(0) \left(\frac{\lambda_0}{\mu_0} \right)^l = \frac{\mu_0}{\mu_0 - \lambda_0} \left(\frac{\lambda_0}{\mu_0} \right)^l = \frac{\beta\mu}{\beta\mu - (\alpha + \beta)\lambda} \left(\frac{\alpha + \beta}{\beta} \cdot \frac{\lambda}{\mu} \right)^l.$$

Elementary calculation shows that under (1) with $p = 1$

$$\gamma_1 = \frac{2\lambda}{\lambda + \beta + \alpha + \mu - \sqrt{s_1}} \geq \frac{\alpha + \beta}{\beta} \cdot \frac{\lambda}{\mu}.$$

It means for big k , that π_0 is stochastically greater than π . We also note that for π_0 only the ratio of $\alpha + \beta$ and β matters, but this is not true for π .

Remark: Limits as the breaking probability α goes to 0. The limit of γ_1 as $\alpha \rightarrow 0$ has twofold nature. It depends on the sign of the difference $\mu - (\lambda + \beta)$:

$$\lim_{\alpha \rightarrow 0} \gamma_1 = \frac{2\lambda}{\beta + \mu + \lambda - \sqrt{(\mu - \lambda - \beta)^2}} = \begin{cases} \frac{\lambda}{\mu} & \text{if } \mu < \lambda + \beta. \\ \frac{\lambda}{\lambda + \beta} & \text{if } \mu > \lambda + \beta, \end{cases}$$

Thus, to calculate the limits of the constants $C^{(1)}(Up)$ and $C^{(1)}(Down)$ as $\alpha \rightarrow 0$, we consider two cases separately:

- $\mu < \lambda + \beta$

$$\begin{aligned} \lim_{\alpha \rightarrow 0} G &= \lambda + \beta - \mu, & \lim_{\alpha \rightarrow 0} \tilde{d}^{(1)} &= \frac{\mu - \lambda}{C}, \\ \lim_{\alpha \rightarrow 0} C^{(1)}(Up) &= \frac{\eta^{(1)}C}{\mu - \lambda}, & \lim_{\alpha \rightarrow 0} C^{(1)}(Down) &= 0. \end{aligned} \tag{2}$$

- $\mu > \lambda + \beta$

$$\begin{aligned} \lim_{\alpha \rightarrow 0} G &= \frac{\beta(\mu - \lambda - \beta)}{\lambda + \beta}, & \lim_{\alpha \rightarrow 0} \tilde{d}^{(1)} &= \frac{\lambda + \beta}{C}, \\ \lim_{\alpha \rightarrow 0} C^{(1)}(Up) &= 0, & \lim_{\alpha \rightarrow 0} C^{(1)}(Down) &= 0. \end{aligned}$$

Of course, from Proposition 2.1 we always have:

$$\lim_{\alpha \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\pi(k, Up)}{C^{(1)}(Up)\gamma_1^k} = 1.$$

Furthermore, if $\mu < \lambda + \beta$, then via (2) we have

Corollary 2.2. *Assume (1) with $p = 1$ and $\mu < \lambda + \beta$. Then for Model 1 we have*

$$\lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow 0} \frac{\pi(k, Up)}{C^{(1)}(Up)\gamma_1^k} = 1.$$

Note, that in this case $\lim_{\alpha \rightarrow 0} \gamma_1 = \frac{\lambda}{\mu}$, thus $(\frac{\lambda}{\mu})^k$ is the asymptotic for the “limiting system”. However, if $\mu > \lambda + \beta$, then $\lim_{\alpha \rightarrow 0} \gamma_1 = \frac{\lambda}{\lambda + \beta}$, but $(\frac{\lambda}{\lambda + \beta})^k$ is not a correct asymptotic, because both constants $C^{(1)}(Up)$ and $C^{(1)}(Down)$ have limits 0. In this case the asymptotic for the “limiting system” cannot be recovered from Proposition 2.1.

However, using matrix geometric approach, we have the following result.

Proposition 2.3. *Assume (1) with $p = 1$ and $\mu > \lambda + \beta$. Then for Model 1 we have*

$$\pi(k, Up) \sim C^{(1)}(Up)\gamma_1^k + C(Up)\gamma^k$$

and

$$\lim_{k \rightarrow \infty} \lim_{\alpha \rightarrow 0} \frac{\pi(k, Up)}{C(Up)\gamma^k} = 1,$$

where

$$\gamma = \frac{2\lambda}{\lambda + \beta + \mu + \alpha + \sqrt{s_1}}, \quad C(Up) > 0.$$

Note, that γ and γ_1 differ only by the sign at $\sqrt{s_1}$ and that for $\mu > \lambda + \beta$ we also have $\lim_{\alpha \rightarrow 0} \gamma = \frac{\lambda}{\mu}$, so that the asymptotic for small α is still $(\frac{\lambda}{\mu})^k$ (we do not have any information about constant a $C(Up)$).

Remark: Large deviation path. Propositions 2.1 and 2.3 suggest two different large deviations paths for small α . The way a large deviation path appears depends on the sign of the difference $\mu - (\lambda + \beta)$:

- For $\mu < \lambda + \beta$ the most probable path leading to a big queue is to be more often in the Up status, and to accumulate a lot of customers, because service rate is not big enough. This is exactly the way it appears in standard M/M/1 queue.
- For $\mu > \lambda + \beta$ the service rate μ is big enough, so that large deviation path does not appear in the standard way: in this case the most probable situation is, that a lot of customers join the queue, when the server is almost entirely in the $Down$ status.

We illustrate this behaviour in Figure 3 below: The plots are for both cases: $\mu < \lambda + \beta$ and $\mu > \lambda + \beta$; x -axis is the step number, y -axis is the number of customers, 'dot' denotes that the server was in Up status and 'cross' denotes that the server was in $Down$ status. For each case there are two plots: one with steps ranging from 0 to 70000 and second with steps chosen in such the way, so that a large deviation path is well depicted. In case $\mu < \lambda + \beta$ there is also depicted a line with slope of the large deviation path given by $\tilde{d}^{(1)}$ in (11).

2.3 Results for Model 2

For the general Model 2 with $p \in (0, 1)$ we have the following result about exact asymptotic, although we do not have knowledge about the constants.

Proposition 2.4. *Assume that (1) holds. For Model 2 we have*

$$\begin{aligned} \pi(k, y, Up) &\sim C(Up, y)\gamma_p^k, \\ \pi(k, y, Down) &\sim C(Down, y)\gamma_p^k, \end{aligned}$$

where $C(Up, y) > 0$, $C(Down, y) > 0$.

Remark: Limits as the breaking probability α goes to 0. The limit of γ_p as $\alpha \rightarrow 0$ has again twofold nature, it depends on the sign of the difference $\mu p - (\lambda + \beta)$.

$$\lim_{\alpha \rightarrow 0} \gamma_p = \frac{2\lambda}{\beta + \mu p + \lambda - \sqrt{(\mu p - \lambda - \beta)^2}} = \begin{cases} \frac{\lambda}{\mu p} & \text{if } \mu p < \lambda + \beta. \\ \frac{\lambda}{\lambda + \beta} & \text{if } \mu p > \lambda + \beta, \end{cases}$$

Unfortunately, we do not have any information about constants $C(U_p, y), C(Down, y)$. In particular, we do not know if the limits of them are positive (in Model 1 in one case the constant $C^{(1)}(U_p)$ was positive, while in the other it was 0). It means that from Proposition 2.4 we cannot recover the asymptotic for the “limiting system”.

For Model 2 with $p = 1$ (which is the tandem of reliable and unreliable servers) we have the following exact asymptotic result.

Proposition 2.5. *Assume that (1) holds with $p = 1$. For the tandem system with unreliable server 1 (i.e. Model 2 with $p = 1$) we have*

$$\begin{aligned} \pi(k, y, U_p) &\sim C^{(2)}(U_p) \left(\frac{\lambda}{\mu}\right)^y \gamma_1^k, \\ \pi(k, y, Down) &\sim C^{(2)}(Down) \left(\frac{\lambda}{\mu}\right)^y \gamma_1^k, \end{aligned}$$

where

$$\begin{aligned} C^{(2)}(U_p) &= \frac{\eta^{(2)}}{\tilde{d}^{(2)}} \frac{1}{G} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} \cdot B \neq 0, & C^{(2)}(Down) &= \frac{\eta^{(2)}}{\tilde{d}^{(2)}} \frac{\alpha}{G} \cdot B \neq 0, \\ B &= 1 - \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2\mu}, \end{aligned}$$

$\tilde{d}^{(2)}$ is defined in (12) and $\eta^{(2)}$ is equal to η in (6) defined for appropriate process.

Remark: Limits as the breaking probability α goes to 0. Note that $C^{(1)}(U_p), C^{(2)}(U_p)$, and $C^{(1)}(Down), C^{(2)}(Down)$ differ only by a factor of B and $\frac{\eta^{(1)}}{\tilde{d}^{(1)}}$ or $\frac{\eta^{(2)}}{\tilde{d}^{(2)}}$. We can rewrite $C^{(2)}(U_p) = \frac{\eta^{(2)}}{\tilde{d}^{(2)}} \frac{\tilde{d}^{(1)}}{\eta^{(1)}} \cdot B \cdot C^{(1)}(U_p)$ and similarly with $C^{(2)}(Down)$. Moreover, $\tilde{d}^{(1)}$ and $\tilde{d}^{(2)}$ are different, but they have the same limits as $\alpha \rightarrow 0$. Thus, based on results for Model 1 and calculating the limit of B , we have two cases:

- $\mu < \lambda + \beta$

$$\begin{aligned} \lim_{\alpha \rightarrow 0} B &= 0, & \lim_{\alpha \rightarrow 0} \tilde{d}^{(2)} &= \frac{\mu - \lambda}{C}, \\ \lim_{\alpha \rightarrow 0} C^{(1)}(U_p) &= \frac{\eta^{(1)} C}{\mu - \lambda}, & \lim_{\alpha \rightarrow 0} C^{(1)}(Down) &= 0. \\ \lim_{\alpha \rightarrow 0} C^{(2)}(U_p) &= 0, & \lim_{\alpha \rightarrow 0} C^{(2)}(Down) &= 0. \end{aligned}$$

- $\mu > \lambda + \beta$

$$\begin{aligned} \lim_{\alpha \rightarrow 0} B &= \frac{\mu - (\alpha + \beta)}{\mu}, & \lim_{\alpha \rightarrow 0} \tilde{d}^{(2)} &= \frac{\lambda + \beta}{C}, \\ \lim_{\alpha \rightarrow 0} C^{(1)}(Up) &= 0, & \lim_{\alpha \rightarrow 0} C^{(1)}(Down) &= 0. \\ \lim_{\alpha \rightarrow 0} C^{(2)}(Up) &= 0, & \lim_{\alpha \rightarrow 0} C^{(2)}(Down) &= 0. \end{aligned}$$

It means that from Proposition 2.5 we cannot recover the asymptotic for the “limiting system”. For $\mu > \lambda + \beta$ it is because limits of both constants $C^{(1)}(Up)$ and $C^{(1)}(Down)$ (and therefore $C^{(2)}(Up)$ and $C^{(2)}(Down)$) are 0. In case $\mu < \lambda + \beta$ although the limit of $C^{(1)}(Up)$ is strictly positive, the limit of $C^{(2)}(Up)$ is again 0, because of the limit of B .

3 Proofs

3.1 Uniformization and stability

We find it more convenient to work with the embedded discrete-time Markov chain. We denote its kernel by \mathbf{P} . Of course it has the same stationary distribution π . We make uniformization by fixing some C such that $C \geq \lambda + \mu + \alpha + \beta$.

Lemma 3.1. *Model 1 and Model 2 with $p = 1$ are ergodic if and only if $\lambda < \frac{\beta}{\alpha + \beta}\mu$. Moreover, condition $\lambda < \frac{\beta}{\alpha + \beta}\mu p$ is sufficient for stability of Model 2 with $p \in (0, 1)$.*

Proof.

- Model 1:

If we order states in the following way

$$(0, U) \prec (0, D) \prec (1, U) \prec (1, D) \prec (2, U) \prec (2, D) \prec \dots$$

we can rewrite

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1^{(0)} & \mathbf{P}_0 & & & & & \\ \mathbf{P}_2 & \mathbf{P}_1 & \mathbf{P}_0 & & & & \\ & \mathbf{P}_2 & \mathbf{P}_1 & \mathbf{P}_0 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots \end{pmatrix}, \quad (3)$$

where

$$\begin{aligned} \mathbf{P}_1^{(0)} &= \begin{pmatrix} 1 - \frac{\alpha + \lambda}{C} & \frac{\alpha}{C} \\ \frac{\beta}{C} & 1 - \frac{\lambda + \beta}{C} \end{pmatrix}, \quad \mathbf{P}_0 = \begin{pmatrix} \frac{\lambda}{C} & 0 \\ 0 & \frac{\lambda}{C} \end{pmatrix}, \\ \mathbf{P}_2 &= \begin{pmatrix} \frac{\mu}{C} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{P}_1 = \begin{pmatrix} 1 - \frac{\mu + \lambda + \alpha}{C} & \frac{\alpha}{C} \\ \frac{\beta}{C} & 1 - \frac{\lambda + \beta}{C} \end{pmatrix}. \end{aligned} \quad (4)$$

Therefore \mathbf{P} is *quasi-birth-and-death process* (QBD process) with inter-level generator

$$\mathbf{G} = \mathbf{P}_0 + \mathbf{P}_1 + \mathbf{P}_2 = \begin{pmatrix} 1 - \frac{\alpha}{C} & \frac{\alpha}{C} \\ \frac{\beta}{C} & 1 - \frac{\beta}{C} \end{pmatrix}.$$

From Neuts [9] (Theorem 3.1.1), we have that if inter-level generator matrix \mathbf{G} is irreducible, then the process is positive recurrent if and only if

$$\rho \cdot \mathbf{P}_0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} < \rho \cdot \mathbf{P}_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where ρ is the stationary probability vector of \mathbf{G} .

We have $\rho = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$, $\rho \cdot \mathbf{P}_0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \cdot \frac{1}{C}$ and $\rho \cdot \mathbf{P}_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\beta}{\alpha+\beta} \mu \cdot \frac{1}{C}$ which finishes the proof.

- Model 2 with $p = 1$:

The server 2 is stable if and only if $\lambda < \mu$. The output of server 2 is the Poisson process with intensity λ (Burke's Theorem, see Burke [2] for details). In previous case we proved that unreliable server 1 with arrival rate λ and service rate μ is stable if and only if $\lambda < \frac{\beta}{\alpha+\beta} \mu$. Of course the second condition implies first.

- Model 2 with $p \in (0, 1)$:

Later, in Section 3.5.2, the harmonic function of the (so-called) free process is derived. By Proposition 3.2 it gives the following asymptotic (actually this was given in Proposition 2.4), for any $k \in \mathbb{N}$ and $\sigma \in \{U, D\}$

$$\pi(k, y, \sigma) \sim C(\sigma, y) \gamma_p^k.$$

It is enough to show, that $\lambda < \frac{\beta}{\alpha+\beta} \mu p$ implies $\sum_k \pi(k, y, \sigma) < 1$, or equivalently, that $\gamma_p < 1$. It can be easily checked, that $\gamma_p < 1$ if and only if $\lambda < \frac{\beta}{\alpha+\beta} \mu p$, thus this is a sufficient condition. □

Remark. Consider system similar to Model 2, but with 2 reliable servers (i.e. standard Jackson network) with service rate at server 2: $\mu_2 = \mu$ and service rate at server 1: $\mu_1 = \frac{\beta}{\alpha+\beta} \mu$ (which is the effective service rate of the unreliable server). Then, solving traffic equation and using standard stability conditions for Jackson networks, we have that the system is stable if and only if $\frac{\lambda}{\mu_1 p} < 1$, i.e. $\lambda < \frac{\beta}{\alpha+\beta} \mu p$. It suggests that (1) is the necessary stability condition for Model 2 with $p \in (0, 1)$.

3.2 Markov Additive Structure and result of Adan, Foley and McDonald [1]

Tools used in this paper fall into the framework of Adan, Foley and McDonald [1], where Markov additive structure is needed. Let $Z_n = (X_n, Y_n)$ be a Markov process with state space $\mathbb{Z}^k \times \mathcal{E}$, where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. If the transitions are invariant with respect to the translations on $x \in \mathbb{Z}^k$, i.e.:

$$\mathbf{P}((x, y), (x', y')) = \mathbf{P}((0, y), (x' - x, y')) \quad \text{for all } x, x' \in \mathbb{Z}^k \text{ and } y, y' \in \mathcal{E},$$

then it is called a Markov additive process, X_n is its additive part, Y_n is a Markovian part.

Processes $\mathbf{Z}^{(1)}$ and $\mathbf{Z}^{(2)}$ defined in Section 2.1 are Markov additive if we remove the boundaries and let the transitions to be shift invariant relative to the first coordinate. Abusing notation, we denote state spaces of these processes with the same symbols, respectively, $E^{(1)} = \mathbb{Z} \times \{U, D\}$ and $E^{(2)} = \mathbb{Z} \times \mathbb{N} \times \{U, D\}$.

By harmonic function of Markov chain with transition matrix \mathbf{P} we mean the right eigenvector h associated with eigenvalue 1, i.e. such that $\mathbf{P}h = h$. From [1] we can deduce the following.

Proposition 3.2. *Consider Markov process $\{X_t\}_{t \geq 0}$ with stationary distribution π and state space $E = \{(k, A) : k \in \mathbb{Z}, A \in \mathbb{Z}^n\}$. Let $\Delta \subset E$ and let \mathbf{K}^∞ be the kernel of the free process, which is shift invariant relative to first coordinate. Let*

$$\mathbf{K}((k, A), (k', A')) = \mathbf{K}^\infty((k, A), (k', A')) h(k', A') / h(k, A)$$

be the kernel of so-called twisted free process, where h is the harmonic function of \mathbf{K}^∞ . If

$$\sum_{(k,A) \in \Delta} \pi(k,A)h(k,A) < \infty, \quad (5)$$

then

$$\pi(l,A) \sim \frac{\eta\varphi(A)}{\tilde{d} h(l,A)},$$

where \tilde{d} is the stationary horizontal drift and

$$\eta \equiv \sum_{(x',A') \in \Delta} \pi(x',A')h(x',A')\mathcal{H}(x',A'). \quad (6)$$

$\mathcal{H}(x',A')$ is the probability that twisted free process starting from (x',A') never hits $(E \setminus \Delta)^C$.

3.3 Proof of Proposition 2.1

3.3.1 The free process.

We have to define $\Delta \subset E^{(1)}$ and a Markov additive process embedded in original one, so that it is shift invariant outside the boundary Δ . We want the process to be additive in the first coordinate and we want the second coordinate to be the Markovian part. Thus, as a boundary we can take $\Delta = \{(0,Up) \cup (0,Down)\}$. Let us denote the transition kernel of this process by \mathbf{K}^∞ . Being Markov additive in the first coordinate means $\mathbf{K}^\infty((m,\sigma), (z+m,\sigma')) = \mathbf{K}^\infty((0,\sigma), (z,\sigma'))$, where

$$\mathbf{K}^\infty((0,\sigma), (z,\sigma')) = \begin{cases} \frac{\lambda}{C} & \text{for } z = 1 \text{ and } \sigma' = \sigma \\ \frac{\mu}{C} & \text{for } z = -1 \text{ and } \sigma' = \sigma = U \\ \frac{\alpha}{C} & \text{for } z = 0, \sigma = U \text{ and } \sigma' = D \\ \frac{\beta}{C} & \text{for } z = 0, \sigma = D \text{ and } \sigma' = U \\ 1 - \frac{\lambda+\beta}{C} & \text{for } z = 0, \text{ and } \sigma' = \sigma = D \\ 1 - \frac{\alpha+\mu+\lambda}{C} & \text{for } z = 0, \text{ and } \sigma' = \sigma = U \end{cases}$$

Since we have removed the boundary, the free process walks over all $\mathbb{Z} \times \{Up, Down\}$.

3.3.2 Feynman-Kac kernel

With the free process we associate the following Feynman-Kac kernel:

$\mathbf{K}_\theta(\sigma,\sigma') = \sum_z \mathbf{K}^\infty((0,\sigma), (z,\sigma'))e^{\theta z}$, where $\sigma,\sigma' \in \{U,D\}$. We have

$$\mathbf{K}_\theta = \begin{pmatrix} \frac{\lambda}{C}e^\theta + 1 - \frac{\alpha+\mu+\lambda}{C} + \frac{\mu}{C}e^{-\theta} & \frac{\alpha}{C} \\ \frac{\beta}{C} & \frac{\lambda}{C}e^\theta + 1 - \frac{\lambda+\beta}{C} \end{pmatrix}.$$

\mathbf{K}_θ has two eigenvalues

$$k_{1,2}(\theta) := \frac{1}{C} \left(C - \frac{\alpha}{2} - \frac{\beta}{2} - \frac{\mu}{2} - \lambda + \frac{\mu e^{-\theta}}{2} + \lambda e^\theta \pm \frac{1}{2} \sqrt{(\mu e^{-\theta} - \alpha - \beta - \mu)^2 - 4\mu\beta(1 - e^{-\theta})} \right),$$

We are interested in the larger eigenvalue, i.e. we only consider k_1 . We want the largest eigenvalue to be equal to 1, i.e. $k_1(\theta) = 1$. Set: $t = e^\theta$. It means

$$C - \frac{1}{2}(\alpha + \beta + \mu) - \lambda + \frac{\mu}{2t} + \lambda t + \frac{1}{2}\sqrt{\left(\frac{\mu}{t} - \alpha - \beta - \mu\right)^2 - 4\mu\beta\left(1 - \frac{1}{t}\right)} = C.$$

Equivalently,

$$\sqrt{\left(\frac{\mu}{t} - \alpha - \beta - \mu\right)^2 - 4\mu\beta\left(1 - \frac{1}{t}\right)} = \alpha + \beta + \mu + 2\lambda - \frac{\mu}{t} - 2\lambda t \quad (7)$$

To find the solution of the above equation, we have to solve

$$W(t) := \lambda^2 t^3 - \lambda(\beta + \alpha + 2\lambda + \mu)t^2 + (\lambda(\alpha + 2\mu + \beta + \lambda) + \mu\beta)t - \mu(\alpha + \beta) = 0. \quad (8)$$

Of course $W(1) = 0$, thus

$$W(t) = (t - 1)(\lambda^2 t^2 - \lambda(\beta + \alpha + \lambda + \mu)t + \mu(\lambda + \beta)).$$

We obtain two solutions:

$$t_1 = \frac{\lambda + \beta + \mu + \alpha + \sqrt{s_1}}{2\lambda}, \quad t_2 = \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2\lambda}. \quad (9)$$

Note, that $t_2 = \gamma_1^{-1}$. We want the right hand side of (7) to be positive, what is equivalent to

$$2\lambda t^2 - (\alpha + \beta + \mu + 2\lambda)t + \mu < 0.$$

However, one can check (noting, that $s_1 = (\mu + \lambda + \beta + \alpha)^2 - 4\mu(\lambda + \beta)$) that t_1 is not the solution of (7), because then the right hand side of the equation is negative.

3.3.3 The harmonic function of the free process

Lemma 3.3. *The harmonic function of the free process is the following:*

$$h(x, U) = \left(\frac{1}{\gamma_1}\right)^x, \quad h(x, D) = \left(\frac{1}{\gamma_1}\right)^x \cdot \frac{2\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}.$$

Proof. We want to find the harmonic function for free process of the form $h(z, \sigma) = t_2^z e^{\theta\sigma}$, where t_2 is such that the largest eigenvalue of Feynman-Kac kernel is equal to one, i.e.

$$h(z, \sigma) = \left(\frac{1}{\gamma_1}\right)^z e^{\theta\sigma}.$$

For h to be the harmonic function for free process we have to have

$$\forall(z \in \mathbb{Z}) \quad \sum_{x, \sigma} \mathbf{K}^\infty((z, U), (x, \sigma))h(x, \sigma) = h(z, U) \quad \text{and} \quad \sum_{x, \sigma} \mathbf{K}^\infty((z, D), (x, \sigma))h(x, \sigma) = h(z, D). \quad (10)$$

First part of (10) means

$$\sum_{x, \sigma} \mathbf{K}^\infty((z, U), (x, \sigma))h(x, \sigma) = \frac{\lambda}{C} \left(\frac{1}{\gamma_1}\right)^{(z+1)} e^{\theta_U} + \frac{\mu}{C} \left(\frac{1}{\gamma_1}\right)^{(z-1)} e^{\theta_U} + \frac{\alpha}{C} t_2^z e^{\theta_D}$$

$$+ \left(1 - \frac{\lambda + \mu + \alpha}{C}\right) \left(\frac{1}{\gamma_1}\right)^z e^{\theta_U} = h(z, U) = \left(\frac{1}{\gamma_1}\right)^z e^{\theta_U}$$

and equivalently

$$e^{\theta_U} \left[1 - \frac{\lambda}{C} \frac{1}{\gamma_1} - \frac{\mu}{C} \gamma_1 - \left(1 - \frac{\lambda + \mu + \alpha}{C}\right)\right] = \frac{\alpha}{C} e^{\theta_D},$$

i.e.

$$e^{\theta_U} [\lambda + \mu + \alpha - \lambda \frac{1}{\gamma_1} - \mu \gamma_1] = \alpha e^{\theta_D}.$$

Second part of (10) means

$$\begin{aligned} \sum_{x, I} \mathbf{K}^\infty((z, D), (x, I)) h(x, I) &= \frac{\lambda}{C} \left(\frac{1}{\gamma_1}\right)^{(z+1)} e^{\theta_D} + \frac{\beta}{C} \left(\frac{1}{\gamma_1}\right)^z e^{\theta_U} + \left(1 - \frac{\lambda + \beta}{C}\right) \left(\frac{1}{\gamma_1}\right)^z e^{\theta_D} \\ &= h(z, D) = e^{\theta_D} \left(\frac{1}{\gamma_1}\right)^z \end{aligned}$$

and equivalently

$$e^{\theta_D} \left[1 - \frac{\lambda}{C} \frac{1}{\gamma_1} - \left(1 - \frac{\lambda + \beta}{C}\right)\right] = \frac{\beta}{C} e^{\theta_U},$$

i.e.

$$e^{\theta_D} \left[\lambda + \beta - \lambda \left(\frac{1}{\gamma_1}\right)\right] = \beta e^{\theta_U}.$$

Putting these conditions together we have:

$$\begin{cases} e^{\theta_U} [\lambda + \mu + \alpha - \lambda \frac{1}{\gamma_1} - \mu \gamma_1] = \alpha e^{\theta_D}, & (i) \\ e^{\theta_D} [\lambda + \beta - \lambda \frac{1}{\gamma_1}] = \beta e^{\theta_U}. & (ii) \end{cases}$$

One of e^{θ_U} or e^{θ_D} can be arbitrary, set $e^{\theta_U} = 1$. From (ii) we have

$$e^{\theta_D} = \frac{\beta}{\lambda + \beta - \lambda \frac{1}{\gamma_1}} = \frac{2\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}.$$

□

3.3.4 The twisted free process.

With the harmonic function of the free process we can define the h -transform (or twisted kernel) in the following way: $\mathcal{K}((m, \sigma), (z + m, \sigma')) = \mathcal{K}((0, \sigma), (z, \sigma')) = \mathbf{K}^\infty((0, \sigma), (z, \sigma')) \frac{h(z, \sigma')}{h(0, \sigma)}$, i.e.

$$\mathcal{K}((0, \sigma), (z, \sigma')) = \begin{cases} \frac{\lambda}{C} \frac{h(1, U)}{h(0, U)} & = \frac{1}{C} \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2} & \text{for } z = 1, \\ \frac{\mu}{C} \frac{h(-1, U)}{h(0, U)} & = \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} & \text{for } z = -1 \text{ and } \sigma = \sigma' = U \\ \frac{\alpha}{C} \frac{h(0, D)}{h(0, U)} & = \frac{1}{C} \frac{2\alpha\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}} & \text{for } z = 0, \sigma = U \text{ and } \sigma' = D, \\ \frac{\beta}{C} \frac{h(0, U)}{h(0, D)} & = \frac{1}{C} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} & \text{for } z = 0, \sigma = D \text{ and } \sigma' = U, \\ \left(1 - \frac{\lambda + \beta}{C}\right) \frac{h(0, D)}{h(0, D)} & = 1 - \frac{\lambda + \beta}{C} & \text{for } z = 0, \text{ and } \sigma = \sigma' = D, \\ \left(1 - \frac{\lambda + \alpha + \mu}{C}\right) \frac{h(0, U)}{h(0, U)} & = 1 - \frac{\lambda + \alpha + \mu}{C} & \text{for } z = 0, \text{ and } \sigma = \sigma' = U. \end{cases}$$

The transition diagram is simply a reweighting of the transitions in Figure 1.

Now we are interested in the stationary distribution of the Markovian part of the twisted free process, call it \mathcal{K}_2 , which state space is $\{U, D\}$. We have:

$$\begin{aligned}\mathcal{K}_2(U, D) &= \mathcal{K}((0, U), (0, D)), \\ \mathcal{K}_2(D, U) &= \mathcal{K}((0, D), (0, U)), \\ \mathcal{K}_2(U, U) &= \mathcal{K}((0, U), (0, U)) + \mathcal{K}((0, U), (-1, U)) + \mathcal{K}((, U), (1, U)) = 1 - \mathcal{K}_2(U, D), \\ \mathcal{K}_2(D, D) &= \mathcal{K}((0, D), (0, D)) + \mathcal{K}((0, D), (1, D)) = 1 - \mathcal{K}_2(D, U).\end{aligned}$$

For 2-states Markov chain with transition matrix $\begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix}$ the stationary distribution is $\pi(1) = p_2/(p_1 + p_2)$, $\pi(2) = 1 - \pi(1) = p_1/(p_1 + p_2)$.

Let φ be the stationary distribution of \mathcal{K}_2 . We have

$$\varphi(U) = \frac{\mathcal{K}_2(D, U)}{\mathcal{K}_2(D, U) + \mathcal{K}_2(U, D)}, \quad \varphi(D) = \frac{\mathcal{K}_2(U, D)}{\mathcal{K}_2(D, U) + \mathcal{K}_2(U, D)}.$$

Note that $G = C(\mathcal{K}_2(D, U) + \mathcal{K}_2(U, D))$ and rewrite

$$\begin{aligned}\varphi(U) &= \frac{1}{G} \mathcal{K}_2(D, U) = \frac{1}{G} \cdot \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2}, \\ \varphi(D) &= \frac{1}{G} \mathcal{K}_2(U, D) = \frac{1}{G} \cdot \frac{2\alpha\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}.\end{aligned}$$

Next we have to compute the stationary horizontal drift of the twisted free process:

$$\begin{aligned}\tilde{d}^{(1)} &= \varphi(U)[\mathcal{K}((x, U), (x+1, U)) - \mathcal{K}((x, U), (x-1, U))] + \varphi(D)\mathcal{K}((x, D), (x+1, D)) \\ &= \frac{1}{C} \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2} (\varphi(U) + \varphi(D)) - \varphi(U) \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \\ &= \frac{1}{C} \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2} \cdot 1 - \frac{1}{G} \cdot \frac{1}{C} \cdot \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \\ &= \frac{1}{C} \left(\frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2} - \frac{1}{G} \cdot \lambda\mu \cdot \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \right).\end{aligned}\tag{11}$$

The assertion of Proposition 2.1 follows from the Proposition 3.2, because condition (5) is obviously fulfilled, since the boundary Δ consists only of two states.

3.4 Proof of Proposition 2.3

We use the **matrix geometric approach** following Neuts, [9]. For a discrete time QBD process as one given in (3), Theorem 1.2.1 of Neuts implies that

$$\pi(k, Up) = w_2 \left(e^{\theta_2} \right)^k + w_3 \left(e^{\theta_3} \right)^k,$$

where $e^{\theta_2} \geq e^{\theta_3}$ are the eigenvalues of matrix \mathbf{R} described below. Note that θ_2, θ_3 and w_2, w_3 depend on α . For any $w_2 > 0$ we have that for k big enough the term $(e^{\theta_2})^k$ dominates $(e^{\theta_3})^k$. However, when $\alpha \rightarrow 0$, then $w_2 \rightarrow 0$ (see Remark on page 5), so that $w_3(e^{\theta_3})^k$ is the leading term.

For matrices $\mathbf{P}_2, \mathbf{P}_1, \mathbf{P}_0$ defined in (4) we want to find a matrix $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ fulfilling

$$\mathbf{R} = \mathbf{R}^2 \mathbf{P}_2 + \mathbf{R} \mathbf{P}_1 + \mathbf{P}_0.$$

We have:

$$\begin{aligned} & \mathbf{R}^2 \mathbf{P}_2 + \mathbf{R} \mathbf{P}_1 + \mathbf{P}_0 \\ = & \begin{bmatrix} \frac{(r_{11}^2 + r_{12} r_{21})}{C} \mu & 0 \\ \frac{(r_{21} r_{11} + r_{22} r_{21})}{C} \mu & 0 \end{bmatrix} + \begin{bmatrix} r_{11} \left(1 - \frac{\mu + \lambda + \alpha}{C}\right) + \frac{r_{12} \beta}{C} & \frac{r_{11} \alpha}{C} + r_{12} \left(1 - \frac{\lambda + \beta}{C}\right) \\ r_{21} \left(1 - \frac{\mu + \lambda + \alpha}{C}\right) + \frac{r_{22} \beta}{C} & \frac{r_{21} \alpha}{C} + r_{22} \left(1 - \frac{\lambda + \beta}{C}\right) \end{bmatrix} + \begin{bmatrix} \frac{\lambda}{C} & 0 \\ 0 & \frac{\lambda}{C} \end{bmatrix}, \end{aligned}$$

i.e.

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \frac{(r_{11}^2 + r_{12} r_{21})}{C} \mu + r_{11} \left(1 - \frac{\mu + \lambda + \alpha}{C}\right) + \frac{r_{12} \beta}{C} + \frac{\lambda}{C} & \frac{r_{11} \alpha}{C} + r_{12} \left(1 - \frac{\lambda + \beta}{C}\right) \\ \frac{(r_{21} r_{11} + r_{22} r_{21})}{C} \mu + r_{21} \left(1 - \frac{\mu + \lambda + \alpha}{C}\right) + \frac{r_{22} \beta}{C} & \frac{r_{21} \alpha}{C} + r_{22} \left(1 - \frac{\lambda + \beta}{C}\right) + \frac{\lambda}{C} \end{bmatrix}.$$

One can check that the solution is

$$\mathbf{R} = \begin{bmatrix} \frac{\lambda}{\mu} & \frac{\alpha \lambda}{\mu(\lambda + \beta)} \\ \frac{\lambda}{\mu} & \frac{(\alpha + \mu) \lambda}{(\lambda + \beta) \mu} \end{bmatrix} = \frac{\lambda}{\mu} \begin{bmatrix} 1 & \frac{\alpha}{\lambda + \beta} \\ 1 & \frac{\alpha + \mu}{\lambda + \beta} \end{bmatrix}.$$

Eigenvalues of \mathbf{R} are

$$\begin{aligned} e^{\theta_2} &:= \frac{\lambda + \mu + \alpha + \beta + \sqrt{s_1}}{2(\lambda + \beta)} \cdot \frac{\lambda}{\mu}, \\ e^{\theta_3} &:= \frac{\lambda + \mu + \alpha + \beta - \sqrt{s_1}}{2(\lambda + \beta)} \cdot \frac{\lambda}{\mu}. \end{aligned}$$

It is easy to check that $e^{\theta_2} = \gamma_1$ (what we already have had) and $e^{\theta_3} = \gamma$. Now, as $\alpha \rightarrow 0$ we have $w_2 \rightarrow 0$ and thus $\lim_{\alpha \rightarrow 0} w_3 > 0$ (because both limits cannot be equal to 0). The leading term is $w_3 (e^{\theta_3})^k$, thus the asymptotic of $\pi(k, Up)$ is $C(Up) \gamma^k$, where $C(Up) = w_3$. This finishes the proof.

Remark. Note, that this method does not give us constant w_3 (nor w_2 , but we already have it, it is $C(Up)$).

Remark. While looking for parameter θ in Section 3.3.2 for which the largest eigenvalue of the Feynman-Kac kernel is equal to 1, we encountered equation (8). This equation has two solutions: t_1 and t_2 given in (9). It turns out, that t_1 is not the solution for Feynman-Kac kernel, because the right hand side of (7) (and (8) is simply obtained from (7) by squaring both sides) is negative. However, t_2 is exactly the second term in spectral expansion of π , what we derived in Section 3.4 using matrix geometric approach. We conjecture that this can always be the case for QBD processes.

3.5 Proof of Proposition 2.4

The asymptotic without constants is obtained via Proposition 3.2 by calculating the harmonic function of the free process and by verifying that condition (5), what is done in Section 3.6.2.

3.5.1 The free process.

For Model 2 as the boundary we can take $\Delta = \{(0, y, \sigma), y \in \mathbb{N}, \sigma \in \{Up, Down\}\}$. Then the process outside Δ is shift invariant relative to first coordinate. Define free process $\mathbf{K}^\infty((m, y, \sigma), (z + m, y', \sigma')) = \mathbf{K}^\infty((0, y, \sigma), (z, y', \sigma'))$, where

$$\mathbf{K}^\infty((0, y, \sigma), (z, y', \sigma')) = \begin{cases} \frac{\lambda}{C} & \text{for } z = 0, y' = y + 1, \sigma' = \sigma \\ \frac{\mu}{C}p & \text{for } z = -1, y' = y \text{ and } \sigma' = \sigma = U \\ \frac{\mu}{C}(1-p) & \text{for } z = -1, y' = y + 1 \text{ and } \sigma' = \sigma = U \\ \frac{\mu}{C} & \text{or } z = 1, y' = y - 1 \geq 0 \text{ and } \sigma' = \sigma \\ \frac{\alpha}{C} & \text{for } z = 0, y' = y, \sigma = U \text{ and } \sigma' = D \\ \frac{\beta}{C} & \text{for } z = 0, y' = y, \sigma = D \text{ and } \sigma' = U \\ 1 - \frac{\lambda + \beta}{C} & \text{for } z = 0, y' = y = 0 \text{ and } \sigma' = \sigma = D \\ 1 - \frac{\lambda + \mu + \alpha}{C} & \text{for } z = 0, y' = y = 0 \text{ and } \sigma' = \sigma = U \\ 1 - \frac{\lambda + \mu + \beta}{C} & \text{for } z = 0, y' = y \geq 1 \text{ and } \sigma' = \sigma = D, \\ 1 - \frac{\lambda + 2\mu + \alpha}{C} & \text{for } z = 0, y' = y \geq 1 \text{ and } \sigma' = \sigma = U. \end{cases}$$

After removing the boundary, the free process walks over all $\mathbb{Z} \times \mathbb{N} \times \{Up, Down\}$.

3.5.2 The harmonic function of the free process

Lemma 3.4. *The harmonic function of the free process is following:*

$$h(x, y, U) = \left(\frac{1}{\gamma_p}\right)^{x+y}, \quad h(x, y, D) = \left(\frac{1}{\gamma_p}\right)^{x+y} \frac{2\beta}{\lambda + \beta - \mu p - \alpha + \sqrt{s_p}}.$$

Proof. For the free process we want to find the harmonic function of form $h(x, y, \sigma) = e^{\theta_1 x} e^{\theta_2 y} e^{\theta \sigma}$. For h to be the harmonic function for free process we must have

$$\forall (y \in \mathbb{N}, \sigma \in \{U, D\}) \quad \sum_{x', y', \sigma'} \mathbf{K}^\infty((0, y, \sigma), (x', y', \sigma')) h(x', y', \sigma') = h(0, y, \sigma).$$

For $y = 0, \sigma = U$ we have

$$\begin{aligned} & \mathbf{K}^\infty((0, 0, U), (0, 1, U))h(0, 1, U) + \mathbf{K}^\infty((0, 0, U), (-1, 0, U))h(-1, 0, U) \\ & + \mathbf{K}^\infty((0, 0, U), (-1, 1, U))h(-1, 1, U) + \mathbf{K}^\infty((0, 0, U), (0, 0, D))h(0, 0, D) \\ & + \mathbf{K}^\infty((0, 0, U), (0, 0, U))h(0, 0, U) = h(0, 0, U), \\ & \frac{\lambda}{C}e^{\theta_2}e^{\theta U} + \frac{\mu}{C}pe^{-\theta_1}e^{\theta U} + \frac{\mu}{C}(1-p)e^{-\theta_1}e^{\theta_2}e^{\theta U} + \frac{\alpha}{C}e^{\theta D} + \left(1 - \frac{\lambda + \mu + \alpha}{C}\right)e^{\theta U} = e^{\theta U}, \end{aligned}$$

i.e.

$$e^{\theta U}[\lambda + \mu + \alpha - \lambda e^{\theta_2} - \mu p e^{-\theta_1} - \mu(1-p)e^{-\theta_1}e^{\theta_2}] = \alpha e^{\theta D}$$

Similarly, considering cases $y \geq 1, \sigma = U$; $y = 0, \sigma = D$ and $y \geq 1, \sigma = D$ we obtain following four equations:

$$\begin{cases} e^{\theta_U}[\lambda + \mu + \alpha - \lambda e^{\theta_2} - \mu p e^{-\theta_1} - \mu(1-p)e^{-\theta_1}e^{-\theta_2}] & = \alpha e^{\theta_D}, \\ e^{\theta_U}[\lambda + 2\mu + \alpha - \lambda e^{\theta_2} - \mu p e^{-\theta_1} - \mu(1-p)e^{-\theta_1}e^{-\theta_2} - \mu e^{\theta_1}e^{-\theta_2}] & = \alpha e^{\theta_D}, \\ e^{\theta_D}[\lambda + \beta - \lambda e^{\theta_2}] & = \beta e^{\theta_U}, \\ e^{\theta_D}[\lambda + \mu + \beta - \lambda e^{\theta_2} - \mu e^{\theta_1}e^{-\theta_2}] & = \beta e^{\theta_U}. \end{cases}$$

First two imply that $e^{\theta_1} = e^{\theta_2}$ and then last two are equivalent. We are left with 2 equations and 3 variables, thus we can set $e^{\theta_U} = 1$. Denoting $t = e^{\theta_1} (= e^{\theta_2})$ we have

$$\begin{cases} \lambda + \mu + \alpha - \lambda t - \mu p \frac{1}{t} - \mu(1-p) & = \alpha e^{\theta_D}, & (i) \\ \lambda + \beta - \lambda t & = \frac{\beta}{e^{\theta_D}}. & (ii) \end{cases}$$

Comparing e^{θ_D} from both equations we have

$$\frac{\lambda + \mu + \alpha - \lambda t - \mu p \frac{1}{t} - \mu(1-p)}{\alpha} = \frac{\beta}{\lambda + \beta - \lambda t},$$

$$(\lambda + \mu + \alpha - \lambda t - \mu p \frac{1}{t} - \mu(1-p))(\lambda + \beta - \lambda t) = \alpha\beta.$$

Multiplying both sides by t and noting that $t - 1$ is one of the solutions, we can rewrite it as

$$(t - 1)(\lambda^2 t^2 - \lambda(\mu p + \lambda + \alpha + \beta)t + \mu p(\lambda + \beta)) = 0.$$

Recall that $s_p = (\mu p - \lambda - \beta - \alpha)^2 + 4\alpha\mu p$. The solutions are

$$t_1 = \frac{\mu p + \lambda + \alpha + \beta + \sqrt{s_p}}{2\lambda}, \quad t_2 = \frac{\mu p + \lambda + \alpha + \beta - \sqrt{s_p}}{2\lambda}.$$

Noting that $s_p = (\mu p + \lambda + \beta + \alpha)^2 - 4\mu p(\lambda + \beta)$, it can be easily check $t_1 > 1 \iff \lambda > \frac{\beta}{\alpha + \beta}\mu p$ and $t_2 > 1 \iff \lambda < \frac{\beta}{\alpha + \beta}\mu p$, i.e. only t_2 (which is equal to $\frac{1}{\gamma_p}$) is a valid solution.

From (ii) we have

$$e^{\theta_D} = \frac{\beta}{\lambda + \beta - \lambda t_2} = \frac{2\beta}{\lambda + \beta - \mu p - \alpha + \sqrt{s_p}}.$$

□

3.6 Proof of Proposition 2.5

Since Model 2 with $p = 1$ is the special case of general Model 2, we already have the harmonic function given in Lemma 3.4. We can proceed with the twisted free process.

3.6.1 The twisted free process.

Define the twisted kernel in the following way: $\mathcal{K}((m, y, \sigma), (z+m, y', \sigma')) = \mathcal{K}((0, y, \sigma), (z, y', \sigma')) = \mathbf{K}^\infty((0, y, \sigma), (z, y', \sigma')) \frac{h(z, y', \sigma')}{h(0, y, \sigma)}$

$$= \begin{cases} \frac{\lambda}{C} \frac{h(0, y+1, \sigma)}{h(0, y, \sigma)} & = \frac{1}{C} \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2} & \text{for } z = 0, y' = y + 1, \sigma' = \sigma, \\ \frac{\mu}{C} \frac{h(-1, y, U)}{h(0, y, U)} & = \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} & \text{for } z = -1, y' = y \text{ and } \sigma' = \sigma = U, \\ \frac{\mu}{C} \frac{h(0, y, \sigma)}{h(-1, y+1, \sigma)} & = \frac{\mu}{C} & \text{for } z = 1, y' = y - 1 \geq 0 \text{ and } \sigma' = \sigma, \\ \frac{\alpha}{C} \frac{h(0, y, D)}{h(0, y, U)} & = \frac{1}{C} \frac{2\alpha\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}} & \text{for } z = 0, y' = y, \sigma = U \text{ and } \sigma' = D, \\ \frac{\beta}{C} \frac{h(0, y, U)}{h(0, y, D)} & = \frac{1}{C} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} & \text{for } z = 0, y' = y, \sigma = D \text{ and } \sigma' = U, \\ (1 - \frac{\lambda + \beta}{C}) \frac{h(0, 0, D)}{h(0, 0, D)} & = 1 - \frac{\lambda + \beta}{C} & \text{for } z = 0, y' = y = 0 \text{ and } \sigma' = \sigma = D, \\ (1 - \frac{\lambda + \mu + \alpha}{C}) \frac{h(0, 0, U)}{h(0, 0, U)} & = 1 - \frac{\lambda + \mu + \alpha}{C} & \text{for } z = 0, y' = y = 0 \text{ and } \sigma' = \sigma = U, \\ (1 - \frac{\lambda + \mu + \beta}{C}) \frac{h(0, 0, D)}{h(0, 0, D)} & = 1 - \frac{\lambda + \mu + \beta}{C} & \text{for } z = 0, y' = y \geq 1 \text{ and } \sigma' = \sigma = D, \\ (1 - \frac{\lambda + 2\mu + \alpha}{C}) \frac{h(0, 0, U)}{h(0, 0, U)} & = 1 - \frac{\lambda + 2\mu + \alpha}{C} & \text{for } z = 0, y' = y \geq 1 \text{ and } \sigma' = \sigma = U. \end{cases}$$

The transitions of twisted free process are reweighted transitions of the free process.

We are interested in the stationary distribution of the Markovian part of the twisted free process, call it \mathcal{K}_2 , which state space is $\mathbb{N} \times \{U, D\}$.

Denote:

$$\lambda' = \frac{1}{C} \frac{\lambda + \beta + \mu + \alpha - \sqrt{s_1}}{2}, \mu' = \frac{\mu}{C}, \alpha' = \frac{1}{C} \frac{2\alpha\beta}{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}, \beta' = \frac{1}{C} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2}.$$

The transition of \mathcal{K}_2 are

$$\mathcal{K}_2((y, \sigma), (y', \sigma')) = \begin{cases} \lambda' & \text{for } y' = y + 1 \text{ and } \sigma' = \sigma, \\ \mu' & \text{for } y' = y - 1 \geq 0 \text{ and } \sigma' = \sigma, \\ \alpha' & \text{for } y' = y, \sigma = U \text{ and } \sigma' = D, \\ \beta' & \text{for } y' = y, \sigma = D \text{ and } \sigma' = U, \\ 1 - (\lambda' + \mu' + \alpha') & \text{for } y' = y \geq 1 \text{ and } \sigma' = \sigma = U, \\ 1 - (\lambda' + \mu' + \beta') & \text{for } y' = y \geq 1 \text{ and } \sigma' = \sigma = D, \\ 1 - (\lambda' + \alpha') & \text{for } y' = y = 0 \text{ and } \sigma' = \sigma = U, \\ 1 - (\lambda' + \beta') & \text{for } y' = y = 0 \text{ and } \sigma' = \sigma = D. \end{cases}$$

The stationary distribution of \mathcal{K}_2 is given by:

$$\varphi(y, U) = B \cdot \left(\frac{\lambda'}{\mu'}\right)^y \frac{\beta'}{\alpha' + \beta'}, \quad \varphi(y, D) = B \cdot \left(\frac{\lambda'}{\mu'}\right)^y \frac{\alpha'}{\alpha' + \beta'}, \quad B = 1 - \frac{\lambda'}{\mu'} = 1 - \frac{\lambda + \beta + \mu + \alpha - \sqrt{s}}{2\mu}.$$

Marginally $\mathcal{K}_2(y, \cdot)$ is a birth and death process with birth rate λ' and death rate μ' , the stationary distribution of it is geometric: probability of having k customers equals to $B_1 \left(\frac{\lambda'}{\beta'}\right)^k$ (B_1 is a

normalisation constant). Similarly, $\mathcal{K}_2(\cdot, \sigma)$ is a Markov chain with two states, the stationary distribution of which is: $\frac{\beta'}{\alpha' + \beta'}$ of being in *Up* and $\frac{\alpha'}{\alpha' + \beta'}$ of being in *Down* status. Process $\mathcal{K}_2(y, \sigma)$ is not a product of its marginals, but its stationary distribution is of a product form. This can be checked directly, for example for $y \geq 1$ we have:

$$\varphi(y, U) = \sum_{y', \sigma'} \varphi(y', \sigma') \mathcal{K}_2((y', \sigma'), (y, U))$$

since

$$\begin{aligned} \varphi(y, U) &= \varphi(y-1, U) \mathcal{K}_2((y-1, U), (y, U)) + \varphi(y+1, U) \mathcal{K}_2((y+1, U), (y, U)) \\ &\quad + \varphi(y, D) \mathcal{K}_2((y, D), (y, U)) + \varphi(y, U) \mathcal{K}_2((y, U), (y, U)), \end{aligned}$$

$$\varphi(y, U) = \varphi(y-1, U) \lambda' + \varphi(y+1, U) \mu' + \varphi(y, D) \beta' + \varphi(y, U) (1 - (\lambda' + \mu' + \alpha')),$$

$$\varphi(y, U) (\lambda' + \mu' + \alpha') = \varphi(y-1, U) \lambda' + \varphi(y+1, U) \mu' + \varphi(y, D) \beta',$$

$$B \cdot \left(\frac{\lambda'}{\mu'}\right)^y \frac{\beta'}{\alpha' + \beta'} (\lambda' + \mu' + \alpha') = B \cdot \left(\frac{\lambda'}{\mu'}\right)^{y-1} \frac{\beta'}{\alpha' + \beta'} \lambda' + B \cdot \left(\frac{\lambda'}{\mu'}\right)^{y+1} \frac{\beta'}{\alpha' + \beta'} \mu' + B \cdot \left(\frac{\lambda'}{\mu'}\right)^y \frac{\alpha'}{\alpha' + \beta'} \beta',$$

$$\beta' (\lambda' + \mu' + \alpha') = \frac{\mu'}{\lambda'} \beta' \lambda' + \frac{\lambda'}{\mu'} \beta' \mu' + \alpha' \beta',$$

$$\beta' (\lambda' + \mu' + \alpha') = \beta' (\lambda' + \mu' + \alpha').$$

Next we have to compute the stationary horizontal drift of the twisted free process:

$$\tilde{d}^{(2)} =$$

$$\varphi(0, U) [0 - \mathcal{K}((0, y, U), (-1, y, U))] + \sum_{y=1}^{\infty} \varphi(y, U) [\mathcal{K}((0, y, U), (1, y-1, U)) - \mathcal{K}((0, y, U), (-1, y, U))]$$

$$+ \sum_{y=1}^{\infty} \varphi(y, D) [\mathcal{K}((0, y, D), (1, y-1, D))]$$

$$\begin{aligned} \varphi(0, U) \left[0 - \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \right] + \left[\frac{\mu}{C} - \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \right] \sum_{y=1}^{\infty} \varphi(y, U) + \frac{\mu}{C} \sum_{y=1}^{\infty} \varphi(y, D) \\ = \frac{\mu}{C} \left(\sum_{y=1}^{\infty} \varphi(y, U) + \sum_{y=1}^{\infty} \varphi(y, D) \right) - \frac{1}{C} \frac{2\lambda\mu}{\lambda + \beta + \mu + \alpha - \sqrt{s_1}} \sum_{y=0}^{\infty} \varphi(y, U). \end{aligned}$$

$$\text{We have } \sum_{y=1}^{\infty} \varphi(y, U) + \sum_{y=1}^{\infty} \varphi(y, D) = 1 - \varphi(0, U) - \varphi(0, D) = 1 - B \cdot \frac{\beta'}{\alpha' + \beta'} - B \cdot \frac{\alpha'}{\alpha' + \beta'} = 1 - B = \frac{\lambda'}{\mu'}$$

and $\sum_{y=0}^{\infty} \varphi(y, U) = \frac{\beta'}{\alpha' + \beta'}$. Using the definitions of α' and β' we arrive finally at

$$\tilde{d}^{(2)} = \frac{1}{C} \left(\frac{\lambda + \mu + \beta + \alpha - \sqrt{s_1}}{2} - \frac{2\lambda\mu(\lambda + \beta - \mu - \alpha + \sqrt{s_1})^2}{(\lambda + \beta + \mu + \alpha - \sqrt{s_1})(4\alpha\beta + (\lambda + \beta - \mu - \alpha + \sqrt{s_1})^2)} \right). \quad (12)$$

Now we make use of the Proposition 3.2. We postpone verifying the condition (5) to Section 3.6.2. In our case $A = \{(y, \sigma), y \in \mathbb{N}, \sigma \in \{U, D\}\}$.

For $\sigma = U$ we have

$$\pi(k, y, Up) \sim \frac{\eta^{(2)}\varphi(y, Up)}{\tilde{d}^{(2)}h(k, y, Up)} = \frac{\eta^{(2)}}{\tilde{d}^{(2)}} B\left(\frac{\lambda'}{\mu'}\right)^y \frac{\beta'}{\alpha' + \beta'} \gamma_1^k \gamma_1^y = \frac{\eta^{(2)}}{\tilde{d}^{(2)}} B\left(\frac{\lambda'}{\mu'}\gamma_1\right)^y \frac{\beta'}{\alpha' + \beta'} \gamma_1^k.$$

Noting that $\frac{\lambda'}{\mu'}\gamma_1 = \frac{\lambda}{\mu}$ and $G = \frac{1}{C}(\alpha' + \beta')$ we have

$$\pi(k, y, Up) \sim \frac{\eta^{(2)}}{\tilde{d}^{(2)}} \frac{1}{G} \frac{\lambda + \beta - \mu - \alpha + \sqrt{s_1}}{2} B\left(\frac{\lambda}{\mu}\right)^y \gamma_1^y = C^{(2)}(Up) \left(\frac{\lambda}{\mu}\right)^y \gamma_1^y.$$

Similarly for $\sigma = D$ we have

$$\pi(k, y, Down) = \frac{\eta^{(2)}\varphi(y, Down)}{\tilde{d}^{(2)}h(k, y, Down)} = \frac{\eta^{(2)}}{\tilde{d}^{(2)}} \frac{\alpha}{G} B\left(\frac{\lambda}{\mu}\right)^y \gamma_1^k = C^{(2)}(Down) \left(\frac{\lambda}{\mu}\right)^y \gamma_1^k.$$

3.6.2 Verification of the assumption of Proposition 3.2.

For Propositions 2.5 and 2.4 to hold, condition (5) must be verified. We show this for a general $p \in (0, 1]$. We consider similar network to Model 2, but we do not allow a customer to join the queue at server 1 when the server is in *Down* status; in this case customer is rerouted again to the queue at server 1. This is a case of unreliable network with rerouting (“the loss regime”, customer is lost to server in *Down* status, but it is not lost to the network) introduced by Sauer and Daduna (see Sauer, Daduna [11] or Sauer [10]). Namely, when server is in *Up* status it operates as classical Jackson network, but when it is in *Down* status the routing is changed, so that with probability 1 customer stays at server 2. This is so-called RS-RD (Random Selection - Random Destination) principle for rerouting. They showed, that then the stationary distribution (say $\pi^{(S)}$) is a product form of the stationary distribution of pure Jackson network and of the stationary distribution of being in *Up* or *Down* status. For the above introduced system, the traffic equation is:

$$\eta_1 = \eta_2, \quad \eta_2 = \lambda + \eta_1(1 - p).$$

The solution is $\eta_1 = \eta_2 = \frac{\lambda}{p}$. Finally,

$$\pi^{(S)}(x, y, Up) = C^{(S)} \cdot \left(\frac{\lambda}{\mu p}\right)^{x+y} \frac{\beta}{\alpha + \beta}, \quad \pi^{(S)}(x, y, Down) = C^{(S)} \cdot \left(\frac{\lambda}{\mu p}\right)^{x+y} \frac{\alpha}{\alpha + \beta},$$

where $C^{(S)}$ is a normalisation constant. It also can be checked directly, that the above is the correct stationary distribution, by checking that balance equation holds.

Described network differs from Model 2 only by one movement: for $x > 0$ and $y > 0$ there is a possible transition from $(x, y, Down)$ to $(x + 1, y - 1, Down)$ for Model 2, but there is no such transition in the above model. Obviously, the stationary distribution $\pi^{(S)}(0, \cdot, \sigma)$ is stochastically greater than $\pi(0, \cdot, \sigma)$, the stationary distribution of Model 2. This can be seen for example by constructing a coupling such that both networks move in the same way, whenever it is possible (when one of the processes is to make forbidden transition - like leaving the state space - it makes no move then), except one transition: when process of Model 2 goes from $(x, y, Down)$ to $(x + 1, y - 1, Down)$, then the other makes no move.

Now, for Model 2 as boundary we have $\Delta = \{(0, y, \sigma) : y \in \mathbf{N}, \sigma \in \{Up, Down\}\}$ and the harmonic function (given in Lemma 3.4) is $h(x, y, \sigma) = C(\sigma) \left(\frac{1}{\gamma_p}\right)^{x+y}$. In the condition (5) we have:

$$\sum_{(x,A) \in \Delta} \pi(x, A)h(x, A) = \sum_{y=0}^{\infty} \pi(0, y, U)h(0, y, U) + \sum_{y=0}^{\infty} \pi(0, y, D)h(0, y, D)$$

$$=: E_\pi[h(0, Y, U)] + E_\pi[h(0, Y, D)] \leq E_\pi^{(S)}[h(0, Y, U)] + E_\pi^{(S)}[h(0, Y, D)]$$

since h is increasing w.r.t. second coordinate and

$$\pi(0, \cdot, \sigma) <_{st} \pi^{(S)}(0, \cdot, \sigma), \quad \sigma \in \{Up, Down\}.$$

And for $\pi^{(S)}$ we have

$$E_\pi^{(S)}[h(0, Y, U)] + E_\pi^{(S)}[h(0, Y, D)] = \sum_{y=0}^{\infty} c_1 \left(\frac{\lambda}{\mu p}\right)^y \left(\frac{1}{\gamma p}\right)^y + \sum_{y=0}^{\infty} c_2 \left(\frac{\lambda}{\mu p}\right)^y \left(\frac{1}{\gamma p}\right)^y$$

with appropriate constants c_1 and c_2 . Of course it is finite if $\frac{\lambda}{\mu p} < \gamma p$. It can easily be checked, that it holds for any set of parameters. Thus, the condition (5) holds.

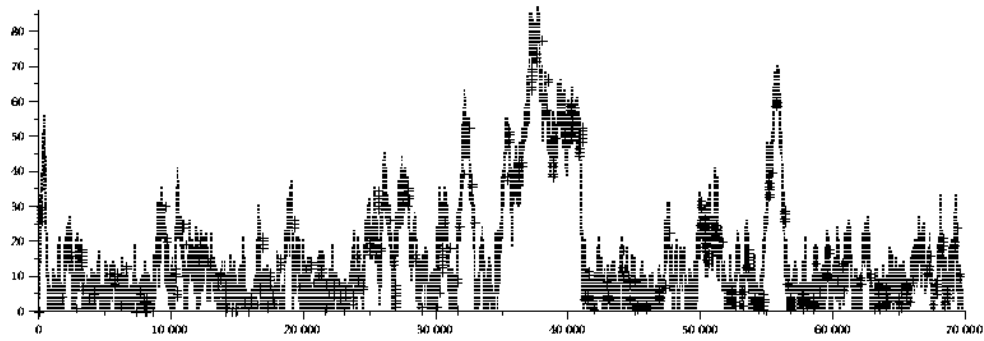
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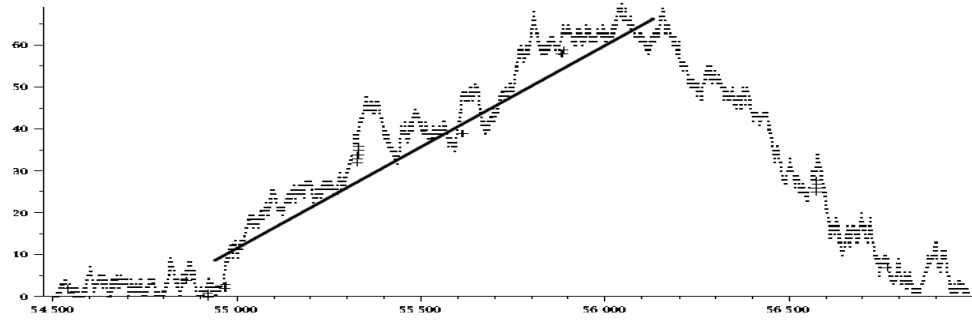
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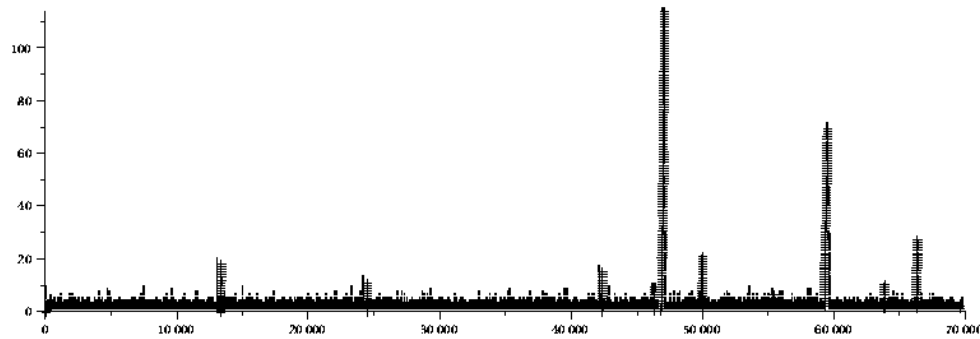


Steps : from 0 to 70000

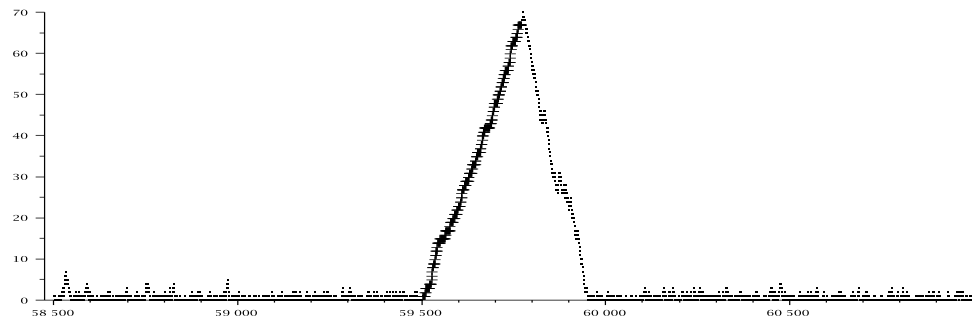


Steps : from 54550 to 57000

a) $\mu < \lambda + \beta$: $\alpha = 0.1, \beta = 10, \lambda = 10, \mu = 11$.



Steps : from 0 to 70000



Steps : from 58500 to 61000

b) $\mu > \lambda + \beta$: $\alpha = 0.01, \beta = 1, \lambda = 20, \mu = 60$.

Figure 3: Two different large deviation paths