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FINITE TIME ASYMPTOTICS OF FLUID AND RUIN MODELS: MULTIPLEXED FRACTIONAL BROWNIAN MOTIONS CASE

Abstract. Motivated by applications in queueing fluid models and ruin theory, we analyze the asymptotics of

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^{n}\lambda_i B_{H_i}(t)-ct\Big)>u\Big),$$

where $\{B_{H_i}(t) : t \geq 0\}$, $i = 1,\dots,n$, are independent fractional Brownian motions with Hurst parameters $H_i \in (0,1]$ and $\lambda_1,\dots,\lambda_n > 0$. The asymptotics takes one of three different qualitative forms, depending on the value of $\min_{i=1,\dots,n} H_i$.

1. Introduction. Let $\{B_{H_i}(t) : t \geq 0\}$, $i = 1,\dots,n$, be independent fractional Brownian motions with Hurst parameters $H_i \in (0,1]$, i.e. centered Gaussian processes with stationary increments, continuous sample paths a.s., and variance functions $\sigma^2_{H_i}(t) = t^{2H_i}$, $i = 1,\dots,n$.

This paper focuses on the analysis of the tail distribution of

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^{n}\lambda_i B_{H_i}(t)-ct\Big)>u\Big), \tag{1}$$

with $\lambda_1,\dots,\lambda_n > 0$. Apart from theoretical interest in (1), our motivation comes from applications of (1) to some problems arising in:

• *Gaussian queueing models.* A vast literature on analysis of traffic in large communication networks focuses on models where the traffic is assumed to be a Gaussian process. There are at least two reasons why Gaussian processes are an appropriate choice here. On the one hand, the class of

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Gaussian processes delivers a broad range of correlation structures, which is convenient from the modeling point of view. On the other (theoretic-level) hand, it has been proven that under *heavy traffic parameterization*, a large number of i.i.d. 0-1 alternating renewal processes (regarded as a natural model of input to the network) can be approximated by a Gaussian process; see, e.g., [3, 8, 10]. Importantly, the statistical measurements that showed the presence of *long-range dependence* and *self-similarity* of the traffic, turned the attention of researchers to the class of *fractional Brownian motions*, [11]. Let us consider a fluid queue with infinite buffer capacity, with the accumulated input over the time interval $[0,t)$ modeled by superposition of a number of independent fractional Brownian motions $\sum_{i=1}^n \lambda_i B_{H_i}(t)$ and drained with a constant rate $c>0$. Let $\{Q(t) : t \geq 0\}$ be the buffer content process. Then, providing that $Q(0)=0$ a.s. and invoking Reich [14], the probability that the *transient* buffer content $Q(T)$ at time T exceeds a level $u>0$ equals (1). The *steady-state* analog of the above problem, i.e. the asymptotics of $\mathbb{P}(\sup_{t\geq 0}(\sum_{i=1}^n \lambda_i B_{H_i}(t) - ct) > u)$, was analyzed in, e.g., [15, 5]. We refer to [11, 6, 4] and references therein for a selection of works that deal with the case of a single fractional Brownian motion source.

• *Ruin models.* The tail probability (1) has a natural interpretation in the context of ruin problems. Using the fact that $\{B_H(t)\} =_{\mathrm{d}} \{-B_H(t)\}$, (1) can be rewritten as the finite-time ruin probability

$$\mathbb{P}\Big(\inf_{t\in[0,T]}\Big(u+ct-\sum_{i=1}^n \lambda_i B_{H_i}(t)\Big)<0\Big)$$

for the ruin model with claims modeled by $\sum_{i=1}^n \lambda_i B_{H_i}(t)$, with initial capital u and premium rate c. We refer to [9] for the limit-theoretic model that justifies approximation of the claims by fractional Brownian motion.

Contribution. The aim of this paper is to give the exact asymptotics of (1) as $u\to\infty$. It appears that the asymptotics takes one of three different quantitative forms, depending on the value of $\min_{i=1,\ldots,n} H_i$. Additionally, under the condition that $\min_{i=1,\ldots,n} H_i \geq 1/2$ (i.e. the increments of fractional Brownian motions are nonnegatively correlated) we obtain uniform upper and lower bounds for (1), which (up to a constant) are asymptotically consistent.

Notation. Let $\Psi(u)=\mathbb{P}(\mathcal{N}>u)$, where $\mathcal{N}$ denotes the standard normal random variable. Pickands's constants $\mathcal{H}_H$, which appear in the exact asymptotics, are defined by the following limit:

$$\mathcal{H}_H := \lim_{T\to\infty}\frac{\mathbb{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}B_H(t)-t^{2H}))}{T}.$$

We refer to [12] for the analysis of the properties of $\mathcal{H}_H$.

Organization. The main results of the paper are presented in Section 2. The proofs are deferred to Section 3.

2. Main results. In this section we provide the asymptotics and estimates for (1). Since for given $H_1 = H_2 = H$ we have $\lambda_1 B_{H_1}(t)+\lambda_2 B_{H_2}(t) =_{\mathrm{d}} \sqrt{\lambda_1^2+\lambda_2^2}\, B_H(t)$, we assume that

$$H_1 < \cdots < H_n.$$

In the following theorem we give the exact asymptotics of (1).

THEOREM 2.1. *Let $\{B_{H_i}(t) : t \geq 0\}$, $i = 1, \ldots, n$, be independent fractional Brownian motions and let $\lambda_i > 0$, $i = 1, \ldots, n$.*

(i) *If $H_1 < 1/2$, then as $u \to \infty$,*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^n \lambda_i B_{H_i}(t) - ct\Big) > u\Big) = \mathcal{H}_{H_1}\bigg(\frac{u+cT}{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}\bigg)^{(1-2H_1)/H_1} \frac{[\lambda_1^2/2]^{1/2H_1}}{\sum_{i=1}^n H_i \lambda_i^2 T^{2H_i-1}} \times \Psi\bigg(\frac{u+cT}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\bigg)(1+o(1)).$$

(ii) *If $H_1 = 1/2$, then as $u \to \infty$,*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^n \lambda_i B_{H_i}(t) - ct\Big) > u\Big) = \bigg[1+\frac{\lambda_1^2/2}{\lambda_1^2/2+\sum_{i=2}^n H_i\lambda_i^2 T^{2H_i-1}}\bigg]\Psi\bigg(\frac{u+cT}{\sqrt{\lambda_1^2 T+\sum_{i=2}^n \lambda_i^2 T^{2H_i}}}\bigg)(1+o(1)).$$

(iii) *If $H_1 > 1/2$, then as $u \to \infty$,*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^n \lambda_i B_{H_i}(t) - ct\Big) > u\Big) = \Psi\bigg(\frac{u+cT}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\bigg)(1+o(1)).$$

The proof of Theorem 2.1 is given in Section 3.

REMARK 2.2. Theorem 2.1 generalizes the results of [9] and [4] where models with a single fractional Brownian motion ($n = 1$) were considered.

REMARK 2.3. The qualitative type of the asymptotics obtained in Theorem 2.1 differs from the one for an infinite time horizon. In particular in [15] it was proved that if $2H_2 > 1 + H_1$, then

$$\mathbb{P}\Big(\sup_{t\in[0,\infty)}(B_{H_1}(t)+B_{H_2}(t)-ct) > u\Big) = \mathbb{P}\Big(\sup_{t\in[0,\infty)}(B_{H_2}(t)-ct) > u\Big)(1+o(1))$$

as $u \to \infty$. From Theorem 2.1 one can observe that this is not the case for a finite time horizon, where each process contributes to the asymptotics.

In the following theorem we present an upper and a lower estimate for (1).

THEOREM 2.4. *Let $\{B_{H_i}(t) : t \geq 0\}$, $i = 1, \dots, n$, be independent fractional Brownian motions and let $\lambda_i > 0$, $i = 1, \dots, n$.*

(i) *For each $T, u \geq 0$,*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^{n} \lambda_i B_{H_i}(t) - ct\Big) > u\Big) \geq \Psi\bigg(\frac{u + cT}{\sqrt{\sum_{i=1}^{n} \lambda_i^2 T^{2H_i}}}\bigg).$$

(ii) *If $H_1 \geq 1/2$, then for each $T, u \geq 0$,*

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\Big(\sum_{i=1}^{n} \lambda_i B_{H_i}(t) - ct\Big) > u\Big)$$
$$\leq \Psi\bigg(\frac{u + cT}{\sqrt{\sum_{i=1}^{n} \lambda_i^2 T^{2H_i}}}\bigg) + \exp\bigg(\frac{-2cTu}{\sum_{i=1}^{n} \lambda_i^2 T^{2H_i}}\bigg)\Psi\bigg(\frac{u - cT}{\sqrt{\sum_{i=1}^{n} \lambda_i^2 T^{2H_i}}}\bigg).$$

The proof of Theorem 2.4 is given in Section 3.

REMARK 2.5. If $H_1 > 1/2$, then the estimates in Theorem 2.4 are asymptotically consistent, up to a constant of 2, with the asymptotics of Theorem 2.1. The lower bound (i) is asymptotically exact in this case.

3. Proofs. To prove Theorem 2.1 we introduce some notation. Let

$$\widetilde{B}(t) = \sum_{i=1}^{n} \lambda_i B_{H_i}(t).$$

Note that $\widetilde{B}(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_{\widetilde{B}}^2(t) = \sum_{i=1}^{n} \lambda_i^2 t^{2H_i}$. A bar will always indicate a standardized process, that is, $\overline{X}(t) := X(t)/\sigma_X(t)$ for some Gaussian process $X(t)$. Let

$$m_u(t) := \frac{u + ct}{\sigma_{\widetilde{B}}(t)} \quad \text{and} \quad \mathcal{F}_{\alpha}^{R} := \lim_{S\to\infty} \mathbb{E}\exp\Big(\sup_{t\in[0,S]} (B_{\alpha/2}(t) - (1 + R)t^{\alpha})\Big)$$

for $\alpha \in (0, 2]$ and $R > 0$.

The proof of Theorem 2.1 is based on an appropriate use of Theorem 1 of Piterbarg and Prisyazhnyuk [13] (see also Theorem 2.2 of Konstant and Piterbarg [7]), which we present in a form suitable for our application.

THEOREM 3.1. *Let $(\xi(t))_{t\in[0,T]}$ be a centered Gaussian process with continuous sample paths a.s. and variance function $\sigma_{\xi}^2(\cdot)$ such that the maximum* *of $\sigma_\xi(\cdot)$ on $[T/2,T]$ is attained at the unique point $t=T$ with $\sigma_\xi(T)=1$. Assume that:*

(a) *there exist $A,\beta>0$ such that*
$$\sigma_\xi(t)=1-A|T-t|^\beta(1+o(1)) \quad \text{as } t\to T;$$

(b) *there exist $D,\alpha>0$ such that*
$$1-\mathbb{C}\mathrm{ov}(\bar{\xi}(t),\bar{\xi}(s))=D|t-s|^\alpha+o(|t-s|^\alpha) \quad \text{as } s,t\to T;$$

(c) *there exist $C,\alpha_1>0$ such that, for $s,t\in[T/2,T]$,*
$$\mathbb{E}(\xi(t)-\xi(s))^2\le C|t-s|^{\alpha_1}.$$

Then:

(i) *for $\beta>\alpha$ and $\mathcal{G}_{\alpha,\beta}:=\mathcal{H}_{\alpha/2}\Gamma(1/\beta)D^{1/\alpha}\beta^{-1}A^{-1/\beta}$, as $u\to\infty$,*
$$\mathbb{P}\Big(\sup_{t\in[T/2,T]}\xi(t)>u\Big)=\mathcal{G}_{\alpha,\beta}u^{2/\alpha-2/\beta}\Psi(u)(1+o(1));$$

(ii) *for $\beta=\alpha$ and $R:=A/D$, as $u\to\infty$,*
$$\mathbb{P}\Big(\sup_{t\in[T/2,T]}\xi(t)>u\Big)=\mathcal{F}_\alpha^R\Psi(u)(1+o(1));$$

(iii) *for $\beta<\alpha$, as $u\to\infty$,*
$$\mathbb{P}\Big(\sup_{t\in[T/2,T]}\xi(t)>u\Big)\sim\Psi(u).$$

3.1. Proof of Theorem 2.1. Observe that
$$\mathbb{P}\Big(\sup_{t\in[0,T]}(\widetilde{B}(t)-ct)>u\Big)\ge\pi(u),$$
$$\mathbb{P}\Big(\sup_{t\in[0,T]}(\widetilde{B}(t)-ct)>u\Big)\le\mathbb{P}\Big(\sup_{t\in[0,T/2]}(\widetilde{B}(t)-ct)>u\Big)+\pi(u),$$

where

(2)
$$\begin{aligned}\pi(u)&:=\mathbb{P}\Big(\sup_{t\in[T/2,T]}(\widetilde{B}(t)-ct)>u\Big)\\&=\mathbb{P}\Big(\sup_{t\in[T/2,T]}\overline{\widetilde{B}}(t)\frac{m_u(T)}{m_u(t)}>m_u(T)\Big).\end{aligned}$$

Since
$$1-\frac{m_u(T)}{m_u(t)}=\frac{\sigma_{\widetilde{B}}(T)-\sigma_{\widetilde{B}}(t)}{\sigma_{\widetilde{B}}(T)}+\frac{\sigma_{\widetilde{B}}(t)c(t-T)}{(u+ct)\sigma_{\widetilde{B}}(T)},$$

for each $\varepsilon>0$ and $t\in[T/2,T]$ we have

(3)
$$1-\frac{\sigma_{\widetilde{B}}(T)-\sigma_{\widetilde{B}}(t)}{\sigma_{\widetilde{B}}(T)}\le\frac{m_u(T)}{m_u(t)}\le1-(1-\varepsilon)\frac{\sigma_{\widetilde{B}}(T)-\sigma_{\widetilde{B}}(t)}{\sigma_{\widetilde{B}}(T)}$$

for u sufficiently large. Let

$$X_\varepsilon(t) := \overline{\widetilde{B}}(t)\left(1-(1-\varepsilon)\frac{\sigma_{\widetilde{B}}(T)-\sigma_{\widetilde{B}}(t)}{\sigma_{\widetilde{B}}(T)}\right)$$

for $\varepsilon \in [0,1)$. Then, from (3), for each $\varepsilon \in (0,1)$ and u sufficiently large,

$$\pi(u) \le \pi_1(u) := \mathbb{P}\Big(\sup_{t\in[T/2,T]} X_\varepsilon(t) > m_u(T)\Big),$$
$$\pi(u) \ge \pi_2(u) := \mathbb{P}\Big(\sup_{t\in[T/2,T]} X_0(t) > m_u(T)\Big).$$

Let us focus on the analysis of $\pi_1(u)$. Let $\varepsilon \in (0,1)$. Then $\sigma_{X_\varepsilon}(t)$ attains its unique maximum over $[T/2,T]$ at $t=T$, with $\sigma_{X_\varepsilon}(T)=1$. Moreover

$$\begin{aligned}\sigma_{X_\varepsilon}(t) &= 1-(1-\varepsilon)\frac{\sigma_{\widetilde{B}}(T)-\sigma_{\widetilde{B}}(t)}{\sigma_{\widetilde{B}}(T)}\\ &= 1-(1-\varepsilon)\frac{\sum_{i=1}^n H_i\lambda_i^2T^{2H_i-1}}{\sum_{i=1}^n \lambda_i^2T^{2H_i}}|T-t|+o(|T-t|)\end{aligned}$$

as $t\uparrow T$, and

$$\begin{aligned}\mathbb{C}\mathrm{ov}(\overline{X_\varepsilon}(s),\overline{X_\varepsilon}(t)) &= \mathbb{C}\mathrm{ov}(\overline{\widetilde{B}}(s),\overline{\widetilde{B}}(t))\\ &= 1-\frac{1}{2}\left[\frac{\lambda_1^2}{\sum_{i=1}^n \lambda_i^2T^{2H_i}}\right]|s-t|^{2H_1}+o(|s-t|^{2H_1})\end{aligned}$$

as $s,t\to T$, and

$$\begin{aligned}\mathbb{E}(X_\varepsilon(s)-X_\varepsilon(t))^2 &= \mathbb{E}\left(\varepsilon(\overline{B}(s)-\overline{B}(t))+\frac{1-\varepsilon}{\sigma_{\widetilde{B}}(T)}(\widetilde{B}(s)-\widetilde{B}(t))\right)^2\\ &\le 2\varepsilon\mathbb{E}(\overline{B}(s)-\overline{B}(t))^2+\frac{2(1-\varepsilon)^2}{\sigma^2_{\widetilde{B}}(T)}\mathbb{E}(\widetilde{B}(s)-\widetilde{B}(t))^2\\ &\le \left(\frac{8\varepsilon^2\sigma^2_{\widetilde{B}}(T)}{\sigma^2_{\widetilde{B}}(T/2)}+\frac{2(1-\varepsilon)^2}{\sigma^2_{\widetilde{B}}(T)}\right)\mathbb{E}(\widetilde{B}(s)-\widetilde{B}(t))^2\\ &\le C|s-t|^{2H_1}\end{aligned}$$

for $s,t\in[T/2,T]$ and some positive constant C. Thus the process $X_\varepsilon(t)$ satisfies the conditions of Theorem 3.1 with

$$A=(1-\varepsilon)\frac{\sum_{i=1}^n H_i\lambda_i^2T^{2H_i-1}}{\sum_{i=1}^n \lambda_i^2T^{2H_i}},\qquad D=\frac{1}{2}\left[\frac{\lambda_1^2}{\sum_{i=1}^n \lambda_i^2T^{2H_i}}\right],$$

$\alpha=2H_1$ and $\beta=1$, which straightforwardly implies that

(i) if $H_1 < 1/2$, then as $u \to \infty$,

$$\pi_1(u) = (1-\varepsilon)^{-1}\mathcal{H}_{H_1}\left(\frac{u+cT}{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}\right)^{(1-2H_1)/H_1} \frac{[\lambda_1^2/2]^{1/2H_1}}{\sum_{i=1}^n H_i\lambda_i^2 T^{2H_i-1}} \times \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\right)(1+o(1));$$

(ii) if $H_1 = 1/2$, then as $u \to \infty$,

$$\pi_1(u) = \left[1 + \frac{\lambda_1^2/2}{(1-\varepsilon)(\lambda_1^2/2 + \sum_{i=2}^n H_i\lambda_i^2 T^{2H_i-1})}\right] \times \Psi\left(\frac{u+cT}{\sqrt{\lambda_1^2 T + \sum_{i=2}^n \lambda_i^2 T^{2H_i}}}\right)(1+o(1)); \tag{4}$$

(iii) if $H_1 > 1/2$, then as $u \to \infty$,

$$\pi_1(u) = \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\right)(1+o(1)).$$

In (4) we used the fact that $\mathbb{E}\exp(\sup_{t\in[0,\infty)} \sqrt{2}B_{1/2}(t)-(1+b)t) = (1+b)/b$ for $b > 0$, which directly follows from the distribution of $\sup_{t\in[0,\infty)} \sqrt{2}B_{1/2}(t) - (1+b)t)$ being exponential with parameter $1+b$.

Hence, letting $\varepsilon \to 0$, we get the asymptotic upper bound for $\pi(u)$ which is consistent with the conclusion of Theorem 2.1.

For $\pi_2(u)$ the argument is the same and thus we omit it.

Finally we observe that due to Borell's inequality (see, e.g., Adler [1]), for some constant C_1,

$$\mathbb{P}\Big(\sup_{t\in[0,T/2]} (\widetilde{B}(t) - ct) > u\Big) \le 2\exp\left(-\frac{(u+C_1)^2}{2\sigma^2_{\widetilde{B}}(T/2)}\right) = o(\pi(u))$$

as $u \to \infty$. This completes the proof of Theorem 2.1. ■

3.2. Proof of Theorem 2.4. (i) It suffices to observe that for each u, T we have

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \Big(\sum_{i=1}^n \lambda_i B_{H_i}(t) - ct\Big) > u\Big) \ge \mathbb{P}\Big(\sum_{i=1}^n \lambda_i B_{H_i}(T) - cT > u\Big) = \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n \lambda_i^2 T^{2H_i}}}\right).$$

This completes the proof of (i).

(ii) Define a Gaussian process $\{Y(t) : t \geq 0\}$ by

$$Y(t) = B_{1/2}\Big(\sum_{i=1}^{n} \lambda_i^2 t^{2H_i}\Big),$$

where $\{B_{1/2}(t) : t \geq 0\}$ is a standard Brownian motion. We have

$$E[Y(t)] = 0 = E[\widetilde{B}(t)], \qquad \sigma_Y^2(t) = \sum_{i=1}^{n} \lambda_i^2 t^{2H_i} = \sigma_{\widetilde{B}}^2(t).$$

Since $H_1 \geq 1/2$, $\sigma_{\widetilde{B}}^2(t)$ is convex. Thus for $0 \leq s \leq t$ we have

$$\begin{aligned}
\mathbb{E}[\widetilde{B}(s)\widetilde{B}(t)] &= \mathbb{E}\Big[\Big(\sum_{i=1}^{n} \lambda_i B_{H_i}(s)\Big)\Big(\sum_{i=1}^{n} \lambda_i B_{H_i}(t)\Big)\Big] \\
&= \sum_{i=1}^{n} \lambda_i^2 E[B_{H_i}(s) B_{H_i}(t)] = \sum_{i=1}^{n} \frac{\lambda_i^2}{2}[s^{2H_i} + t^{2H_i} - (t-s)^{2H_i}] \\
&\geq \sum_{i=1}^{n} \lambda_i^2 s^{2H_i} = E[Y(s)Y(t)].
\end{aligned}$$

Hence, in view of Slepian's inequality (see, e.g., Adler [1]), we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \Big(\sum_{i=1}^{n} \lambda_i B_{H_i}(t) - ct\Big) > u\Big) \\
&\quad \leq \mathbb{P}\Big(\sup_{t\in[0,T]} (Y(t) - ct) > u\Big) = \mathbb{P}\Big(\sup_{t\in[0,T]} (B_{1/2}(\sigma_Y^2(t)) - ct) > u\Big) \\
&\quad = \mathbb{P}\Big(\sup_{t\in[0,\sigma_{\widetilde{B}}^2(T)]} (B_{1/2}(t) - c(\sigma_{\widetilde{B}}^2)^{-1}(t)) > u\Big),
\end{aligned}$$

where $(\sigma_{\widetilde{B}}^2)^{-1}(t)$ is the inverse function of $\sigma_{\widetilde{B}}^2(t)$. Since $(\sigma_{\widetilde{B}}^2)^{-1}(t)$ is concave, we have $(\sigma_{\widetilde{B}}^2)^{-1}(t) \geq (T/\sigma_{\widetilde{B}}^2(T))t$ for $t \in [0, \sigma_{\widetilde{B}}^2(T)]$, which implies

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,\sigma_{\widetilde{B}}^2(T)]} (B_{1/2}(t) - c(\sigma_{\widetilde{B}}^2)^{-1}(t)) > u\Big) \\
&\qquad\qquad \leq \mathbb{P}\Big(\sup_{t\in[0,\sigma_{\widetilde{B}}^2(T)]} \Big(B_{1/2}(t) - c\frac{T}{\sigma_{\widetilde{B}}^2(T)}t\Big) > u\Big).
\end{aligned}$$

Finally, using the formula for the distribution of $\sup_{t\in[0,T]}(B_{1/2}(t) - At)$ (see

Baxter and Donsker [2]), we have

$$\mathbb{P}\Big(\sup_{t\in[0,\sigma^2_{\widetilde{B}}(T)]}\Big(B_{1/2}(t)-c\frac{T}{\sigma^2_{\widetilde{B}}(T)}t\Big)>u\Big) = \Psi\Big(\frac{u+cT}{\sigma_{\widetilde{B}}(T)}\Big)+\exp\Big(-2c\frac{T}{\sigma^2_{\widetilde{B}}(T)}u\Big)\Psi\Big(\frac{u-cT}{\sigma_{\widetilde{B}}(T)}\Big).$$

This completes the proof of (ii). ■

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