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Report on the doctoral dissertation of Zeineb Ghardallou
"Positive Solutions to some Nonlinear Elliptic Problems"
"Dodatnie rozwiązania pewnych nieliniowych problemów eliptycznych"

The dissertation, supervised by Professor Ewa Damek and Professor Mohamed Sifi, has about 150 pages divided into Introduction, 7 Chapters, and Bibliography. It concerns the existence, regularity and growth properties of nonnegative continuous distributional solutions u of the semilinear elliptic equation

$$Lu(x) = \pm\varphi(x, u(x)), \qquad x \in \Omega \subset \mathbb{R}^d, \quad d \geqslant 3.$$

Here L is a second order elliptic operator with smooth coefficients defined in a Greenian, possibly unbounded, domain Ω, $L1 \leqslant 0$, $\varphi \geqslant 0$, and $u \mapsto \pm\varphi(x, u)$ (on the right-hand-side) is generally assumed to be nondecreasing. Kato conditions on $x \mapsto \varphi(x, u)$ and specific limiting behavior of u at $\partial\Omega$ are also assumed as a rule. In geometrically regular relatively compact subdomains D of Ω the solution satisfies the implicit relation

$$u(x) = H_D f(x) \mp G_D(\varphi(\cdot, u(\cdot)))(x),$$

where $f = u \geqslant 0$ on ∂D, H_D is the integration with respect to the harmonic measure and G_D is the Green operator of L for D. The result is discussed in Chapter 3, after intruductory Chapter 1 devoted to the maximum principle and the Green function, and after Chapter 2 on the Kato condition.

General Greenian domains Ω (i.e. those possessing finite Green function) are discusses in Chapter 4. Here of particular interest is the situation when the boundary limits of u are zero at $\partial\Omega$ but $u > 0$. The choice of the sign of the right-hand side of the semilinear equation has much influence on the arguments, as the two cases have distinctive features, depending on the sub- and super-harmonicity of the solutions u and on the corresponding comparison principles. Chapter 5 gives Harnack inequality for the solutions under conditions of sublinear growth of $u \mapsto \varphi(x, u)$. Chapter 6 describes the Green function of the Laplace-Beltrami operator on NA groups for the whole space, which may be considered as a special case of the subject of the dissertation. When $x \mapsto \varphi(x, u)$ is radial, Chapter 7 gives conditions on the existence of bounded and unbounded radial solutions in the whole space for $\Omega = \mathbb{R}^d$ with the Laplacian Δ (recovering the results of El Mabrouk and Hansen, 2007) and for $\Omega = NA$ with the Laplace-Beltrami operator. The thesis also contains a comprehensive bibliography of the subject, and in the Introduction it gives a useful overview of main results.

The methods of the dissertation rely on the Schauder fixed-point theorem (to actually solve for u), the maximum principle for L (to estimate u on D and Ω), and, last but not least, on explicit estimates for the considered Green functions. As such the utilized methods are classical. What distinguishes the paper from other contributions in the field are the broad assumptions on the operator L, the domain Ω, and the function φ. Also, the results on NA groups nicely complement those obtained in the literature for Δ in $\Omega = \mathbb{R}^d$.

The above-mentioned monotonicity of $\pm\varphi$ is the most important structural assumption for the considered semilinear equation, because it allows for the convergence and control of approximations to the solutions. The thesis helps recognize the many available generalizations to other operators, local and nonlocal, conjugate/conditioned operators and non-uniformly elliptic operators, other unbounded domains etc. I expect such progress in near future, and they should mostly depend on the properties of the corresponding Green kernels. The work is impressive in scope and volume, and should be very useful. Part of the results are already published (at Springerlink.com):

Zeineb Ghardallou, Positive Solution to a Nonlinear Elliptic Problem (24 pages). Potential Analysis, 2015, DOI 10.1007/s11118-015-9509-y.

Here are several critical remarks and confusions of minor importance.

1. It would be more economical to build Chapter 3 on the estimates of the Green functions of the regular ($C^{1,1}$) domains given in Theorem 96. For instance the boundary decay of Green potentials in regular domains would follow easily.

2. (3.1.1) is repeated in (3.1.3), Theorem 91 is repeated in Theorem 103, etc.

3. Last line of page 38 contains redundant φ.

4. Lemma 83 needs adding some assumption of integrability/finiteness and its consequences should be examined.

5. I would start Section 5.1 with Lemma 95, which is 3G inequality.

6. Lipschitz condition should be called on page 63.

7. There are frequent minor problems with the English text, e.g. "Let..., then..." rather that the more correct "If..., then..." or "Let.... Then...".

8. There are some problems with the formatting of the references [10], [26], [46], [47], [50].

Coming to the conclusion, I find this direction of research very valuable and interesting. The dissertation indicates the proper approach to semilinear equations, in which Potential Theory is indispensable, that is the approach relying on the properties of the Green functions, like domain monotonicity and reproducing property. When reading the thesis I got convinced by the general architecture of the work and the intention of the author. The results are deep and complete, illustrated with nontrivial examples, and they give further perspectives of development. My evaluation is very positive, in that the dissertation fulfills all the legal and customary requirements for the doctoral thesis in mathematics in Poland, and I recommend proceeding to the doctoral defense.

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