Institut für Mathematische Strukturtheorie

Graz University of Technology

Institut für Mathematische Strukturtheorie (Math C)

Univ.-Prof.
Dipl.-Ing. Dr. Wolfgang Woess

Steyrergasse 30
A-8010 Graz

Tel. +43(0)316 873-7130
Fax +43(0)316 873-4507

woess@TUGraz.at
www.math.tugraz.at/~woess

June 8, 2015

Prof. Dr. Krzystof Dębicki
Mathematical Institute
Wrocław University
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Report on the PhD thesis by Wojciechj CYGAN

This PhD thesis studies certain random processes and their probabilistic properties from a potential theoretic viewpoint and with central use of analytic methods.

The central objects of investigation are *subordinated random walks* on the integer grids $\mathbb{Z}^d$. The "mother" random process is simple random walk S_n, and subordinating means that one considers a sequence of sums τ_n of i.i.d. positive integer random variables (times) and watches the random walk S_{τ_n} whose behaviour will be different from the one of simple random walk, according to the moment conditions satisfied by the increments of τ_n.

In order to apply refined analytic methods, the times τ_n are chosen to be induced by a so-called *Bernstein function* ψ on $\mathbb{R}_+$. These are increasing, infinitely differentiable functions whose derivatives have alternating signs. The transition operator P_ψ of the subordinated random walk is given by $I - P_\psi = \psi(I - P)$.

After Introduction (§1) and Preliminaries (§2), the core of the thesis begins in §3 with a thorough discussion of subordination and properties of Bernstein functions, leading to the result that the ψ-subordinated random walk belongs to the domain of attraction of the symmetric α-stable law when ψ has regular variation of exponent $\alpha/2 < 1$ at 0, one of the new results of this PhD work.

§4 is then devoted to a careful study of the Green function of ψ-subordinated random walk. This is of course linked with renewal theory, and the chapter begins with an analysis of the renewal sequence associated with τ_n. It ends with an asymptotic evaluation of the Green function at infinity under the assumption that the renewal sequence has the *strong renewal property* – another one among the various new contributions comprised in this thesis.

§5 concerns important notions from the Potential theory of transient random walks (Markov chains). A *massive,* or in more probabilistic language *recurrent* subset of the state space is a set that is visited infinitely often with probability 1 from any starting point. For a transient random walk, only infinite sets can be massive, and one would like to understand how "thin" such massive sets may be.

For this purpose, one of the best tools is *capacity.* A test (= criterion) for massiveness using capacity is given in the spirit of Wiener. In brief, the state space is subdivided into annuli of radii between two consecutive powers of 2, the set in question is intersected with those annuli, and Green kernel estimates combined with capacity leads to necessary and sufficient criteria. The corresponding result (Theorem 5.7) follows the spirit of Spitzer's famous book.

The next step is to provide capacity estimates for ψ-subordinated simple random walk on $\mathbb{Z}^d$ by comparing its capacities with those of ψ-subordinated Brownian motion on $\mathbb{Z}^d$, see the – again new – Theorem 5.11 plus Corollary 5.12.

Subsequently, a capacity estimate is given via a Nash-type inequality using Dirichlet forms, and it is shown how such an inequality descends to a subordinated Dirichlet form, resp. the associated "Laplace" operator (Theorem 5.13 plus Corollary 5.14).

The last Chapter 6 then harvests concrete fruits from the previously developed theory. As mentioned above, one wants to see thin infinite sets which are recurrent (massive). A *thorn* in $\mathbb{Z}^d$, resp. $\mathbb{R}^d$ is given by a non-negative, increasing function t on the positive reals, resp integers and consists of all points $(x_1, \ldots, x_d)$ in upper half space such that $\|(x_1, \ldots, x_{d-1})\| \leq t(x_d)$.

Massiveness of thorns is studied here for the α-stable random walks, i.e., those where the Bernstein function is $\psi(s) = s^\alpha$ $(0 < \alpha < 2)$. A first result is that in dimension $d \geq 3$ the thorn is massive when $\limsup t(n)/n > 0$. The – more difficult – Theorem 6.7 presents a sharp criterion in terms of the function t for massivenenss when $\lim t(n)/n = 0$. This also holds in dimension $d = 2$ (Theorem 6.10). In that case, the mother random walk is recurrent, while the α-stable walk is transient for $\alpha < 2$. Nevertheless, the thinnest thorn – with $t \equiv 0$ – is massive if and only if $\alpha \geq 1$ (Proposition 6.9). The next step is to consider thorns in $\mathbb{Z}^2$ where a part of its levels is removed; the maintained levels are at heights indexed by some infinite subset $\mathcal{A}$ of the positive integers. Results on massiveness are derived in terms of the behaviour of the function $l(x) = x/|\mathcal{A} \cap (0, x]|$ as $x \to \infty$, see Proposition 6.11.

At last, the one-dimensional case is studied in the spirit of the last proposition mentioned above. Here, α-stable random walks are studied for $0 < \alpha < 1$ (for larger α one has recurrence). Nice massiveness results are derived for subsets of $\mathbb{N}$, including "convex" sequences, primes, and Leitmann primes.

This is a very substantial thesis containing a variety of very interesting, original new results by Cygan and co-authors. The material is very well organised and – given the high amount of subtle analytical techniques that are applied and displayed – well to read. (Some minor flaws in the English do not affect readability at all.) Having read it thoroughly, I really appreciate the work done by Cygan and the way how it is presented.

The thesis is based on three papers by the author (with Bendikov, resp. Grzywny), two of which have already appeared/are accepted in good international journals. The one with Grzywny seems to be still in progress. On arXiv one also finds two more recent papers by Cygan.

Wojiech Cygan has performed very accurate and substantial work with high professional attitude and ambition. It is without doubt that this thesis merits the highest available mark.

With best regards,

Wolfgang Woess